ISSN 0974 - 9373

Vol. 14 No.4 (2010) **Journal of International Academy of Physical Sciences** pp. 385 - 398

On C-Conformal Change in a Finsler Space

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(Received January 10, 2010)

Abstract: The present paper deals with the differential geometry of C-conformal Finsler spaces. It has been shown that the nature of certain kinds of vector fields cannot be inherited to a C-conformal space from its parent space. Also, it has been proved that the nature of these vector fields on the parent space cannot be retained by the corresponding C-conformal vector field on the C-conformal Finsler space. Results of Bagewadi et al. have been revised in the light of above investigations.

Keywords: C-conformal Finsler space, Parallel vector field, Concurrent vector field, Normalized semi-parallel vector field.

2000 AMS Classification No.: 53B40.

1. Introduction

M. S. Knebelman[1] initiated the conformal theory of Finsler spaces in 1929. Several authors including P. N. Pandey[2,3] discussed groups of conformal transformations in conformally related Finsler spaces. M. Hashiguchi[4] introduced a special type of conformal change called C-conformal change. This conformal change is non-homothetic together with a specific condition called C-condition. C. Shibata and H. Azuma[5] obtained certain tensors which are invariant under a C- conformal change. K. Takano[6], P. N. Pandey [7, 8, 9, 10, 11, 12, 13] and others studied contra, concurrent, special concircular, recurrent, concircular and torse-forming vector fields in Riemannian, non- Riemannian and Finsler spaces. M. Matsumoto and K. Eguchi[14] discussed concurrent vector fields in a Finsler space. M. Kitayama[15] studied a Finsler space admitting a parallel vector field.

S. K. Narasimhamurthy and C. S. Bagewadi[16] considered a C- conformal change between two Finslers paces F^n and $\bar{F}^n$ over the same Manifold and obtained a Lemma which states "if X^i is a parallel vector field on Finsler space F^n, the vector field $\bar{X}^i\,(=e^{-\sigma(x)}X^i)$ is parallel on $\bar{F}^n$ if and only if $\delta^i_j\,\sigma_h\,\bar{X}^h-\sigma_j\,X^j=0$." Using this result they obtained nine theorems in which F^n and $\bar{F}^n$ have been taken of special types. In the present paper, we show that if the vector field X^i is parallel on F^n, $\bar{X}^i (= e^{-\sigma(x)} X^i)$ cannot be parallel on the C-conformal space $\bar{F}^n$. In view of this result, none of the nine theorems proved by Narasimhamurthy and Bagewadi[16] is true. In fact, not only in case of parallel vector field but also in case of concurrent and recurrent vector fields we get similar results. This paper also includes the results which confirm that a C-conformal transformation changes the nature of a vector field, i.e. a parallel vector field on F^n cannot be parallel in C-conformal Finsler space $\bar{F}^n$, a concurrent vector field on F^n can not be concurrent on $\bar{F}^n$ and a recurrent vector field on F^n cannot be recurrent on $\bar{F}^n$.

2. Preliminaries

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with the fundamental function $L(x, y)$ satisfying the requsite conditions[17]. The normalized supporting element, the metric tensor, the angular metric tensor and Cartan tensor are defined by $l_i = \dot{\partial}_i L$, $g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2$, $h_{ij} = L\dot{\partial}_i \dot{\partial}_j L$ and $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ respectively. Throughout the paper, we use the symbols $\dot{\partial}_i$ and ∂_i for $\partial/\partial y^i$ and $\partial/\partial x^i$ respectively.The inverse of the metric tensor and the generalized Christoffel symbols are given by

$$g_{ij} g^{jk} = \delta_i^k, \ \gamma_{ijk} = \frac{1}{2}\{\partial_i g_{jk} + \partial_k g_{ij} - \partial_j g_{ik}\}, \ \gamma^i_{jk} = g^{ir}\gamma_{jrk}.$$

The Cartan connection in F^n is given as $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$, where

$$F^i_{jk} = \frac{1}{2} g^{ip}\{\delta_j g_{pk} + \delta_k g_{pj} - \delta_p g_{jk}\}, \quad \delta_j = \partial_j - G^h_j \dot{\partial}_h,$$

$$G^i_j = \dot{\partial}_j G^i, \quad G^i = \frac{1}{2}\gamma^i_{jk} y^j y^k, \quad C^i_{jk} = g^{ir} C_{jrk}.$$

The h- and v-covariant derivatives of a contravariant vector $X^i(x, y)$ with respect to the Cartan connection are respectively given by

(2.1) $$X^i_{|j} = \partial_j X^i - (\dot{\partial}_h X^i) G^h_j + F^i_{rj} X^r,$$

and

(2.2) $$X^i|_j = \dot{\partial}_j X^i + C^i_{rj} X^r.$$

The Cartan connection coefficients F^i_{jk} and the Berwald connection coefficients $G^i_{jk} (= \dot{\partial}_k G^i_j)$ are related by

$$G^i_{jk} - F^i_{jk} = P^i_{jk}, \tag{2.3}$$

where $P^i_{jk} = C^i_{jk|h} y^h$.

3. C-conformal Finsler space

Let $F^n(M^n, L)$ and $\bar{F}^n(M^n, \bar{L})$ be two Finsler spaces over the same manifold M^n. A transformation $F^n \to \bar{F}^n$ is said to be a conformal change if $\bar{L}(x,y) = e^{\sigma(x)} L(x,y)$. The scalar $\sigma(x)$ depends upon the positional coordinates only. M. Hashiguchi[4] called such transformation as C-conformal transformation if

$$C^i_{jk}\sigma^j = 0, \qquad \sigma^j \neq 0, \tag{3.1}$$

where $\sigma^j = g^{ij}\sigma_i$ and $\sigma_i = \partial_i \sigma$. This transformation is in fact, non homothetic. Throughout this paper, the geometric objects associated with $\bar{F}^n$ will be barred. If the Finsler space $\bar{F}^n$ is obtained from F^n by a C-conformal transformation, we have the following [4,5]:

$$\begin{aligned}
&a)\ \bar{l}_i = e^{\sigma} l_i, \quad \bar{l}^i = e^{-\sigma} l^i,\\
&b)\ \bar{h}_{ij} = e^{2\sigma} h_{ij},\\
&c)\ \bar{g}_{ij} = e^{2\sigma} g_{ij}, \quad \bar{g}^{ij} = e^{-2\sigma} g^{ij},\\
&d)\ \bar{C}_{ijk} = e^{2\sigma} C_{ijk}, \quad \bar{C}^i_{jk} = C^i_{jk},\\
&e)\ \bar{P}^i_{jk} = P^i_{jk} + \sigma_0 C^i_{jk},\\
&f)\ \bar{F}^i_{jk} = F^i_{jk} - g_{jk}\sigma^i + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma_0 C^i_{jk}.
\end{aligned} \tag{3.2}$$

The Finsler space $\bar{F}^n$ is called a C-conformal Finsler space.

4. Parallel Vector Field under a C-conformal Change

Let the Finsler space F^n admits a parallel vector field $X^i(x^j)$ characterized by[15]

$$X^i_{|k} := \partial_k X^i - G^h_k \dot{\partial}_h X^i + X^j F^i_{jk} = \partial_k X^i + X^j F^i_{jk} = 0, \tag{4.1}$$

(4.2) $$X^i\Big|_k := \dot{\partial}_k X^i + X^j C^i_{jk} = X^j C^i_{jk} = 0.$$

If this vector field is also parallel in the C-conformal Finsler space $\bar{F}^n$, we have

(4.3) $$X^i_{\bar{1}_k} = \partial_k X^i + X^j \bar{F}^i_{jk} = 0$$

and

(4.4) $$X^i \,\bar{1}_k = X^j \bar{C}^i_{jk} = 0.$$

Substituting the value of $\bar{F}^i_{jk}$ from (3.2f) in (4.3), we get

(4.5) $$\partial_k X^i + X^j\{F^i_{jk} - g_{jk}\sigma^i + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma_0 C^i_{jk}\} = 0,$$

which in view of (4.1) and (4.2), gives

(4.6) $$-X^j g_{jk}\sigma^i + X^i \sigma_k + \delta^i_k \sigma_j X^j = 0.$$

Contracting i and k, we get

(4.7) $$\sigma_j X^j = 0,$$

and therefore (4.6) gives

(4.8) $$X_k \sigma^i = X^i \sigma_k.$$

Transvecting by σ^k and using (4.7), we find $\sigma^k \sigma_k X^i = 0,$ which implies $\sigma^k \sigma_k = 0$ for $X^i \neq 0.$ Since the metric of our space is positive definite, $\sigma^k \sigma_k = 0$ implies $\sigma_k = 0$. This shows that the transformation is homothetic, which is not true. Thus we have:

Theorem 4.1. *A parallel vector field $X^i(x^j)$ in a Finsler space F^n cannot be parallel in a C-conformal Finsler space $\bar{F}^n$.*

Narasimhamurthy and Bagewadi[16] defined a vector field $\bar{X}^i$ on the C-conformal Finsler space $\bar{F}^n$ by

(4.9) $$\bar{X}^i = e^{-\sigma(x)} X^i,$$

and called it C-conformal vector field. They established therein the following:

Lemma 4.2. *Let $\bar{F}^n$ be C-conformal space of F^n. If X^i is a parallel vector field on F^n, then C-conformal vector field $\bar{X}^i$ on $\bar{F}^n$ is parallel if*

(4.10) $\delta^i_j \sigma_h \bar{X}^h - \sigma_j \bar{X}^i = 0.$

Using this lemma they proved nine theorems in the paper[16]. But we show that the condition $\delta^i_j \sigma_h \bar{X}^h - \sigma_j \bar{X}^i = 0$ implies $\sigma_j = 0$ or $X^i = 0$. The proof is as follows:

Contracting i and j in (4.10), we get $(n-1)\sigma_h \bar{X}^h = 0$, which implies $\sigma_h \bar{X}^h = 0$ for $n \neq 1$. Using $\sigma_h \bar{X}^h = 0$ in (4.10) we find $\sigma_j \bar{X}^i = 0$. This implies $\bar{X}^i = 0$ for $\sigma_j \neq 0$. $\bar{X}^i = 0$ implies $X^i =$ 0, a contradiction. Therefore the above lemma may be modified as:

Theorem 4.3. *Let $\bar{F}^n$ be C-conformal Finsler space of F^n. If X^i is a parallel vector field on F^n, then C-conformal vector field $\bar{X}^i$ on $\bar{F}^n$ cannot be parallel.*

In view of Theorem 4.3, all the nine theorems proved by Narasimhamurthy and Bagewadi[16] collapse.

5. Concurrent Vector Field under a C-conformal Change

Let the Finsler space F^n admits a concurrent vector field $X^i(x^j)$ characterized by[14]

(5.1) $X^i_{|_k} := \partial_k X^i - G^h_k \dot{\partial}_h X^i + X^j F^i_{jk} = \partial_k X^i + X^j F^i_{jk} = K\delta^i_k$, $K =$ Constant

and

(5.2) $X^i\big|_k := \dot{\partial}_k X^i + X^j C^i_{jk} = X^j C^i_{jk} = 0.$

If this vector field is also concurrent in the C-conformal Finsler space $\bar{F}^n$, we have

(5.3) $X^i_{\bar{|}_k} = \partial_k X^i + X^j \bar{F}^i_{jk} = \bar{K}\delta^i_k,$

and

(5.4) $X^i \,\bar{|}_k = X^j \bar{C}^i_{jk} = 0.$

Substituting the value of $\bar{F}^i_{jk}$ from (3.2f) in (5.3), we get

(5.5) $\partial_k X^i + X^j \{F^i_{jk} - g_{jk}\sigma^i + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma_0 C^i_{jk}\} = \bar{K}\delta^i_k,$

which, in view of (5.1) and (5.2), reduces to

(5.6) $-X^j g_{jk}\sigma^i + X^i \sigma_k + \delta^i_k \sigma_j X^j = (\bar{K} - K)\delta^i_k.$

Contracting i and k, we get

(5.7) $\sigma_j X^j = \bar{K} - K,$

and therefore (5.6) gives

(5.8) $X_k \sigma^i = X^i \sigma_k.$

From this, we conclude that $X^i = \phi\sigma^i$ for some ϕ. Therefore from (5.1), we have $(\phi\sigma^i)_{|k} = K\delta^i_k$, which implies $\sigma^i_{|k} = \eta_k \sigma^i + \rho\delta^i_k$,

where $\eta_k = -\dfrac{\phi_{|k}}{\phi}$ and $\rho = \dfrac{K}{\phi}$. Therefore σ^i is concircular. Thus we conclude:

Theorem 5.1. *A concurrent vector field $X^i(x^j)$ in a Finsler space F^n cannot be concurrent in a C-conformal Finsler space $\bar{F}^n$ unless the vector field σ^i is concircular.*

Let the C-conformal Finsler space $\bar{F}^n$ admits the concurrent C-conformal vector field $\bar{X}^i$ then

(5.9) $\bar{X}^i_{\bar{|} k} = \bar{K}\delta^i_k, \qquad \bar{K} = \text{constant},$

i.e.

$$\partial_k \bar{X}^i + \bar{X}^j \bar{F}^i_{jk} = \bar{K}\delta^i_k.$$

Using (3.2 f) and (4.9) in the above equation, we get

$$\partial_k (e^{-\sigma(x)} X^i) + e^{-\sigma(x)} X^j \{F^i_{jk} - g_{jk}\sigma^i + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma_0 C^i_{jk}\}$$
$$= \bar{K}\delta^i_k,$$

or

$$e^{-\sigma(x)} (\partial_k X^i + X^j F^i_{jk}) - e^{-\sigma(x)} (X^j g_{jk}\sigma^i - \delta^i_k \sigma_j X^j) = \bar{K}\delta^i_k,$$

which in view of (5.1) reduces to

(5.10) $(Ke^{-\sigma(x)} - \bar{K})\delta^i_k - e^{-\sigma(x)}(X_k\sigma^i - \delta^i_k\sigma_j X^j) = 0.$

Contracting i and k, we get

(5.11) $n(Ke^{-\sigma(x)} - \bar{K}) + (n-1)e^{-\sigma(x)} X_j\sigma^j = 0.$

Equations (5.10) and (5.11) give

(5.12) $$ne^{-\sigma(x)}X_k\sigma^i - e^{-\sigma(x)}X_j\sigma^j\delta^i_k = 0.$$

Transvecting by σ^k, we obtain $X_j\sigma^j = 0$ for $n \neq 1$ and $\sigma^i \neq 0$; and therefore (5.12) gives $X_k\sigma^i = 0$. This implies $X_k = 0$, which is not possible. Thus, we have:

Theorem 5.2. *Let $\overline{F}^n$ be C-conformal space of F^n. If X^i is a concurrent vector field on F^n, then C-conformal vector field $\overline{X}^i$ cannot be concurrent on $\overline{F}^n$.*

6. Recurrent Vector Field under a C-conformal Change

Let the Finsler space F^n admits a recurrent vector field $X^i(x^j)$ characterized by

(6.1) $$X^i_{|_k} = \partial_k X^i - G^h_k \dot{\partial}_h X^i + X^j F^i_{jk} = \partial_k X^i + X^j F^i_{jk} = \mu_k X^i.$$

If X^i is also recurrent in the C-conformal Finsler space $\overline{F}^n$, we have

(6.2) $$X^i_{\overline{|}_k} = \partial_k X^i + X^j \overline{F}^i_{jk} = \overline{\mu}_k X^i.$$

Substituting the value of $\overline{F}^i_{jk}$ from (3.2f), we get

(6.3) $$\partial_k X^i + X^j\{F^i_{jk} - g_{jk}\sigma^i + \delta^i_j\sigma_k + \delta^i_k\sigma_j - \sigma_0 C^i_{jk}\} = \overline{\mu}_k X^i.$$

which in view of (6.1), reduces to

(6.4) $$-X^j g_{jk}\sigma^i + X^i\sigma_k + \delta^i_k\sigma_j X^j - \sigma_0 X^j C^i_{jk} = (\overline{\mu}_k - \mu_k)X^i.$$

Transvecting by $X_i y^k$, we get

$$X_i X^i(\overline{\mu}_k - \mu_k - \sigma_k)y^k = 0,$$

which implies either $X_i X^i = 0$ *or* $(\overline{\mu}_k - \mu_k - \sigma_k)y^k = 0$. Since our metric is positive definite, $X_i X^i = 0$ implies $X^i = 0$, which is not true. Therefore we must have

(6.5) $$(\overline{\mu}_k - \mu_k - \sigma_k)y^k = 0.$$

Differentiating partially with respect to y^j, we get

(6.6) $$(\bar{\mu}_j - \mu_j - \sigma_j) + (\dot{\partial}_j \bar{\mu}_k - \dot{\partial}_j \mu_k) y^k = 0.$$

Differentiating (6.1) and (6.2) partially with respect to y^j, we obtain

$$X^r \dot{\partial}_j F^i_{rk} = X^i \dot{\partial}_j \mu_k, \qquad X^r \dot{\partial}_j \bar{F}^i_{rk} = X^i \dot{\partial}_j \bar{\mu}_k \ ,$$

which after transvection by y^k gives

(6.7) $$X^r P^i_{rj} = X^i y^k \dot{\partial}_j \mu_k \qquad X^r \bar{P}^i_{rj} = X^i y^k \dot{\partial}_j \bar{\mu}_k \, .$$

From (3.2e) and (6.7) we get

(6.8) $$\sigma_0 X^r C^i_{rj} = X^i y^k (\dot{\partial}_j \bar{\mu}_k - \dot{\partial}_j \mu_k).$$

Transvecting by $y_i (= g_{ir} y^r)$ and using $y_i C^i_{rj} = 0,$ we get

$$y_i X^i y^k \ (\dot{\partial}_j \bar{\mu}_k - \dot{\partial}_j \mu_k) = 0,$$

which implies $y^k \ (\dot{\partial}_j \bar{\mu}_k - \dot{\partial}_j \mu_k) = 0$ for $y_i X^i = 0$ *implies* $X^i = 0$ which is not true. Using $y^k \ (\dot{\partial}_j \bar{\mu}_k - \dot{\partial}_j \mu_k) = 0$ in (6.6), we get $\bar{\mu}_j - \mu_j = \sigma_j$. In view of this, the equation (6.4) and (6.8) yield

(6.9) $$- X_k \sigma^i + \delta^i_k \sigma_j X^j = 0.$$

Contracting the indices i *and* k, we get $\sigma_j X^j = 0$ for $n \neq 1$. Using $\sigma_j X^j = 0$ in (6.9), we get either $X_k = 0$ *or* $\sigma^i = 0$, which means $X^k = 0$ *or* $\sigma^i = 0$. But both are not true. Therefore our supposition that the vector field X^i is recurrent in $\bar{F}^n$ is not correct. Hence we have:

Theorem 6.1. *A recurrent vector field* $X^i (x^j)$ *in a FInsler space* F^n *cannot be recurrent in a C-conformal Finsler space* $\bar{F}^n$.

Let the C-conformal Finsler space $\bar{F}^n$ admits the recurrent C-conformal vector field $\bar{X}^i$, then

(6.10) $$\bar{X}^i_{\bar{|} k} = \partial_k \bar{X}^i + \bar{X}^j \bar{F}^i_{jk} = \bar{\mu}_k \bar{X}^i \, .$$

Using (3.2f) and (4.9) in the above equation, we get

$$\partial_k (e^{-\sigma(x)} X^i) + e^{-\sigma(x)} X^j (F^i_{jk} - g_{jk} \sigma^i + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma_0 C^i_{jk}) = \bar{\mu}_k e^{-\sigma(x)} X^i \, ,$$

or

$\partial_k X^i - \sigma_k X^i - \bar{\mu}_k X^i + X^j F^i_{jk} - X_k \sigma^i + X^i \sigma_k + \delta^i_k \sigma_j X^j - \sigma_0 X^j C^i_{jk} = 0$,
which in view of (6.1) reduces to

(6.11) $$(\mu_k - \bar{\mu}_k) X^i = X_k \sigma^i - \delta^i_k \sigma_j X^j + \sigma_0 X^j C^i_{jk}.$$

Transvecting by X_i and y^k, we get

$$(\mu_k - \bar{\mu}_k) y^k X^i X_i = 0.$$

This implies at least one of the following

(6.12) $$a) \quad X^i X_i = 0, \qquad b) \quad (\mu_k - \bar{\mu}_k) y^k = 0.$$

Since the metric of the Finsler space considered is positive definite, (6.12a) implies $X^i = 0$, which is not true. Therefore (6.12b) holds. Differentiating (6.12b) partially with respect to y^j, we get

(6.13) $$\{\dot{\partial}_j \mu_k - \dot{\partial}_j \bar{\mu}_k\} y^k + (\mu_j - \bar{\mu}_j) = 0.$$

Differentiating (6.1) partially with respect to y^j, we obtain,

$$X^r \dot{\partial}_j F^i_{rk} = X^i \dot{\partial}_j \mu_k.$$

Transvecting by y^k gives

$$X^r y^k \dot{\partial}_j F^i_{rk} = X^i y^k \dot{\partial}_j \mu_k,$$

or

$$X^r \{\dot{\partial}_j G^i_r - F^i_{jr}\} = X^i y^k \dot{\partial}_j \mu_k,$$

i.e.

(6.14) $$X^r P^i_{rj} = X^i y^k \dot{\partial}_j \mu_k.$$

Similarly differentiating (6.10) partially with respect to y^j and then transvecting by y^k, we get

$$\bar{X}^r \bar{P}^i_{rj} = \bar{X}^i y^k \dot{\partial}_j \bar{\mu}_k,$$

which in view of (3.2e) and (4.9) gives

(6.15) $$X^r (P^i_{rj} + \sigma_0 C^i_{rj}) = X^i y^k \dot{\partial}_j \bar{\mu}_k,$$

From (6.14) and (6.15), we find

(6.16) $$X^i y^k (\dot{\partial}_j \mu_k - \dot{\partial}_j \bar{\mu}_k) = -\sigma_0 X^r C^i_{jr}.$$

Transvecting by y_i and using $y_i C^i_{jr} = 0$, *we have*

$$(y_i X^i) y^k (\dot{\partial}_j \mu_k - \dot{\partial}_j \bar{\mu}_k) = 0.$$

This gives at least one of the following:

$$y_i X^i = 0 \qquad or \qquad y^k (\dot{\partial}_j \mu_k - \dot{\partial}_j \bar{\mu}_k) = 0.$$

Differentiating $y_i X^i = 0$ partially with respect to y^h, we get $g_{ih} X^i = 0$, which implies $X^i = 0$. Therefore $y_i X^i = 0$ is not possible. Hence

(6.17) $$y^k (\dot{\partial}_j \mu_k - \dot{\partial}_j \bar{\mu}_k) = 0.$$

In view of this, the equation (6.16) gives

(6.18) $$C^i_{jr} X^r = 0.$$

Using (6.17) in (6.13), we have

(6.19) $$\bar{\mu}_k = \mu_k.$$

Using (6.18) and (6.19) in (6.11), we find

(6.20) $$\sigma^i X_k = \sigma_j X^j \delta^i_k.$$

Contracting the indices *i* and *k,* we get $\sigma_i X^i = 0$ *for* $n \neq 1$ and therefore (6.20) gives $\sigma_i X^k = 0$. This gives either $X^i = 0$ *or* $\sigma_i = 0$. Both of these are not true. Therefore our hypothesis is wrong. This leads to

Theorem 6.2. *Let* $\bar{F}^n$ *be C-conformal space of* F^n. *If* X^i *is a recurrent vector field on* F^n, *then C-conformal vector field* $\bar{X}^i$ *cannot be recurrent on* $\bar{F}^n$.

7. Normalized Semi-parallel Vector Field under a C-conformal Change

The semi-parallel vector field in Riemannian manifold has been studied by G. M. Fulton[18]. In Finsler spaces, a semi-parallel vector field $v^i(x^j)$ is characterized by[19]

$$v^i_{|k} = \lambda(\delta^i_k + \varepsilon v^i v_k) \qquad and \qquad v^i|_k = 0.$$

If *v* denotes the length of the vector v^i, then the normalized semi-parallel vector $X^i = v^i / v$ is given by

(7.1) $$X^i_{|k} = K(\delta^i_k - X^i X_k) \qquad and \qquad X^i|_k = 0.$$

Pandey and Diwedi[20] proved that K is a function of position only. Using (2.1) and (2.2), we get

(7.2) $$\partial_k X^i + F^i_{rk} X^r = K(\delta^i_k - X^i X_k),$$

and

(7.3) $$C^i_{rk} X^r = 0.$$

If X^i is also normalized semi-parallel in the C-conformal Finsler space $\bar{F}^n$, we have

(7.4) $$X^i_{|k} = \partial_k X^i + X^j \bar{F}^i_{jk} = \bar{K}(\delta^i_k - X^i X_k),$$

and

(7.5) $$X^i|_k = X^j \bar{C}^i_{jk} = 0.$$

Substituting the value of $\bar{F}^i_{jk}$ from (3.2f) in (7.4), we get

$$\partial_k X^i + X^j \{F^i_{jk} - g_{jk}\sigma^i + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma_0 C^i_{jk}\} = \bar{K}(\delta^i_k - X^i X_k).$$

which in view of (7.2) and (7.3) , reduces to

(7.6) $$-X^j g_{jk}\sigma^i + X^i \sigma_k + \delta^i_k \sigma_j X^j = (\bar{K} - K)(\delta^i_k - X^i X_k).$$

Contracting i and k , we get

(7.7) $$\sigma_j X^j = (\bar{K} - K)(1 - \frac{1}{n}),$$

for X^i is a normalized vector, i.e. $X^i X_i = |X^i|^2 = 1$. Therefore (7.6) gives

$$-X_k \sigma^i + X^i \sigma_k + \delta^i_k (\bar{K} - K)(1 - \frac{1}{n}) = (\bar{K} - K)(\delta^i_k - X^i X_k).$$

i.e.

$$(\bar{K} - K)(X^i X_k - \frac{1}{n}\delta^i_k) - X_k \sigma^i + X^i \sigma_k = 0.$$

Transvecting by X_i , we get

$$(\bar{K} - K)(X_k - \frac{1}{n} X_k) - X_k \sigma^i X_i + \sigma_k = 0.$$

In view of (7.7), the above equation gives $\sigma_k = 0$. This shows that the transformation is homothetic, which is not true. Thus we have:

Theorem 7.1. *A normalized semi-parallel vector field $X^i(x^j)$ in a Finsler space F^n cannot be normalized semi-parallel in a C-conformal Finsler space $\bar{F}^n$.*

Let the C-conformal Finsler space $\bar{F}^n$ admits the normalized semi-parallel C-conformal vector field $\bar{X}^i$ then

(7.8) $$\bar{X}^i_{\bar{|}k} := \partial_k \bar{X}^i + \bar{X}^j \bar{F}^i_{jk} = \bar{K}(\delta^i_k - \bar{X}^i \bar{X}_k).$$

Using (3.2f) and (4.9) in the above equation, we get

$$\partial_k (e^{-\sigma(x)} X^i) + e^{-\sigma(x)} X^j (F^i_{jk} - g_{jk}\sigma^i + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma_0 C^i_{jk}) = \bar{K}(\delta^i_k - e^{-2\sigma(x)} X^i X_k),$$

or

$$e^{-\sigma(x)} (\partial_k X^i + X^j F^i_{jk}) - e^{-\sigma(x)} (X_k \sigma^i - \delta^i_k \sigma_j X^j) = \bar{K}(\delta^i_k - e^{-2\sigma(x)} X^i X_k),$$

which in view of (7.2) reduces to

$$e^{-\sigma(x)} K(\delta^i_k - X^i X_k) - e^{-\sigma(x)} (X_k \sigma^i - \delta^i_k \sigma_j X^j) = \bar{K}(\delta^i_k - e^{-2\sigma(x)} X^i X_k),$$

or

(7.9) $$\delta^i_k (e^{-\sigma(x)} K - \bar{K}) + e^{-\sigma(x)} \left\{ -K X^i X_k - X_k \sigma^i + \delta^i_k \sigma_j X^j \right\} + e^{-2\sigma(x)} \bar{K} X^i X_k = 0$$

Contracting *i* and *k*, we get

(7.10) $$n(e^{-\sigma(x)} K - \bar{K}) + e^{-\sigma(x)} \left\{ -K + (n-1)\sigma_j X^j \right\} + e^{-2\sigma(x)} \bar{K} = 0.$$

Equations (7.9) and (7.10) give

$$\delta^i_k \left[e^{-\sigma(x)} \left\{ K - (n-1)\sigma_j X^j \right\} - e^{-2\sigma(x)} \bar{K} \right] + n e^{-\sigma(x)} \left\{ -K X^i X_k - X_k \sigma^i + \delta^i_k \sigma_j X^j \right\}$$
$$+ n e^{-2\sigma(x)} \bar{K} X^i X_k = 0.$$

Transvecting by $e^{\sigma(x)} X^k X_i$, we get

$$K - (n-1)\sigma_j X^j - e^{-\sigma(x)} \bar{K} - nK - nX_i \sigma^i + n\sigma_j X^j + n e^{-\sigma(x)} \bar{K} = 0,$$

i.e.

(7.11) $$\sigma_j X^j = e^{-\sigma(x)} \bar{K} - K.$$

Putting this in (7.10), we get

$$n(e^{-\sigma(x)} K - \bar{K}) + e^{-\sigma(x)} \left\{ -K + (n-1)\ (e^{-\sigma(x)} \bar{K} - K) \right\} + e^{-2\sigma(x)} \bar{K} = 0,$$

i.e.

(7.12) $$n\bar{K}(e^{-2\sigma(x)} - 1) = 0\,.$$

Since $n\bar{K} \neq 0$, (7.12) implies $\sigma = 0$, which is not possible. This leads to:

Theorem 7.2. *Let $\bar{F}^n$ be C-conformal space of F^n. If X^i is a normalized semi-parallel vector field on F^n, then C-conformal vector field $\bar{X}^i$ can not be normalized semi-parallel on $\bar{F}^n$.*

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