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Finsler Hypersurface given by Generalized β-Conformal Change

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Abstract: In 2009 N. L. Youssef, S. H. Abed and S. G. Elgendi[1] introduced generalized β-conformal change. This transformation combines both β-change and conformal change in general setting. In 1985, M. Mastumoto studied the theory of Finslerian hypersurface[2]. In the present paper, we obtain the relations for Finsler hypersurface given by generalized β-conformal change to be hyperplane of 1st kind , hyperplane of 2nd kind and hyperplane of 3rd kind. Further these Finsler spaces are Landsberg space, Berwald space and locally Minkowskian space. The terminology and notations are referred to the Matsumoto's monograph[3].

Keywords: Finsler hypersurface, hyperplane of 1st kind, hyperplane of 2nd kind and hyperplane of 3rd kind, Berwald space, Landsberg space locally Minkowskian space.
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1. Introduction

The conformal theory of Finsler spaces has been initiated by M. S. Kneblman[4] and has been deeply investigated by many authors[5, 6, 7]. In 1941, G. Randers[8] has introduced Randers change. The geometric properties of Randers change have been studied by several authors[9]. In 1974, M. Matsumoto[10] introduced β-change and this change has been studied by many authors like C. Shibata[11], R. Miron[12].

A change generalizing all the above mentioned changes has been introduced by S. H. Abed[13] in 2006 and named it as conformal β-change. In 2009 N. L. Youssef, S. H. Abed and S. G. Elgendi[1] introduced generalized β-conformal change. They have established the relationship between some important tensors associated with Finsler space and the corresponding tensors associated with Finsler space given by generalized β-conformal change.

In 1985, M. Matsumoto[2] has studied the theory of Finsler hypersurfaces. He treated various types of Finsler hypersurfaces and they are called hyperplane of 1st kind, hyperplane of 2nd kind and hyperplane of 3rd kind.

In this paper we consider Finsler space ${}^*F^n = (M, {}^*L)$ and special Finsler hypersurfaces ${}^*F^{n-1}$ of ${}^*F^n$. We discuss the different kinds of Finsler hypersurfaces under generalized β-conformal change. Further we study the geometric properties of Finsler hypersurfaces under some conditions.

2. Priliminaries

Let M^n be an n-dimensional smooth manifold and $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with a fundamental function $L(x, y)$ on M^n. The metric tensor g_{ij} and Cartan's C-tensor C_{ijk} are given by[14]

$$g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2, \quad g^{ij} = (g_{ij})^{-1},$$

$$C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}, \quad C^i_{jk} = \frac{1}{2} g^{im}(\dot{\partial}_k g_{jm}),$$

where $\dot{\partial}_i = \dfrac{\partial}{\partial y^i}$.

The Berwald connection and the Cartan connection of F^n are given by $B\Gamma = (G^i_{jk}, N^i_j, 0)$ and $C\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ respectively.

A hypersurface M^{n-1} of the underlying smooth manifold M^n may be parametrically represented by the equation $x^i = x^i(u^\alpha)$, where u^α are Gaussian co-ordinates on M^{n-1} and Greek indices run from 1 to n-1. Here, we shall assume that the matrix consisting of the projection factors $B^i_\alpha = \partial x^i / \partial u^\alpha$ is of rank (n-1). The following notations are also employed

$$B^i_{\alpha\beta} = \partial^2 x^i / \partial u^\alpha \partial u^\beta, \quad B^i_{0\beta} = v^\alpha B^i_{\alpha\beta}, \quad B^{ij\cdots}_{\alpha\beta\cdots} = B^i_\alpha B^j_\beta \cdots$$

If the supporting element y^i at a point (u^α) of M^{n-1} is assumed to be tangential to M^{n-1}, we may then write $y^i = B^i_\alpha(u)v^\alpha$, so that v^α is thought of as the supporting element of M^{n-1}. Since the function $\underline{L}(u,v) = L(x(u), y(u,v))$ gives rise to a Finsler metric of M^{n-1}, we get a (n-1)-dimensional Finsler space $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$.

At each point (u^α) of F^{n-1}, the unit normal vector $N^i(u,v)$ is defined by

(2.1) $$g_{ij}B^i_\alpha N^j = 0, \quad g_{ij}N^iN^j = 1.$$

If (B^α_i, N_i) is the inverse matrix of (B^i_α, N^i), we have

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha N_i = 0, \quad N^i B^\alpha_i = 0, \quad N^i N_i = 1,$$

and further

(2.2) $$B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j.$$

Making use of the inverse $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get

$$B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta, \quad N_i = g_{ij}N^j.$$

For the induced Cartan connections $IC\Gamma = (N^\alpha_\beta, F^\alpha_{\beta\gamma}, C^\alpha_{\beta\gamma})$ on F^{n-1}, the second fundamental h-tensor $H_{\alpha\beta}$ and the normal curvature tensor H_α are given by

(2.3) $$H_{\alpha\beta} = N_i(B^i_{\alpha\beta} + F^i_{jk}B^{jk}_{\alpha\beta}) + M_\alpha H_\beta,$$
$$H_\alpha = N_i(B^i_{0\alpha} + N^i_j B^j_\alpha),$$

respectively, where $M_\alpha = C_{ijk}B^i_\alpha N^j N^k$ and $B^i_{0\alpha} = B^i_{\beta\alpha}v^\beta$.

Transvecting $H_{\beta\alpha}$ by v^β, we get

(2.4) $$H_{0\alpha} = H_{\beta\alpha}v^\beta = H_\alpha.$$

Further we put

(2.5) $$M_{\alpha\beta} = C_{ijk}B^{ij}_{\alpha\beta}N^k.$$

The Gauss equation with respect to $IC\Gamma$ is written as

(2.6) $$R_{\alpha\beta\gamma\delta} = R_{ijkh}B^{ijkh}_{\alpha\beta\gamma\delta} + P_{ijkh}(B^h_\gamma H_\delta - B^h_\delta H_\gamma)B^{ij}_{\alpha\beta}N^k + (H_{\alpha\gamma}H_{\beta\delta} - H_{\alpha\delta}H_{\beta\gamma}).$$

3. Generalized β-conformal change Finsler spaces

Let $F^n = (M, L)$ be an n-dimensional Finsler space with fundamental function $L = L(x,y)$. Consider the following change of Finsler structures which will be referred to as a generalized β-conformal change:

(3.1) $$L(x,y) \to {}^*L(x,y) = f(e^{\sigma(x)}L(x,y), \beta(x,y)),$$

where f is a positive homogeneous function of degree one in $e^\sigma L$ and β where $\beta = b_i(x)y^i$. Assume that ${}^*F^n = (M, {}^*L)$ has the structure of a Finsler

space. Entities related to ${}^{*}F^{n}$ will be denoted by asterisk symbols.
N. L. Youssef, S. H. Abed and S. G. Elgendi[1] defined,

$$f_1 = \frac{\partial f}{\partial \tilde{L}}, \quad f_2 = \frac{\partial f}{\partial \beta}, \quad f_{12} = \frac{\partial^2 f}{\partial \tilde{L} \partial \beta}, \ldots \text{etc.},$$

where $\tilde{L} = e^{\sigma} L$. We use the following notations[1]

$$\text{(3.2)} \quad \begin{aligned} q &= ff_2, & p &= ff_1 / L, \\ q_0 &= ff_{22}, & p_0 &= f_2^2 + q_0, \\ q_{-1} &= ff_{12} / L, & p_{-1} &= q_{-1} + pf_2 / f, \\ q_{-2} &= f(e^{\sigma} f_{11} - f_1 / L) / L^2, & p_{-2} &= q_{-2} + e^{\sigma} p^2 / f^2. \end{aligned}$$

Note that the subscript under the above geometric objects indicates the degree of homogeneity of these objects. We also use the notations:

$$b^i = g^{ij} b_j, \quad m_i = b_i - (\beta / L^2) y_i \neq 0, \quad \sigma_i = \partial_i \sigma, \quad p_{02} = \frac{\partial p_0}{\partial \beta}.$$

The normalized supporting element, the metric tensor, the angular metric tensor of ${}^{*}F^{n}$ are given by[3]

(3.3) (a) ${}^{*}l_i = e^{\sigma} f_1 l_i + f_2 b_i$

(b) ${}^{*}h_{ij} = e^{\sigma} p h_{ij} + q_0 m_i m_j$

(c) ${}^{*}g_{ij} = e^{\sigma} p g_{ij} + p_0 b_i b_j + e^{\sigma} p_{-1}(b_i y_j + b_j y_i) + e^{\sigma} p_{-2} y_i y_j$.

(d) ${}^{*}G^i = G^i + D^i$, ${}^{*}G^i_{jk} = G^i_{jk} + B^i_{jk}$,

(e) ${}^{*}N^i_j = N^i_j + D^i_j$.

We use the following lemma which is given by N. L. Youssef, S. H. Abed and S. G. Elgendi[1].

Lemma(3.1). *Under a generalized β-conformal change $L \rightarrow {}^{*}L = f(e^{\sigma(x)} L, \beta)$, Consider the following two assertions:*

(i) The covariant vector b_i is parallel with respect to the Cartan connection $C\Gamma$.

(ii) The difference tensor D^i_{jk} vanishes identically.

Then, we have

(a) If (i) and (ii) hold, then σ is homothetic.

(b) If σ is homothetic, then (i) and (ii) are equivalent.

Above lemma leads us to ${}^{*}N^i_j = N^i_j$.

We have,

Theorem[1](3.1). *Assume that the covariant vector $b_i(x)$ is Cartan parallel and σ is homothetic. If F^n is locally Minkowskian, then so is the space ${}^*F^n$.*

4. Finsler Hypersurfaces Given by the Generalized β-Conformal Change ${}^*F^{(n-1)}$

We now consider a Finsler hypersurface $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$ of the Finsler space F^n and another Finsler hypersurface ${}^*F^{n-1} = (M^{n-1}, {}^*\underline{L}(u,v))$ of the Finsler space ${}^*F^n$ given by the generalized β-conformal change.

Let $N^i(u,v)$ be a unit normal vector at each point of the F^{n-1} and they are invariant under the generalized β-conformal change. Thus we shall show that a unit normal vector ${}^*N^i(u,v)$ of ${}^*F^{n-1}$ is uniquely determined by

(4.1) $${}^*g_{ij} B^i_\alpha {}^*N^j = 0, \quad {}^*g_{ij} {}^*N^i {}^*N^j = 1.$$

Now transvecting (2.1) by v^α, we get

(4.2) $$y_i N^i = 0$$

Further contracting (3.3(c)) by N^i, N^j and using (2.1) and (4.2), we have

$${}^*g_{ij} N^i N^j = e^\sigma p + p_0 (b_i N^i)^2,$$

which implies that

$${}^*g_{ij}\left(\pm \frac{N^i}{\sqrt{e^\sigma p + p_0 (b_i N^i)^2}}\right)\left(\pm \frac{N^j}{\sqrt{e^\sigma p + p_0 (b_i N^i)^2}}\right) = 1,$$

provided $e^\sigma p + p_0 (b_i N^i)^2 > 0$. Therefore we can put

(4.3) $${}^*N^i = \frac{N^i}{\sqrt{e^\sigma p + p_0 (b_i N^i)^2}}$$

where we have chosen the sign "+" in order to fix orientation.
Using (2.1) and (4.2), the first condition of (4.1), gives us

(4.4) $$(b_j N^j)(p_0 b_i B^i_\alpha + e^\sigma p_{-1} y_i B^i_\alpha) = 0 .$$

Let $(p_0 b_i B^i_\alpha + e^\sigma p_{-1} y_i B^i_\alpha) = 0$, now contracting this by v^α, we find

$$(p_0 \beta + e^\sigma p_{-1} L^2) = 0 .$$

By equation (3.2) this equation leads us to $ff_\beta = 0$. Thus we have $f_\beta = 0$ because $f \neq 0$. This fact means that ${}^*L = f(L)$, which is the contradiction to the definition of Generalized β-conformal change of Finsler metric. Consequently (4.4), gives us

(4.5) $$b_j N^j = 0 .$$

Therefore (4.3) is rewritten as

(4.6) $$^*N^i = N^i / \sqrt{e^\sigma p}\,, \qquad (p > 0)$$

now it is clear that $^*N^i$ satisfies (4.1). Summarizing the above, we state that:

Theorem (4.1). *For a field of linear frame* $(B^i_1, B^i_2, \cdots, B^i_{n-1}, N^i)$ *of* F^n*, there exists a field of linear frame* $(B^i_1, B^i_2, \cdots, B^i_{n-1}, {}^*N^i = N^i / \sqrt{e^\sigma p})$ *of the* $^*F^n$*, such that (4.1) satisfied along* $^*F^{n-1}$.

The quantities $^*B^\alpha_i$ are uniquely defined along $^*F^{n-1}$ by

$$^*B^\alpha_i = {}^*g^{\alpha\beta}\, {}^*g_{ij} B^j_\beta ,$$

where $({}^*g^{\alpha\beta})$ is the inverse matrix of $({}^*g_{\alpha\beta})$.

Let $({}^*B^\alpha_i, {}^*N_i)$ is the inverse matrix of $({}^*B^i_\alpha\, {}^*N^i)$ and then we have

(4.7) $$B^i_\alpha\, {}^*B^\beta_i = \delta^\beta_\alpha , \quad B^i_\alpha\, {}^*N_i = 0 , \quad {}^*N^i\, {}^*B^\alpha_i = 0 , \quad {}^*N^i\, {}^*N_i = 1 .$$

And further

$$B^i_\alpha\, {}^*B^\alpha_i + {}^*N^i\, {}^*N_j = \delta^i_j$$

We also get $^*N_i = {}^*g_{ij}\, {}^*N^j$, i.e.,

(4.8) $$^*N_i = \sqrt{e^\sigma p} N_i$$

If each path of a hypersurface F^{n-1} with respect to the induced connection is also a path of the ambient space F^n, then F^{n-1} is called a hyperplane of the 1st kind. A hyperplane of the 1st kind is characterized by $H_\alpha = 0$.

From (2.3), we have

$$^*H_\alpha = {}^*N_i ({}^*B^i_{0\alpha} + {}^*N^i_j\, {}^*B^j_\alpha) ,$$

From equation (4.8) and lemma 3.1, we get

$$^*H_\alpha = \sqrt{e^\sigma p} H_\alpha .$$

Thus we obtain:

Theorem (4.2). *Let* σ *be homothetic and covariant vector* $b_i(x)$ *is Cartan parallel on* F^n*. Then a hypersurface* F^{n-1} *is a hyperplane of 1st kind if and only if the hypersurface* $^*F^{n-1}$ *is a hyper plane of 1st kind.*

The torsion tensor $^*C_{ijk}$ of $^*F^n$ is given by[1]

(4.9) $$^{*}C_{ijk} = e^{\sigma} pC_{ijk} + \frac{e^{\sigma}}{2} p_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_i) + \frac{p_{02}}{2} m_i m_j m_k ,$$

where

(4.10) $$m_i = b_i - \frac{\beta}{L^2} y_i .$$

Contracting (4.10) by N^i and using (4.2) and (4.5), we get

(4.11) $$m_i N^i = 0 .$$

As for the angular metric tensor $h_{ij} = g_{ij} - l_i l_j$, (2.1) and (4.2) yield

(4.12) $$h_{ij} B^i_{\alpha} N^j = 0 .$$

Transvecting (4.9) by $B^{ij}_{\alpha\beta}$ and paying attention to (4.11) and (4.12), we get

(4.13) $$^{*}C_{ijk} = e^{\sigma} pC_{ijk} .$$

If each h-path of a hypersurface F^{n-1} with respect to the induced connection is also h-path of the ambient space F^n, then F^{n-1} is called a hyperplane of 2nd kind. A hyperplane of 2nd kind is characterized by $H_{\alpha\beta} = 0$.

Now from (2.3), (4.13) and lemma 3.1, we get

$$^{*}H_{\alpha\beta} = \sqrt{e^{\sigma} p} H_{\alpha\beta} ,$$

Thus we state the following:

Theorem (4.3). *Let σ be homothetic and covariant vector $b_i(x)$ is Cartan parallel on F^n. Then a hypersurface F^{n-1} is a hyperplane of 2nd kind if and only if the hypersurface $^{*}F^{n-1}$ is a hyperplane of 2nd kind.*

Using (2.5) and (4.6), the equation (4.13) is rewritten as

(4.14) $$^{*}M_{\alpha\beta} = \sqrt{e^{\sigma} p} M_{\alpha\beta} .$$

If the unit normal vector of F^{n-1} is parallel along each curve of F^{n-1}, then F^{n-1} is called a hyperplane of 3rd kind. A hyperplane of 3rd kind characterized by $H_{\alpha\beta} = M_{\alpha\beta} = 0$.

Thus, from Theorem 4.3 and equation (4.14), we obtain:

Theorem (4.4). *Let σ be homothetic and covariant vector $b_i(x)$ is Cartan parallel. Then a hypersurface F^{n-1} is a hyperplane of 3rd kind if and only if the hypersurface $^{*}F^{n-1}$ is a hyperplane of 3rd kind.*

For hyperplane of 1st kind, the $(v)hv$-torsion tensor is given by[2]

(4.15) $$P^{\alpha}_{\beta\gamma} = B^{\alpha}_i K^i_{\beta\gamma},$$

where $K^i_{\beta\gamma} = P^i_{jk} B^{jk}_{\beta\gamma}$.

Now using (2.2), then (4.15) becomes

(4.16) $$K^i_{\beta\gamma} = B^i_\delta P^\delta_{\beta\gamma} + N^i N_h K^h_{\beta\gamma}.$$

Theorem 3.1 gives us ${}^*K^i_{\beta\lambda} = K^i_{\beta\gamma}$ and then we immediately obtain:

(4.17) $${}^*P^\alpha_{\beta\gamma} = {}^*B^\alpha_i K^i_{\beta\gamma}$$

On substituting (4.16) in (4.17) and using (4.6) and (4.7), we get

$${}^*P^\alpha_{\beta\gamma} = P^\alpha_{\beta\gamma}.$$

Thus we obtain:

Theorem (4.5). *Let σ is homothetic and covariant vector $b_i(x)$ is Cartan parallel on F^n. Then a hyperplane F^{n-1} of 1st kind is a Landsberg space if and only if the hyperplane ${}^*F^{n-1}$ of 1st kind is a Landsberg space.*

For the hyperplane of 1st kind, the Berwald connection coefficients $G^\alpha_{\beta\gamma}$ are given by[2]

(4.18) $$G^\alpha_{\beta\gamma} = B^\alpha_i A^i_{\beta\gamma},$$

where $A^i_{\beta\gamma} = B^i_{\beta\gamma} + G^i_{jk} B^{jk}_{\beta\gamma}$.

Now using (2.2), then (4.18) becomes

(4.19) $$A^i_{\beta\gamma} = B^i_\delta G^\delta_{\beta\gamma} + N^i N_h A^h_{\beta\gamma}.$$

Since lemma 3.1 leads us to ${}^*A^i_{\beta\gamma} = A^i_{\beta\gamma}$, we immediately get

(4.20) $${}^*G^\partial_{\beta\gamma} = {}^*B^\alpha_i A^i_{\beta\gamma}.$$

On substituting (4.19) in (4.20) and using (4.6) and (4.7), we get

$${}^*G^\alpha_{\beta\gamma} = G^\alpha_{\beta\gamma}.$$

Thus we obtain:

Theorem (4.6). *Let σ be homothetic and covariant vector $b_i(x)$ is Cartan parallel on F^n. Then a hyperplane F^{n-1} of 1st kind is a Berwald space if and only if the hyperplane ${}^*F^{n-1}$ of 1st kind is a Berwald space.*

From (2.6), the Gauss equation of hyperplane of 1st kind is rewritten as

$$R_{\alpha\beta\gamma\delta} = R_{ijkh} B^{ijkh}_{\alpha\beta\gamma\delta} + (H_{\alpha\gamma} H_{\beta\delta} + H_{\alpha\delta} H_{\beta\gamma})$$

Then by Theorem 3.1 and ${}^*H_{\alpha\beta} = \sqrt{e^\sigma p} H_{\alpha\beta}$ gives the following:

Theorem (4.7). *Let σ be homothetic and covariant vector $b_i(x)$ be Cartan parallel. Then the curvature tensor R_{ijkh} of a hyperplane F^{n-1} of 1st kind with $R_{ijkh} = 0$ vanishes if and only if the curvature tensor ${}^*R_{\alpha\beta\gamma\delta}$ of the* *hyperplane* ${}^{*}F^{n-1}$ *of the 1st kind of* ${}^{*}F^{n}$ *with* ${}^{*}R_{\alpha\beta\gamma\delta}=0$ *vanishes.*

Further Theorem 4.6 and Theorem 4.7 immediately gives,

Theorem (4.8). *Let* σ *be homothetic and covariant vector* $b_i(x)$ *is Cartan parallel. Then a hyperplane* F^{n-1} *of the 1st kind of* F^{n} *with* $R_{ijkh}=0$ *is a locally Minkowskian if and only if the hyperplane* ${}^{*}F^{n-1}$ *of the 1st kind of* ${}^{*}F^{n}$ *with* ${}^{*}R_{ijkh}=0$ *is locally Minkowskian.*

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