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S. I. S Model for Disease Transmission from Prey to Predator*

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Abstract: A model for disease transmission in a predator-prey system has been proposed and analyzed in this paper. From infected prey, the disease is spreading to susceptible prey and consequently to predator species. The disease does not cause immunity in the prey and predator species. Therefore, SIS (Susceptible-Infected-Susceptible) models are considered for both the species. Conditions for the existence and stability of disease free prey predator system are obtained. The next generation approach is used to obtain the epidemiological threshold quantities R_0 , R_1 for the model system. Conditions for endemic disease in prey species are discussed. The predator may become extinct due to disease in prey. Disease is not found to be endemic in both the prey and predator species.
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1. Introduction

In natural ecosystem, regulatory mechanisms for evolution of biological species are provided by the predator-prey interactions. Not only the disease in the system affects the dynamics of prey predator population, but the prey predator interactions also affect the dynamics of disease[1-4]. Some previous studies of infectious diseases in animal populations focused on the effects of disease-induced mortality or disease-reduced reproduction in the regulation of natural populations (in their natural habitats) [1, 3-5] Reduced population sizes and destabilization of equilibrium into oscillations are caused by the presence of infectious disease in one or both of the populations.

The fact that predators take a disproportionate number of infected preys has been confirmed by earlier studies [1, 3, 6]. Infected prey is more vulnerable to predation on account of the fact that the disease may impair their ability

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to protect themselves and expose them to predators[7]. This enhanced vulnerability of infected prey may lead to persistence of the predator (which would otherwise be starved and become extinct). Further, predation of the infected prey may lead to the disappearance of disease which may otherwise be endemic in prey population[4]. Together with no reproduction in infected prey, as predation on infected prey increases, the infection tends to destabilize prey-predator interactions [1].

Most of the eco-epidemiological studies are restricted to the situations where only prey species can get infection. Few investigations [1, 6, 8] consider the spread of disease from prey to predator through predation of infected prey. For the purpose of eliminating either the disease from predator- prey system or eliminating the menace of the predator by controlling disease in prey, a simplified eco- epidemiological four species prey predator SI model [8] is shown to be useful.

This paper proposes a predator- prey model. The disease is spreading from infected prey to susceptible prey and consequently to the predator species. Both, the prey and the predator species are compartmentalized into susceptible and infected classes. The disease does not cause immunity to the prey and the predator species. Consequently, SIS model is considered for both the prey and the predator.

2. Mathematical Model

Let $N_1(t) = S_1 + I_1$ and $N_2(t) = S_2 + I_2$ be the density of prey and predator species respectively where S_i, I_i denote the density of susceptible and infected population of specific population. It is assumed that only susceptible prey is capable of reproducing logistically. The infected prey are weakened due to disease and become easier to catch, while susceptible prey can easily escape the predation. The infected prey and predator do not reproduce but still continue to consume resources. The susceptible predator gets food from infected prey for its survival. It may become infected due to interaction with infected prey. Mortality rate of infected predator is higher than that of susceptible predator. These assumptions lead to the following mathematical model:

$$\frac{dS_1}{dt} = rS_1\left(1 - \frac{S_1 + I_1}{K_1}\right) - a_1S_1(S_2 + I_2) - c_1S_1I_1 + \gamma_1I_1, \tag{1.1}$$

$$\frac{dI_1}{dt} = c_1S_1I_1 - a_2I_1(S_2 + I_2) - d_1I_1 - \gamma_1I_1, \tag{1.2}$$

(1.3) $$\frac{dS_2}{dt} = -a_2 S_2 I_1 + (-d_2 + ka_1 S_1) S_2 + \gamma_2 I_2,$$

(1.4) $$\frac{dI_2}{dt} = a_2 S_2 I_1 - d_3 I_2 - \gamma_2 I_2,$$

Various positive parameters in the model are described as

a_1, a_2 : Predation rate on the susceptible and infected prey respectively $(a_2 > a_1)$

k : Feeding efficiency of predator

r : Intrinsic growth rate constant of susceptible prey

K_1 : Carrying capacity of environment with respect to prey species

c_1 : Disease incidence rate

γ_1, γ_2 : The recovery rate of infected prey and predator to susceptible respectively

d_1 : Net death rate of infected prey

d_2, d_3 : Death rate of susceptible and infected predator respectively $(d_3 > d_2)$

Initial conditions associated with the system are: $S_1(0) \geq 0$, $S_2(0) \geq 0$, $I_1(0) \geq 0$ and $I_2(0) \geq 0$.

Mathematical Analysis:

Theorem 2.1 *All the solutions of the system (1) which initiate in* R_+^4 *are uniformly bounded.*

Proof: By theorems of Nagumo [9], it can be easily proved that S_1, I_1, S_2 and I_2 remain positive.

Let $W_1 = S_1 + I_1$

Add (1.1) and (1.2) and its simplification gives

$$\frac{d(S_1 + I_1)}{dt} \leq rS_1\left(1 - \frac{S_1}{K_1}\right) - d_1 I_1.$$

For arbitrarily chosen η_1, this simplifies to

$$\frac{d(S_1 + I_1)}{dt} + \eta_1(S_1 + I_1) \leq (r + \eta_1)S_1 - \frac{rS_1^2}{K_1} - (d_1 - \eta_1)I_1,$$

$$\frac{dW_1}{dt}+\eta_1 W_1 \le \frac{K_1(r+\eta_1)^2}{4r}-\frac{r}{K_1}\left(S_1^2-(r+\eta_1)\frac{K_1}{r}S_1+\frac{K_1^2(r+\eta_1)^2}{4r^2}\right)-(d_1-\eta_1)I_1,$$

$$\frac{dW_1}{dt}+\eta_1 W_1 \le \frac{K_1(r+\eta_1)^2}{4r}-\frac{r}{K_1}\left(S_1-(r+\eta_1)\frac{K_1}{2r}\right)^2-(d_1-\eta_1)I_1,$$

Choosing $\eta_1 < d$ and applying the results of (Birkhoff and Rota, 1982 **[10]**), yields

$$0 < W_1(S_1(t), I_1(t)) \le \frac{L_1}{\eta_1}(1-e^{-\eta_1 t})+W_1(S_1(0), I_1(0))e^{-\eta_1 t},$$

as $t \to \infty$, it gives

$$0 < S_1 + I_1 \le \frac{L_1}{\eta_1}, \quad \text{Where } L_1 = \frac{K_1(r+\eta_1)^2}{4r},$$

$\Rightarrow$ $W_1 = S_1 + I_1$ is bounded.

$\Rightarrow$ From positivity of S_1 and I_1, $0 \le S_1 \le \frac{L_1}{\eta_1}$; $0 \le I_1 \le \frac{L_1}{\eta_1}$.

Simillarly defining another positive definite function W as

$$W = S_1 + I_1 + S_2 + I_2.$$

Proceeding in the similar manner and choosing $\eta = \min(d_1, d_2, d_3)$ yields

$$0 < W(S_1, I_1, S_2.I_2) \le \frac{L_1(r+\eta)}{\eta\eta_1}(1-e^{-\eta t})+W(S_1(0), I_1(0), S_2(0), I_2(0))e^{-\eta t}.$$

As $t \to \infty$, it gives

$$0 < W \le \frac{L_1(r+\eta)}{\eta\eta_1} = L, \text{ where } L = \frac{L_1(r+\eta)}{\eta\eta_1}.$$

$\Rightarrow$ $W = S_1 + I_1 + S_2 + I_2$ is bounded.

$\Rightarrow$ Hence all solutions of (2.1) that initiate in R_+^4 are confined in the region

$$B = \left\{(S_1, I_1, S_2, I_2) \in R_+^4 : S_i \le L;\ I_i \le L;\ \ i = 1, 2\right\},$$

The following equilibrium points exist for the system (1):

The trivial equilibrium point $E_0(0,0,0,0)$ and the axial equilibrium point $E_1(K_1,0,0,0)$ always exist.

The planar equilibrium point $E_2(S_1^{'}, I_1^{'}, 0, 0)$ in $S_1 - I_1$ plane is obtained as

$$S_1^{'} = \frac{d_1+\gamma_1}{c_1},\ I_1' = \frac{rS_1'(c_1K_1-d_1-\gamma_1)}{(c_1d_1K_1+rd_1+r\gamma_1)},\ S_1^{'}+I_1^{'} < K_1.$$

The equilibrium point $E_2(S_1^{'}, I_1^{'}, 0, 0)$ exists provided

(2.1) $$(c_1K_1 - d_1 - \gamma_1) > 0.$$

In this case, the predator species is eliminated and the disease persists in the prey species.

Another disease free planar equilibrium point $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ on $S_1 - S_2$ plane exists provided

(2.2) $$K_1 > \frac{d_2}{ka_1}.$$

The equilibrium level densities of susceptible prey and predator are

$$S_1^{''} = \frac{d_2}{ka_1} \text{ and, } S_2^{''} = \frac{r(ka_1K_1 - d_2)}{ka_1^2K_1}.$$

For non-zero equilibrium point $E^*(S_1^*, I_1^*, S_2^*, I_2^*)$ of the system (1), S_1^* is given by the positive root of the quadratic

(2.3a) $$f(s) = AS^2 - BS + C = 0,$$

where $$A = ra_2d_3 + a_1c_1d_3K_1 + ka_1(d_3 + \gamma_2)(r + c_1K_1) > 0,$$

$$B = [ra_2d_3K_1 + a_1K_1d_3(d_1 + \gamma_1) + (d_3 + \gamma_2)\{(r + c_1K_1)d_2 + \gamma_1ka_1K_1\}],$$

$$C = \gamma_1d_2K_1(d_3 + \gamma_2),$$

A, B and C being positive, the quadratic has two positive roots if $B^2 > 4AC$. Also

(2.3b) $$I_1^* = \frac{ka_1(d_3 + \gamma_2)(S_1^* - S_1^{''})}{a_2d_3},$$

(2.3c) $$S_2^* = \frac{c_1d_3(S_1^* - S_1^{'})}{a_2(d_3 + (S_1^* - S_1^{''})ka_1)},$$

(2.3d) $$I_2^* = \frac{ka_1(S_1^* - S_1^{''})S_2^*}{d_3}$$

It is clear from (2.3b) and (2.3c) that $S_1^* \geq S_1^{'} \text{ and } S_1^{''}$. Further, $S_1^* + I_1^* \leq K_1$, which implies that $S_1^* \leq K_1 - I_1^*$. Hence, $S_1^* \in (\bar{S}, \hat{S})$ where $\bar{S} = \max\left(\frac{d_2}{ka_1}, \frac{d_1 + \gamma_1}{c_1}\right)$ and $\hat{S} = \frac{K_1a_2d_3 + (d_3 + \gamma_2)d_2}{a_2d_3 + (d_3 + \gamma_2)ka_1}$. Further, the root S_1^* of the quadratic is unique provided $f(\bar{S})f(\hat{S}) < 0$. This will also ensure the uniqueness of positive equilibrium point $E^*(S_1^*, I_1^*, S_2^*, I_2^*)$.

The Basic Reproduction Number is defined as the average number of secondary cases when one infective is introduced into a completely susceptible host population. Let R_0 is the basic reproduction number for isolated prey population and R_1 is the basic reproduction in the prey population when both prey and predator are present in the system. Using next generation approach [11], the epidemiological threshold quantities R_0 and R_1 for the system (1) are computed as

$$R_0 = \frac{c_1 K_1}{d_1 + \gamma_1} \text{ and } R_1 = \frac{c_1 S_1^{''}}{d_1 + \gamma_1 + a_2 S_2^{''}},$$

It may be noted that $R_1 < R_0$. Also, the greater vulnerability of prey will reduce R_1.

The variational matrix V of given system (1) is given by

$$V = \begin{pmatrix} a_{11} & -\frac{rS_1}{K_1} - c_1 S_1 + \gamma_1 & -a_1 S_1 & -a_1 S_1 \\ c_1 I_1 & a_{22} & -a_2 I_1 & -a_2 I_1 \\ k a_1 S_2 & -a_2 S_2 & a_{33} & \gamma_2 \\ 0 & a_2 S_2 & a_2 I_1 & -d_3 - \gamma_2 \end{pmatrix},$$

where

$$a_{11} = r - \frac{2rS_1}{K_1} - a_1(S_2 + I_2) - c_1 I_1 - \frac{rI_1}{K_1}$$

$$a_{22} = c_1 S_1 - d_1 - \gamma_1 - a_2(S_2 + I_2)$$

$$a_{33} = -a_2 I_1 + (-d_2 + k a_1 S_1)$$

Theorem 2.2 *The trivial equilibrium point* $E_0(0,0,0,0)$ *is a saddle point with unstable manifold along* S_1 *direction.*

Proof: From variational matrix V, the characteristic equation about $E_0(0,0,0,0)$ is obtained as

$$(\lambda - r)(\lambda + d_1 + \gamma_1)(\lambda + d_2)(\lambda + d_3 + \gamma_2) = 0.$$

All the eigenvalues are negative except the one along S_1 i.e. $\lambda = r$.

Hence result.

Theorem 2.3 *The axial equilibrium* $E_1(K_1,0,0,0)$ *is locally asymptotically stable provided*

(2.4) $R_0 < 1$ *and* $ka_1K_1 < d_2$.

Proof: The characteristic equation about $E_1(K_1,0,0,0)$ is obtained as

$$(\lambda + r)(\lambda - c_1K_1 + d_1 + \gamma_1)(\lambda + d_2 - ka_1K_1)(\lambda + d_3 + \gamma_2) = 0.$$

All the eigenvalues are negative provided

$c_1K_1 < d_1 + \gamma_1$ and $ka_1K_1 < d_2$

Hence result.

When the perturbations are taken in S_1 - I_1 plane only, then the equilibrium point $E_1(K_1,0,0,0)$ is locally asymptotically stable for $R_0 < 1$. However in the presence of predator population the stability depends upon the dynamics of predator also as is evident from (2.4). In fact, when the feeding efficiency of predator is sufficiently low then the disease will die out from both the prey and predator under condition (2.4) and the equilibrium point $E_1(K_1,0,0,0)$ is locally asymptotically stable.

The equilibrium $E_1(K_1,0,0,0)$ is saddle point when

(2.5) $$K_1 > \max\left(\frac{d_1+\gamma_1}{c_1}, \frac{d_2}{ka_1}\right)$$

Further, when $E_1(K_1,0,0,0)$ is locally asymptotically stable, it is observed from (2.1) and (2.2) that the equilibrium points $E_2(S_1^{'},I_1^{'},0,0)$ and $E_3\left(S_1^{''},0,S_2^{''},0\right)$ do not exist. Further, the stability of E_1 also excludes the existence of $E^*(S_1^*,I_1^*,S_2^*,I_2^*)$ ..

Theorem 2.4 *If* $R_0 > 1$, *then* $E_2(S_1^{'},I_1^{'},0,0)$ *locally asymptotically stable provided*

(2.6) $$ka_1S_1^{'} - \frac{a_2d_3}{(d_3+\gamma_2)}I_1^{'} - d_2 < 0.$$

Proof: From variational matrix V, the characteristic equation about $E_2(S_1^{'},I_1^{'},0,0)$ is obtained as

$$\left\{\lambda^2 + \left(\frac{rS_1^{'}}{K_1} + \frac{\gamma_1I_1^{'}}{S_1^{'}}\right)\lambda + c_1I_1^{'}\left(\frac{rS_1^{'}}{K_1} + c_1S_1^{'} - \gamma_1\right)\right\}\left(\lambda^2 + P\lambda + Q\right) = 0,$$

where

$$P = a_2 I_1^{'} - k a_1 S_1^{'} + d_2 + d_3 + \gamma_2,$$

$$Q = (d_2 - k a_1 S_1)(d_3 + \gamma_2) + d_3 a_2 I_1^{'}.$$

Since the expression $\left(c_1 S_1^{'} - \gamma_1\right) > 0$ is always positive, the first quadratic factor gives eigenvalues with negative real part. Therefore, the system around $E_2(S_1^{'}, I_1^{'}, 0, 0)$ is locally asymptotically stable provided (2.6) is satisfied.

It may be concluded from theorem 2.4 that the disease may be endemic in prey population subject to the condition (2.6) when the basic reproduction number R_0 is above the threshold. In this case, the predator species goes to extinction.

The condition of instability of $E_2(S_1^{'}, I_1^{'}, 0, 0)$ if it exists, is given by

$$k a_1 S_1^{'} - \frac{a_2 d_3}{(d_3 + \gamma_2)} I_1^{'} - d_2 > 0. \tag{2.7}$$

Theorem 2.5 *The disease free equilibrium state* $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$, *is locally asymptotically stable provided*

$$c_1 \hat{S}_1 - a_2 \hat{S}_2 - d_1 - \gamma_1 < 0.$$

Proof: From variational matrix V, the characteristic equation about $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ is obtained as

$$\left\{ \lambda^2 + \left(\frac{r S_1^{''}}{K_1} + a_1 S_2^{''} \right) \lambda + k a_1^2 S_1^{''} S_2^{''} \right\} (c_1 S_1^{''} - a_2 S_2^{''} - d_1 - \gamma_1 - \lambda)(-d_3 - \lambda) = 0.$$

Since the quadratic factor always yields two eigenvalues with negative real part, the condition for stability is

$$c_1 \hat{S}_1 - a_2 \hat{S}_2 - d_1 - \gamma_1 < 0. \tag{2.8}$$

This completes the proof.

Its simplification gives $R_1 < 1.$

If the feeding efficiency of predator population is sufficiently high so as $K_1 > \frac{d_2}{k a_1}$ then the predator population persist. When the basic reproduction number R_1 is below the threshold, then the disease dies out and prey and

predator population go to their usual persistent equilibrium values and the equilibrium point $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ is stabilized.

The greater vulnerability of prey to predation may be responsible for persistence of disease free prey predator population.

The disease free equilibrium state $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ if it exist, is unstable if

$$c_1\hat{S}_1 - a_2\hat{S}_2 - d_1 - \gamma_1 > 0 \tag{2.9}$$

Theorem 2.6 *The non-zero equilibrium point* $E^*(S_1^*, I_1^*, S_2^*, I_2^*)$ *, if exists, is not locally asymptotically stable.*

Proof: The characteristic equation about E^* is obtained as

$$\lambda^4 + A_0\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \tag{2.10}$$

where

$$A_0 = d_3 + \gamma_2 + \frac{\gamma_1 I_1}{S_1} + \frac{rS_1}{K_1} + \frac{\gamma_2 I_2}{S_2},$$

$$A_1 = \left(d_3 + \gamma_2 + \frac{\gamma_2 I_2}{S_2}\right)\left(\frac{\gamma_1 I_1}{S_1} + \frac{rS_1}{K_1}\right) + \left(cS_1 - \gamma_1 + \frac{rS_1}{K_1}\right) + ka_1^{\,2} S_1 S_2,$$

$$A_2 = c_1\left(c_1 S_1 - \gamma_1\right)\left(d_3 + \gamma_2 + \frac{\gamma_2 I_2}{S_2}\right) I_1 + \left\{c_1\left(d_3 + \gamma_2\right) - \gamma_2 a_2\right\}\frac{rS_1 I_1}{K_1}$$
$$+ \left[\frac{r\gamma_2 S_1 I_2}{K_1 S_2} + ka_1^2 S_1 S_2 + \left\{\left(a_1 - c\right) - \frac{r}{K_1}\right\} ka_1 S_1 I_2 - \frac{d_3 a_2^2 I_1 I_2}{d_3 + \gamma_2}\right]\left(d_3 + \gamma_2\right)$$
$$+ \frac{c_1\gamma_2 rS_1 I_1 I_2}{K_1 S_2} + (\gamma_1 ka_1 - d_3 a_2) I_1 S_2,$$

$$A_3 = -\left\{\left(c_1 d_3 a_1 + \frac{d_3 r a_2}{K_1} + \frac{rka_1 a_2 I_1^*}{K_1} + \frac{Drka_1}{K_1}\right) S_2^* + \left(\frac{ra_2}{K_1} + c_1 a_1\right) I_2^*\right\} a_2 S_1^* I_1^*$$
$$- \frac{\gamma_1 a_2^{\,2} d_3 I_1^{*2}(S_2 + I_2)}{S_1} - (cS_1^* - \gamma_1)\left\{a_2 I_1^* + (d_3 + \gamma_2)\right\} ka_1 a_2 I_1^* S_2^*.$$

As the constant term A_3 is always negative, at least one eigenvalue of the equation (2.10) will be positive, therefore $E^*(S_1^*, I_1^*, S_2^*, I_2^*)$ is not locally asymptotically stable.

It is observed that $E_2(S_1^{'}, I_1^{'}, 0, 0)$, if it exist, is locally asymptotically stable provided $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ is unstable and vice-versa. However, it is possible for some choice of parameters that both $E_2(S_1^{'}, I_1^{'}, 0, 0)$ and $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ exist and locally stable.

It is also observed that when both $E_2(S_1^{'}, I_1^{'}, 0, 0)$ and $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ are stable then non-zero equilibrium point $E^*(S_1^*, I_1^*, S_2^*, I_2^*)$ always exist.

3. Numerical simulations

Numerical simulations have been carried out to investigate dynamics of the proposed model in the 3-D system. Computer simulations have been performed on MATLAB for different set of parameters.

Consider the following set of parametric values:

$$r = 0.1, K_1 = 50,\ \gamma_1 = 0.25, \gamma_2 = 0.3, d_1 = 0.7,\ d_2 = 0.6,$$
$$d_3 = 0.6, c_1 = 0.01, a_1 = 0.8, a_2 = 0.9, k = 0.012. \tag{3.1}$$

The system (1) has equilibrium point $E_1(50, 0, 0, 0)$ for the data set (3.1). It is locally asymptotically stable as the computed value of R_0 is $0.5882 < 1$ and the value of feeding efficiency k of predator population is sufficiently low and the condition $K_1 < \frac{d_2}{ka_1}$ is satisfied. The solution trajectories in phase space $S_1 - I_1 - S_2$ are drawn in Fig 1 for different initial values.

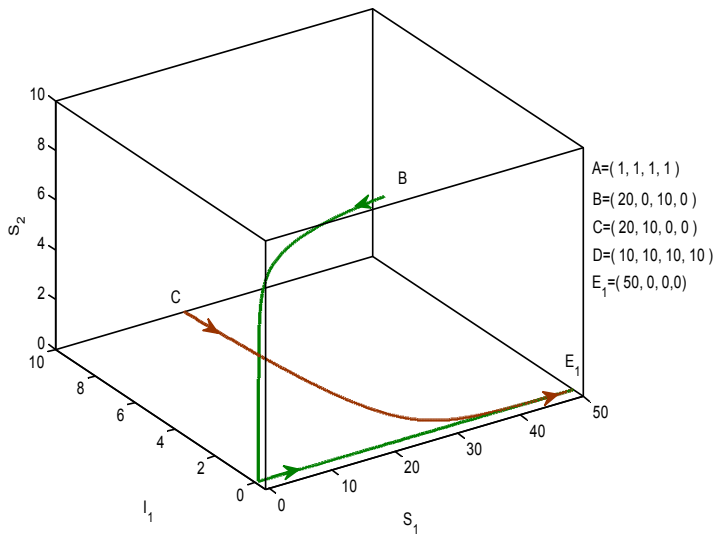

Fig 1a: Phase plot depicting the stability of equilibrium point E_1 in phase space S_1 - I_1 - S_2

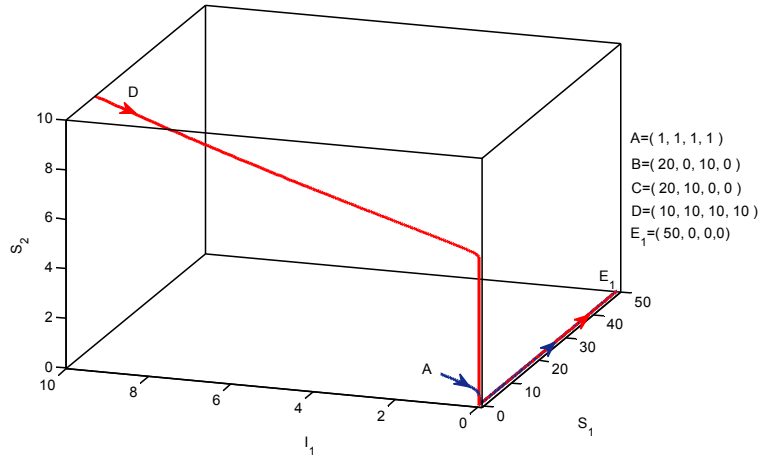

Fig 1b: Phase plot depicting the stability of equilibrium point E_1 in phase space S_1 - I_1 - S_2

Now consider the following set of parametric values:

$$r = 0.59, K_1 = 250,\ \gamma_1 = 0.27, \gamma_2 = 0.3, d_1 = 0.7,\ d_2 = 0.45,$$
$$d_3 = 0.6, c_1 = 0.001, a_1 = 0.07, a_2 = 0.09, k = 0.1. \tag{3.2}$$

For this set, the point $E_1(250, 0, 0, 0)$ is unstable and $E_3(64.2857,\ 0,\ 6.2612,\ 0)$ is locally asymptotically stable as feeding efficiency $k = 0.1$ is sufficiently large such that $K_1 > \dfrac{d_2}{ka_1} (= 64.2857)$ and $R_1 = 0.0419 < 1$. Fig. 2 shows the stability of the equilibrium point $E_3(64.2857,\ 0,\ 6.2612,\ 0)$. Further, the equilibrium point $E_2(S_1^{'}, I_1^{'}, 0, 0)$ does not exist in this case as $K_1 < \dfrac{(d_1 + \gamma_1)}{c_1} (= 970)$.

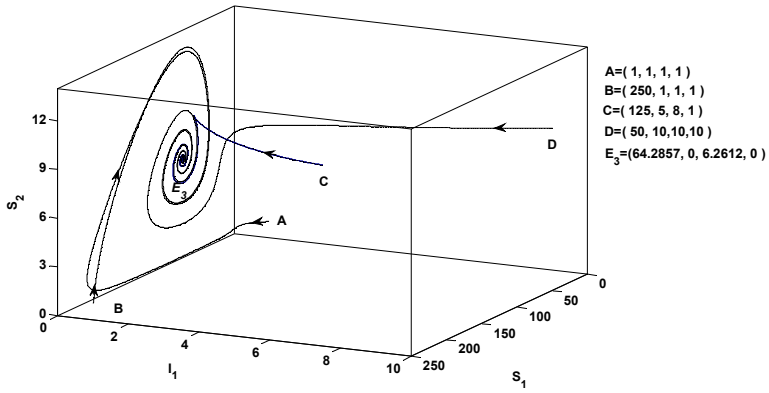

Fig 2: Phase plot depicting the stable behavior of the equilibrium point E_3

Consider the following data with other parameter as in (3.2):

(3.3) $$K_1 = 500,\ c_1 = 0.025, a_1 = 0.01, a_2 = 0.03, k = 0.1.$$

The equilibrium point $E_1(500,0,0,0)$ is unstable and $E_2(38.8,28.3133,0,0)$ exists as infection rate c_1 in prey population is high enough so that $K_1 > \dfrac{(d_1+\gamma_1)}{c_1}$ and $E_2(38.8,28.3133,0,0)$ becomes locally stable as condition (2.5) is satisfied and $R_0 = 12.8866 > 1$. The stability of the equilibrium point $E_2(38.8,28.3133,0,0)$ is shown in Fig 3. Further, it was observed that the equilibrium point $E_3(\hat{S}_1,0,\hat{S}_2,0)$ does not exist in this case as $K_1 < \dfrac{d_2}{ka_1}(=4500)$.

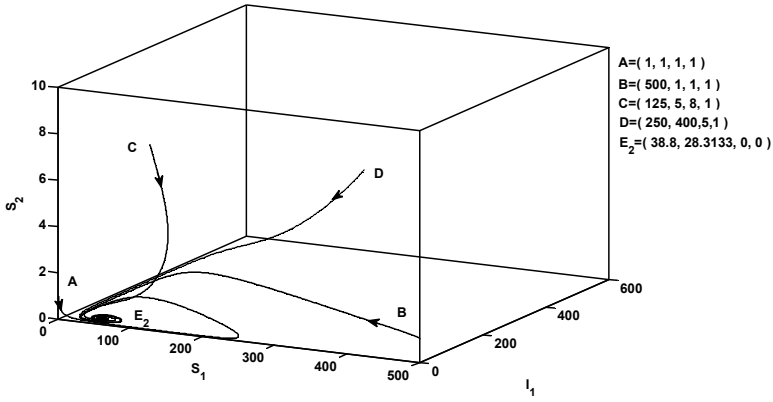

Fig 3**:** Phase plot depicting the stable behavior of the equilibrium point E_2

Now we examine the case when E_2 is stable and E_3 is unstable. For this take following data set of parameters.

$$r = 0.02, K_1 = 100, \gamma_1 = 0.01, \gamma_2 = 0.3, d_1 = 0.01, d_2 = 0.11, \quad (3.4)$$
$$d_3 = 0.15, c_1 = 0.002, a_1 = 0.01, a_2 = 0.02, k = 0.33.$$

As K_1 is greater than computed value 33.3333 of $\frac{d_2}{ka_1}$ and 10 of $\frac{(d_1+\gamma_1)}{c_1}$ in this case, the equilibrium point $E_1(100,0,0,0)$ is unstable and both the planar equilibrium points $E_2(10, 15, 0, 0)$ and $E_3(33.3333, 0, 1.3333, 0)$ exists. The equilibrium point $E_3(33.3333, 0, 1.3333, 0)$ is found to be unstable as it satisfies the condition (2.8) i.e. $c_1 S_1'' - a_2 S_2'' - d_1 - \gamma_1 = 0.02 > 0$ and $E_2(10, 15, 0, 0)$ is found to be locally asymptotically stable as condition (2.6) is satisfied i.e. $ka_1 S_1' - \frac{a_2 d_3}{(d_3+\gamma_2)} I_1' - d_2 = -0.1770 < 0$ and $R_0 = 10 > 1$. The solution trajectories in phase space $S_1 - I_1 - S_2$ are drawn in Fig 4 for different initial values. All these trajectories converge to $E_2(10, 15, 0, 0)$. Starting with initial value in the neighborhood of the point $E_3(33.3333, 0, 1.3333, 0)$, the solution trajectory approaches to the equilibrium point $E_2(10, 15, 0, 0)$. This confirms the instability of $E_3(33.3333, 0, 1.3333, 0)$.

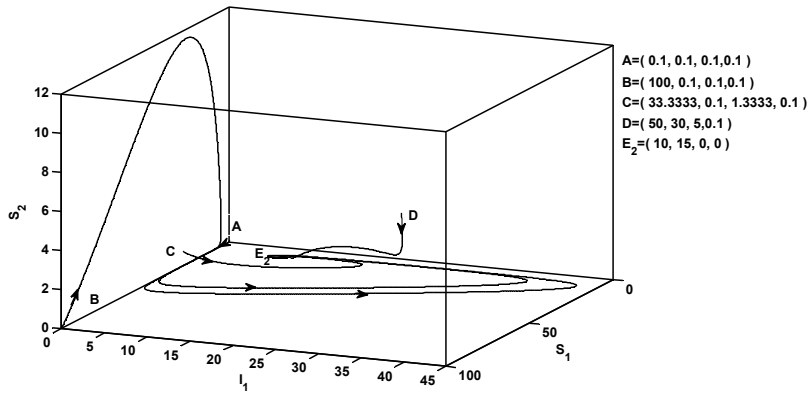

Fig 4: Phase plot depicting the stability of E_2 and instability of E_3

Now we take the case when E_3 is stable and E_2 unstable, for this consider following set of data

$$r = 0.75, K_1 = 45, \gamma_1 = 0.29, \gamma_2 = 0.3, d_1 = 0.6, d_2 = 0.9, \quad (3.5)$$
$$d_3 = 0.95, c_1 = 0.02, a_1 = 0.3, a_2 = 0.5, k = 0.73.$$

As K_1 is greater than computed value 4.3379 of $\frac{d_2}{ka_1}$ and 44.5 of $\frac{(d_1+\gamma_1)}{c_1}$ in this case, the equilibrium point $E_1(45,0,0,0)$ is unstable and system (1) has two planar equilibrium points, say $E_2(44.5,0.2764,0,0)$ and $E_3(4.3379,0,2.2590,0)$. The equilibrium point $E_3(4.3379,0,2.2590,0)$ is found to be stable as it satisfies the condition (2.8)) i.e. $c_1 S_1'' - a_2 S_2'' - d_1 - \gamma_1 = -1.9327 < 0$ and $E_2(44.5,0.2764,0,0)$ is found to be unstable as $R_0 = 1.0112 > 1$ and condition (2.6) is not satisfied i.e. $ka_1 S_1' - \frac{a_2 d_3}{(d_3+\gamma_2)} I_1' - d_2 = 8.6899 > 0$. The solution trajectories in phase space S_1 - I_1 - S_2 are drawn in Fig 5 for different initial values. All these trajectories approach to $E_3(4.3379,0,2.2590,0)$. Convergence of solution trajectory to equilibrium $E_3(4.3379,0,2.2590,0)$ with initial value near $E_2(44.5,0.2764,0,0)$ also confirms the instability of $E_2(44.5,0.2764,0,0)$.

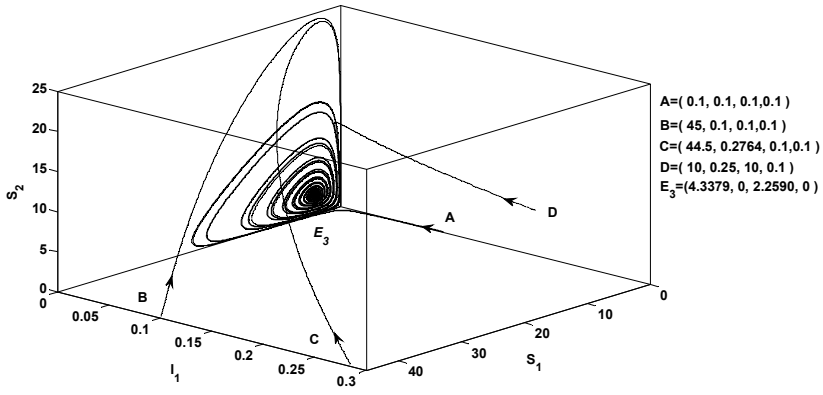

Fig 5: Phase plot depicting the stability of E_3 and instability of E_2

The following data is selected for which all the equilibrium points exist:

$$r = 0.6, K_1 = 50,\ \gamma_1 = 0.2, \gamma_2 = 0.3, d_1 = 0.6,\ d_2 = 0.3,$$
$$d_3 = 0.35, c_1 = 0.2, a_1 = 0.2, a_2 = 0.3, k = 0.25. \quad (3.6)$$

In this case, the equilibrium point $E_1(50,0,0,0)$ is unstable. It is observed that both the equilibrium points $E_2(4.0,\ 3.4074,\ 0,\ 0)$ and $E_3(6.0,\ 0,\ 2.64,\ 0)$ are locally asymptotically stable. The point $E_2(4.0,3.4074,0,0)$ is stable because of $ka_1S_1' - \dfrac{a_2 d_3}{d_3+\gamma_2} I_1' - d_2 = -0.6504 < 0$ and $R_0 = 12.5 > 1$. Also, $E_3(6.0,0,2.64,0)$ is locally asymptotically stable as $R_1 = 0.7538 < 1$. Both the equilibrium points have their own domains of attractions. Trajectories with different initial conditions converge to different equilibrium points (See Fig 6).

The instability of non-zero equilibrium point $E^*(7.2908,0.3995,1.8523,0.3416)$ can be seen from Fig 6.

It is also noticed that when even a very small perturbation is taken from $E^*(7.2908,0.3995,1.8523,0.3416)$ to $E(7.2908,0.3995,1.85,0.3416)$, along z-direction, the trajectory converges to $E_2(4.0,\ 3.4074,\ 0,\ 0)$. Whereas, when the perturbation is taken to $F(7.2908,0.3995,1.86,0.3416)$, the trajectory of the system converges to $E_3(6.0,0,2.64,0)$.

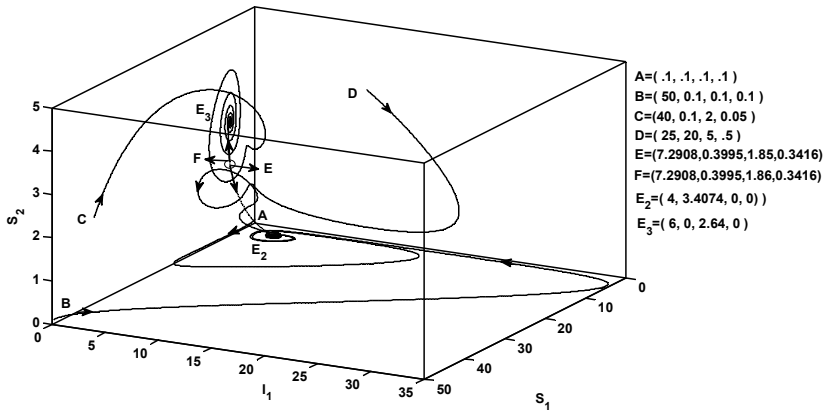

Fig 6: Phase plot depicting the stable behaviour of E_2 , E_3

5. Discussion

A predator-prey model has been proposed and analyzed in this paper. From the infected prey, the disease is spreading to susceptible prey and consequently to the predator species. The predator-prey species are categorized into susceptible and infected classes. It is assumed that the disease does not cause immunity in the prey and predator species. The SIS model system admits four boundary equilibrium points and one non-zero interior equilibrium point. The dynamic behavior of the system around each equilibrium point has been studied and threshold values for basic Reproduction numbers R_0 and R_1 are computed. It has been observed the predator population becomes extinct and the disease dies out from the prey population for sufficiently small feeding efficiency of predator when the basic reproduction number R_0 in prey population is below the threshold. If the basic reproduction number R_0 in the prey population is above the threshold, the disease in prey population approaches endemic level and predator population becomes extinct. Since the logistic growth for infected prey is not considered, prey population, $S_1^{'} + I_1^{'}$ will never reach its carrying capacity. Disease in prey may be responsible for elimination of predator.

In case the feeding efficiency of predator population is sufficiently high so that $K_1 > \frac{d_2}{ka_1}$ then the predator population persists. When the basic reproduction number R_1 is below the threshold, then the disease dies out and prey and predator population both go to their persistent equilibrium values

and the equilibrium point $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ is stabilized. The difference in death rates of infected and susceptible species may be responsible for instability of the equilibrium point $E^*(S_1^*, I_1^*, S_2^*, I_2^*)$. Therefore, the disease may not be endemic in prey predator system

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