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On the Shear Flow Instability in the β-Plane

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Abstract: In this chapter, the stability of a parallel zonal flow of an incompressible, inviscid fluid in the β-plane has been critically examined for non-oscillatory modes. Necessary condition for instability and the condition for the temporal growth rate n_i of unstable modes have been obtained. Other important results include the temporal growth rate for unstable modes, condition for the non-existence for non-oscillatory modes and necessary condition for the existence of non-oscillatory unstable modes.

Key Words and Phrases: Shear Flow, β-plane.

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1. Introduction

The study of dynamic instability may be dated back to Helmholtz and Reynolds Lamb[1] and since then has attracted the attention of many workers in hydrodynamics and meteorology. Helmholtz investigated wave motion along a surface of discontinuity with an abrupt change in the wind and density along the vertical and showed that the common surface is unstable for sufficiently short wave disturbances. It was also shown that any finite wind discontinuity will have the effect of destablization. Later on Rayliegh[2] investigated the stability of horizontal parallel flows and tried to approximate the actual current profile by combining a number of belts, each with linear velocity distribution, and showed that the stability of the current will depend upon the shape of the profile. Heisenberg[3] questioned the validity of this method. Later on, most advancement of the study of instability of an inviscid fluid was made by Lin[4] who also gave a physical interpretation in terms of conservation of vorticity.

Since all these studies are restricted to non-rotating systems, while in meteorology the earth's rotation plays a very important role in the dynamics of atmosphere, the results of these investigations can hardly be applied in the field of meteorology except for some special phenomena occuring on a rather small scale. Parallel of this study of stability of wave disturbances was the consideration of the acceleration attained by a particle displaced from its original position, based largely on the assumption of conservation of momentum of the particle, which implies that there will be no change in the pressure field. This consideration was extended into meteorology by Solberg[5] and later on by Kleinschmidt[6] by considering the balance of energy. We should distinguish the inertial instability discaused. Solberg from the instability we are considering which for the case of wave disturbances would depend upon the wave length. Furthermore in view of the emphasis generally given to the theory of momentum transfer in trubulent motion, it seems very desirable to consider the problem from the point of view of vorticity transfer. Besides, most writers in dealing with the problem of instability of wave motion were satisfied by showing that the wave velocity will become complex under certain conditions, very little attention has been given to the order of magnitude of the amplifying factors owing to mathematical difficulties.

It should be pointed out that no attempt has been made to explain all the growth and decay of a very complicated system morely by dynamic instability. The dynamic stability or instability can only give some indication of the possibility for the development of some disturbances, while the actual mechanism of the disturbances itself must be studied from other considerations. Furthermore, since in some cases the general character of the basic current changes so rapidly that the disturbances can hardly be considered as small, and the system is receiving energy from outside, the dynamic stability will not give us the clue for these changes. But for many cases we actually find the main pattern of the flow remaining nearly the same for a long period of time without much change and with a series of waves long or short, moving along it. It is to these cases that our results should apply and if we can show that the order of magnitude of the amplifying factor is the same as in the actual cases, at least we can say the dynamic instability must be one of the main factors that are operating.

A sufficient condition of stability has been obtained for the problems of Kuo[7] Hickernel[8], Shandil[9] governing the linear stability of a parallel zonal flow of an incompressible, inviscid fluid in the β-plane. For the eigen value problem prescribed by the governing equation, Kuo obtained an extension of the celebrated Rayleigh[10] inflexion point criterion inviscid, homogeneous parallel shear flows and proved that a necessary condition of instability of this problem is that $(D^2 U - \beta) = 0$, some where in the flow domain, It is mathematically proved that if the separation between the two boundaries does not exceed a critical value then the flow is stable even when the modified Rayleigh and Fjortoft[11] necessary instability criteria are satisfied. Recently Padmini et.al.[12] obtained bounds on the wave velocity of neutral modes. In the present paper, we extend the results of Shandil for homogeneous shear flow to the problem.

2. Formulation of the problem -perturbation equations

Consider the atmosphere to be barotropic, then the vorticity equation is given by

(1) $$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 \varphi + \left(\beta - U''\right)\frac{\partial \varphi}{\partial x} = 0,$$

where φ = perturbation stream function

and U = U (z), basic zonal velocity.

Let us consider the perturbnations of the form

(2) $$\varphi\ (x, y, t) = \varphi(y)\ e^{i(x+y-ct)},$$

On using equation (2), equation (1) reduces to

(3) $$\left(D^2 - k^2\right)\varphi - \frac{\left(D^2 U - \beta\right)\varphi}{U - c} = 0,$$

and the associated boundary conditions are that ϕ must vanish on the rigid walls, we may recede to $\pm\infty$ in the limiting cases and thus

(4) $$\varphi(z_1) = \varphi(z_2) = 0,$$

where z is the real independent variable such that $z_1 \le z < z_2,\quad D \equiv \frac{d}{dz}$, k is a real constant and denotes the wave number, $c = c_r + ic_i$ is the complex wave velocity, U(z) is a twice continuously differentiable function of z and denotes the prescribed basic velocity distribution while the dependent variable $\varphi(z)$ is, in general a complex valued function of z and denotes the z component of velocity distribution of the parallel flow and the parameter β is the derivative of the Coriolis force in the latitudinal direction. In complex wave velocity $c_r \neq 0$, then the amplitude will be an exponential function of time t, for $c_i > 0$ the amplitude will increase exponentially with time and the wave is said to be amplified, if on the other hand $c_i < 0$ then the amplitude will decrease with time and the wave will be damped. If $c_i = 0$, the amplitude will remain constant and the waves are said to be neutral.

3. Mathematical Analysis

Using the transformation $\varphi = (U-c)^{1/2}\ \psi$ in equation (1), we get

(5) $$D\left[(U-c)D\psi\right] + \left[-\frac{1}{4(U-c)}U'^2 + \frac{U''}{2} - k^2(U-c) - \left(D^2U-\beta\right)\right]\psi = 0.$$

Multiplying equation (5) by ψ^* (the complex conjugate of ψ) throughout and integrating the resulting equation over the boundary conditions (4) we get

(6) $$-\int (U-c)\left(|D\psi|^2 + k^2|\psi|^2\right)dz + \int\left[-\frac{U'^2}{4(U-c)} + \left(\beta - \frac{U''}{2}\right)\right]|\psi|^2\,dz = 0.$$

Now equating the real and imaginary parts on both sides of equation (6), we obtain

(7) $$\begin{aligned} &-\int (U-c_r)\left[|D\psi|^2 + k^2|\psi|^2\right]dz \\ &+\int\left[-\frac{U'^2}{4}\frac{(U-c_r)}{|U-c|^2} + \left(\beta - \frac{1}{2}U''\right)\right]|\psi|^2\,dz = 0, \end{aligned}$$

and

(8) $$c_i\left[\int|D\psi|^2\,dz + \int\left\{k^2 - \frac{U'^2}{4\left[(U-c_r)^2 + c_i^2\right]}\right\}|\psi|^2\,dz\right] = 0.$$

Now we prove the following theorems:

Theorem 1: *The temporal growth rate n_i of unstable modes satisfies the condition*

$$n_i^2 < \frac{1}{4}\left(U'^2\right)_{\max} \quad where \quad n_i = c_i k.$$

Proof : If $c_i \neq 0$, then from equation (8), we have

$$\int |D\psi|^2\, dz + \int \left(k^2 - \frac{U'^2}{4c_i^2} \right) |\psi|^2\, dz \le 0 . \tag{9}$$

For the validity of inequality (9), we must have

$$n_i^2 < \frac{1}{4}\left(U'^2\right)_{\max.} \quad \text{where} \quad n_i = c_i k$$

The above bounds on n_i clearly indicate that the range of n_i increase with the increase in shear velocity. A possible destablizing character of shear is clearly indicated from this result. These bounds are obtained for general types of modes whether oscillatory or non-oscillatory. However if we analyze the instability of non-oscillatory modes then the theorem provides a condition which rules out the existence of non-oscillatory unstable modes.

Theorem 2: *For non-oscillatory modes a necessary condition for instability is* $n_i^2 < \dfrac{U'^2}{4} - k^2\, U^2$, *where* $n_i = c_i k$.

Proof: For non-oscillatory unstable modes ($c_r = 0$), equation (8) reduces to

$$\int |D\psi|^2\, dz + \int \left[\frac{k^2 U^2 + k^2 c_i^2 - \frac{1}{4}U'^2}{\left(U^2 + c_i^2\right)} \right] |\psi|^2\, dz = 0 . \tag{10}$$

Equation (10) shows that a necessary condition for its validity is that

$$n_i^2 < \frac{1}{4}U'^2 - k^2 U^2, \quad \text{where } n_i = c_i k,$$

We conclude from here that

(i) If the condition $\dfrac{k^2 U^2}{U'^2} > \dfrac{1}{4}$ is satisfied everywhere in the flow domain, where U' is nowhere zero, then non-oscillatory unstable mode can not exit.

(ii) The bounds provided by this inequality on n_i for non-oscillatory unstable modes are sharper bounds as compared to the bounds obtained in Theorem 1.

Theorem 3: If $\frac{U''}{2\beta} \geq 1,$ then non-oscillatory modes do not exit.

Proof: Let the modes be non-oscillatory i.e. $c_r = 0$ so that equation (7) reduces to

$$\int U\left(|D\psi|^2 + k^2|\psi|^2\right)dz + \int\left[\frac{UU'^2}{4\left(U^2 + c_i^2\right)} + \frac{1}{2}\left(U'' - 2\beta\right)\right]|\psi|^2 = 0. \tag{11}$$

Now if $\frac{U''}{2\beta} \geq 1,$ everywhere in the flow domain then equation (11) becomes mathematically inconsistent showing thereby that our assumption of the existence of non-oscillatory modes is not correct. In other words c_r cannot be zero under the condition of the theorem.

Theorem 4: If $k^2 > \frac{1}{2}U'' - \beta,$ then non-oscillatory modes do not exist.

Proof: On rewriting equation (11), we have

$$\int U|D\psi|^2\,dz + \int\left[Uk^2 + \frac{1}{2}U'' - \beta\right]|\psi|^2\,dz + \frac{1}{4}\int\frac{UU'^2}{U^2 + c_i^2}|\psi|^2\,dz = 0. \tag{12}$$

Now if $Uk^2 > \frac{1}{2}U'' - \beta,$ then for the validity of equation (12), we must have $c_r \neq 0,$ which shows that the non-oscillatory modes do not exist.

Theorem 5: Neutrally non-oscillatory modes $\left(c_r = 0, c_i = 0\right)$ do not exist if the condition

$$\frac{1}{4} + \frac{U}{U'^2}\left(\frac{1}{2}U'' - \beta\right) > 0,$$

holds everywhere in the flow domain.

Proof : Substituting $c_i = 0$ in equation (11), we have

$$\int U\left(|D\psi|^2 + k^2|\psi|^2\,dz\right) + \int\left[\frac{UU'^2}{4U^2} + \left(\frac{1}{2}U'' - \beta\right)\right]|\psi|^2\,dz = 0\,.$$

Clearly the neutrally non-oscillatory modes do not exist if the condition

$$\frac{1}{4} + \frac{U}{U'^2}\left(\frac{1}{2}U'' - \beta\right) > 0, \tag{13}$$

holds everywhere in the flow domain, where $U' \neq 0$ for all values of z in the flow domain.

A Particular Case: **(Linear Velocity Profile)**: For linear velocity profile condition (13) reduces to

$$\frac{\beta U}{U'^2} < \frac{1}{4}\ .$$

Theorem 6. The necessary condition for unstable non-oscillatory modes is that $U'' - 2\beta < 0$.

Proof : Multiplying equation (6) on both sides by c^* (complex conjugate of c) and integrating over the flow domain, we get

$$\begin{aligned}&-\int\left[c^*U - |c|^2\right]\left[|D\psi|^2 + k^2|\psi|^2\right]dz\\ &+\int\left[-\frac{U'^2}{4}\cdot\frac{c^*U - c^{*2}}{|U-c|^2} + \left(\beta - \frac{1}{2}U''\right)c^*\right]|\psi|^2\,dz = 0.\end{aligned} \tag{14}$$

The real and imaginary parts of (14) must vanish separately and the vanishing of the imaginary part gives.

$$c_i\left[\int U\left(|D\psi|^2 + k^2|\psi|^2\right)dz - \int\left\{\frac{U'^2}{4}\cdot\frac{(2c_r - U)}{|U-c|^2} + \left(\beta - \frac{1}{2}U''\right)\right\}|\psi|^2\,dz\right] = 0 \tag{15}$$

If $c_i \neq 0$ and $c_r = 0$ then equation (15) reduces to

$$\int U\left(|D\psi|^2 + k^2|\psi|^2\right)dz + \int\left[\frac{U'^2}{4}\cdot\frac{U}{(U^2 + c_i^2)} + \left(\frac{1}{2}U'' - \beta\right)\right]|\psi|^2\,dz = 0\,. \tag{16}$$

For the validity of equation (16), we must necessarily have

$$U'' - 2\beta < 0.$$

whereas the above condition holds automatically for linear profile, it will not be so if the velocity profile is parabolic or of any other type satisfying $U''>2\beta$ everywhere in the flow domain.

Theorem 7: The non-oscillatory modes are unstable if the condition

$$\frac{U'^2}{4}+\frac{1}{2}UU''-U\beta<0,$$

holds everywhere in the flow domain.

Proof: If $c_i \neq 0$, then for the consistency of equation (16), we must necessarily have

$$\frac{UU'^2}{4\left(U^2+c_i^2\right)}+\frac{1}{2}U''-\beta<0, \text{ some where in the flow domain.}$$

This provides

$$\frac{U'^2}{4}+\frac{1}{2}UU''-U\beta<0,$$

This condition is violated under the condition

$$\frac{2\beta}{U''}<\frac{1}{2}.$$

where U' is nowhere zero and $U''$is strictly positive in the range of z.

Theorem 8: Non-oscillatory unstable modes exist if

$$S_\beta<\frac{1}{4}, \quad \text{where} \quad S_\beta=\frac{Uk^2\left(U'^2+2k^2U^2\right)}{\left(2\beta-U''\right)\left(U'^2+4k^2U^2\right)}$$

Proof: Let the modes be non-oscillatory and unstables, then using Theorem 1 in equation (11), we have

$$\int U|D\psi|^2\,dz+\int\left[Uk^2+\frac{UU'^2}{4\left(U^2+\frac{U'^2}{4k^2}\right)}+\frac{1}{2}\left(U''-2\beta\right)\right]|\psi|^2\,dz\leq 0,$$

For the validity of this ineqality, we must have

$$\frac{Uk^2\left[4k^2U^2+U'^2\right]+UU'^2k^2}{4k^2U^2+U'^2}+\frac{1}{2}\left(U''-2\beta\right)<0,$$

or

$$\frac{Uk^2\left(U'^2+2k^2U^2\right)}{(2\beta-U")\,(4k^2U^2+U'^2)}<\frac{1}{4}, \tag{17}$$

or $\quad S_\beta<\frac{1}{4}$, where $\quad S_\beta=\dfrac{Uk^2\left(U'^2+2k^2U^2\right)}{(2\beta-U")\left(U'^2+4k^2U^2\right)}$

This proves the theorem.

Remark: **Linear velocity profile**: For linear velocity profile $\left(U"=0\right)$, the condition (17) reduces to

$$S_\beta=\frac{Uk^2\left(U'^2+2k^2U^2\right)}{2\beta\left(U'^2+4k^2U^2\right)}<\frac{1}{4}.$$

It follows from the fact that for $\beta>0$ and for linear velocity profile non-oscillatory unstable mode exist under the condition

$$\frac{U\,k^2}{2\beta}<\frac{1}{4}.$$

Theorem 9: The necessary condition for unstable modes is that $c_r>\frac{1}{2}U_{\max}$ if $U"-2\beta>0$ holds everywhere in the flow domain.

Proof: If $c_i\neq 0,$ then proof follows from equation (15).

Theorem 10: If the condition

$$\frac{Uk^2U'^2}{\left(\beta-\frac{1}{2}U"\right)\left(U'^2+4k^2U^2\right)}>1,$$

holds everywhere in the flow domain, then non-oscillatory unstable modes do not exist.

Proof: If $U"-2\beta<0,$ everywhere in the flow domain, then for the validity of equation (16) the necessary condition is

$$\frac{UU'^2}{4\left(U^2+c_i^2\right)}<\frac{1}{2}\left(2\beta-U"\right),$$

or

(18) $$c_i^2 > \frac{UU'^2}{2(2\beta - U'')} - U^2$$

which provides the lower bounds on c_i^2, also from Theorem 1, we have the upper bound on c_i^2 as

(19) $$c_i^2 < \frac{U'^2}{4k^2}.$$

Combining these inequalities (18) and (19), we get

$$\frac{UU'^2}{2(2\beta - U'')} - U^2 < c_i^2 < \frac{U'^2}{4k^2}.$$

These bounds on c_i^2 do not exist if the condition

$$\frac{4k^2UU'^2}{2(2\beta - U'')\left(U'^2 + 4k^2U^2\right)} > 1$$

or $$\frac{Uk^2U'^2}{\left(\beta - \frac{1}{2}U''\right)\left(U'^2 + 4k^2U^2\right)} > 1,$$

holds everywhere in the flow domain.

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