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On a Quarter-Symmetric Metric Connection in an (ε)-Kenmotsu Manifold

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Abstract: The object of the present paper is to study a quarter-symmetric metric connection in an (ε)-Kenmotsu manifold. We study some curvature properties of an (ε)-Kenmotsu manifold with respect to the quarter-symmetric metric connection.

Keywords: quarter-symmetric metric connection, (ε)-Kenmotsu manifold, locally ϕ-symmetric, ϕ-recurrent.

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1. Introduction

A semi-symmetric linear connection on a differentiable manifold was first introduced by Friedmann and Schouten[1] in 1924. Hayden[2] introduced and studied a semi-symmetric metric connection on a Riemannian manifold. Duggal and Sharma[3] studied a semi-symmetric metric connection on a semi-Riemannian manifold. The quarter-symmetric connection generalizes the semi-symmetric connection. The semi-symmetric metric connection is important in the geometry of Riemannian manifolds having also physical application; for instance, the displacement on the earth surface following a fixed point is metric and semi-symmetric[4]. In 1975, Golab[5] introduced and studied quarter symmetric connection in a differentiable manifold. A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold $\left(M^n, g\right)$ is said to be a quarter-symmetric connection[5] if its torsion tensor $\tilde{T}$ defined by

$$\tilde{T}\left(X, Y\right)=\tilde{\nabla}_X Y-\tilde{\nabla}_Y X-\left[X, Y\right]$$

is of the form

(1.1) $\tilde{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y$.

where η is 1-form and ϕ is a tensor of type (1, 1). In addition, a quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition[6]

(1.2) $(\tilde{\nabla}_X g)(Y,Z) = 0$,

for all $X,Y,Z \in TM^n$, where TM^n is the Lie-algebra of vector fields of the manifold M^n and g be the Riemannian metric, then $\tilde{\nabla}$ is said to be quarter-symmetric metric connection. In particular, if $\phi X = X$ and $\phi Y = Y$, then the quarter symmetric connection reduces to a semi-symmetric connection[5].

After Golab[5], Rastogi[7] continued the symmetric study of quarter-symmetric metric connection. In 1980, Mishra and Pandey[8] studied quarter-symmetric metric connection in a Riemannian, Kaehlerian and Sasakian manifold. In 1982, Yano and Imai[9] studied quarter-symmetric metric connection in Hermitian and Kaehlerian manifolds, Quarter-symmetric connection are also studied by Biswas and De[10], Singh[11], De and Mondal[12], De and De[13], Singh and Pandey[6] and many other.

On the other hand, the study of manifolds with indefinite metrics is of interest from the standpoint of physics and relativity. Manifolds with indefinite metrics have been studied by several authors. In 1993, Bejancu and Duggal[14] introduced the concept of (ε)-Sasakian manifolds and Xupeng and Xiaoli[15] established that these manifolds are real hyper-surfaces of indefinite Kaehlerian manifolds. De and Sarkar[16] introduced (ε)-Kenmotsu manifolds and studied various properties of (ε)-Kenmotsu manifold. A semi-symmetric metric connection in an (ε)-Kenmotsu manifold whose projective curvature tensor satisfies certain curvature conditions have been studied by Singh, Pandey, Pandey and Tiwari[17], Motivated by these studies, in this paper, we study some curvature properties of an (ε)-Kenmotsu manifold with respect to quarter-symmetric metric connection. The present paper is organized as follows:

After preliminaries in section 3, we find the expression for curvature tensor (resp. Ricci tensor) with respect to quarter-symmetric metric connection and established relations between curvature tensor (resp. Ricci tensor) with respect to quarter-symmetric connection and curvature tensor (resp. Ricci tensor) with respect to Levi-Civita connection in an (ε)-Kenmotsu manifold. Section 4 deals with quasi-projectively flat (ε)-Kenmotsu manifold with respect to quarter-symmetry metric connection. Section 5 is devoted to study ϕ-protectively flat (ε)-Kenmotsu manifold with respect to a quarter-symmetric connection. In the last section, we study (ε)-Kenmotsu manifold with respect to a quarter-symmetric connection satisfying $\tilde{P}.\tilde{S}=0$.

2. (ε)-Kenmotsu Manifold

An n-dimensional smooth manifold (M^n, g) is called an (ε)-almost contact metric manifold if [16,17]

(2.1) $$\phi^2 X = -X + \eta(X)\xi,$$

(2.2) $$\eta(\xi) = 1,$$

(2.3) $$g(\xi,\xi) = \varepsilon,$$

(2.4) $$\eta(X) = \varepsilon g(X,\xi),$$

(2.5) $$g(\phi X, \phi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y),$$

where ε is 1 or -1 according as ξ is space-like or time-like and rank ϕ is $n-1$.

It is important to mention that in the above definition ξ is never a light-like vector field. If

(2.6) $$d\eta(X,Y) = g(X,\phi Y),$$

for every $X, Y \in TM^n$, then we say that M^n is an (ε)-contact metric manifold[17]. Also,

(2.7) $$\phi\xi = 0, \quad \eta o \phi = 0 \quad .$$

If an (ε)-contact metric manifold satisfies

(2.8) $$(\nabla_X \phi)(Y) = -g(X,\phi Y)\xi - \varepsilon\eta(Y)\phi X,$$

where ∇ denotes the Riemannian connection of g, then M^n Is called an (ε)-Kenmotsu manifold[16].

An (ε)-almost contact metric manifold is an (ε)-Kenmotsu manifold[16] if and only if

(2.9) $$\nabla_X \xi = \varepsilon\left(X - \eta(X)\xi\right).$$

In an (ε)-Kenmotsu manifold, the following relations hold[16]

(2.10) $$\left(\nabla_X \eta\right)(Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y),$$

(2.11) $$R(X,Y)\xi = \eta(X)Y - \eta(Y)X,$$

(2.12) $$R(\xi,X)Y = \eta(Y)X - \varepsilon g(X,Y)\xi,$$

(2.13) $$R(X,Y)\phi Z = \phi R(X,Y)Z + \varepsilon\{g(Y,Z)\phi X - g(X,Z)\phi Y$$
$$+g(x,\phi Z)Y - g(Y,\phi Z)X\}.$$

(2.14) $$\eta\left(R(X,y)Z\right) = \varepsilon\left[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\right],$$

(2.15) $$S(X,\xi) = -(n-1)\eta(X),$$

(2.16) $$S(\phi X,\phi Y) = S(X,Y) + \varepsilon(n-1)\eta(X)\eta(Y).$$

Definition 2.1. *An (ε)-Kenmotsu manifold M^n is said to be η-Einstein if its Ricci tensor S is of the form*[17]

(2.17) $$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a *and* b *are scalar functions of* ξ. *De and Sarkar*[16] *have given an example of* (ε)*-Kenmotsu manifold*

Example: We consider the three dimensional manifold $M^3 = \{(X,Y,Z) \in R^3$ $Z \neq 0\}$, where (X,y,Z) are the standard co-ordinates in R^3. The vector fields $e_1 = Z\frac{\partial}{\partial X}$, $e_2 = Z\frac{\partial}{\partial X}$, $e_3 = -Z\frac{\partial}{\partial Z}$ are linearly independent at each point of the manifold. Let g be the Riemannian metric defined by

$$g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0$$

and

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = \varepsilon,$$

where $\varepsilon = \pm 1$. Let η be the 1-form defined by

$$\eta(Z) = \varepsilon g(Z,e_3) \text{ for any } Z \in TM^n.$$

Let ϕ be the (1, 1) tensor field defined by

$$\phi(e_1) = e_2, \phi(e_2) = e_1, \phi(e_3) = 0.$$

Then using the linearity property of ϕ and g, we have

$$\eta(e_3) = 1, \phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z,\phi W) = g(Z,W) - \eta(Z)\eta(W) \text{ for any } Z, W \in TM^n.$$

Let ∇ be the Levi-Civita connection with respect to metric g. Then we have

$$[e_1,e_2] = 0, [e_1,e_3] = \varepsilon e_1, [e_2,e_3] = \varepsilon e_2.$$

The Riemannian connection ∇ of the metric g is given by

$$2g(\nabla_X Y,Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z])$$

$$-g\left(Y,[X,Z]\right)+g\left(Z,[X,Y]\right),$$

which is known as Koszul's formula, then Koszul's formula yields

$$\nabla_{e_1}e_3=\varepsilon e_1, \nabla_{e_2}e_3=\varepsilon e_2, \nabla_{e_3}e_3=0,$$

$$\nabla_{e_1}e_2=0, \nabla_{e_2}e_2=-\varepsilon e_3, \nabla_{e_3}e_2=0,$$

$$\nabla_{e_1}e_1=-\varepsilon e_3, \nabla_{e_2}e_1=0, \nabla_{e_3}e_1=0.$$

These result shows that the manifold satisfies

$$\nabla_X\xi=\varepsilon\left(X-\eta(X)\xi\right),$$

for $\xi=e_3$. Hence the manifold under consideration is an (ε)-Kenmotsu manifold of dimension three.

3. Curvature Tensor of an (ε)-Kenmotsu Manifold with respect to Quarter-Symmetric Metric Connection

Let $\tilde{\nabla}$ be the linear connection and ∇ be the Riemannian connection of an almost contact metric manifold such that

(3.1) $$\tilde{\nabla}_X Y=\nabla_X Y+H(X,Y),$$

where H is a tensor field of type (1, 1). For $\tilde{\nabla}$ to be quarter-symmetric metric connection in M^n, we have[5].

(3.2) $$H(X,Y)=\frac{1}{2}\left[\tilde{T}(X,Y)+\tilde{T}'(X,Y)+\tilde{T}'(Y,X)\right]$$

and

(3.3) $$g\left(\tilde{T}'(X,Y),Z\right)=g\left(\tilde{T}'(Z,X),Y\right).$$

In view of equations (1.1) and (3.3), we have

(3.4) $$\tilde{T}'(X,Y)=\eta(X)\phi Y-g(\phi X,Y)\xi .$$

Now using equations (1.1) and (3.4) in equation (3.2), we get

(3.5) $$H(X,Y)=\eta(Y)\phi X-g(\phi X,Y)\xi .$$

Hence in view of equations (3.1) and (3.5) a quarter-symmetric connection $\tilde{\nabla}$ on an (ε)-Kenmotsu manifold is given by

(3.6) $$\tilde{\nabla}_X Y=\nabla_X Y+\eta(Y)\phi X-g(\phi X,Y)\xi .$$

Thus the above equation is the relation between quarter-symmetric metric connection and the Levi-Civita connection. Further, a quarter-symmetric connection is called a quarter-symmetric metric connection[9] if

(3.7) $$\left(\tilde{\nabla}_X g\right)(Y,Z)=0 .$$

The curvature tensor $\tilde{R}$ of M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ is given by

(3.8) $$\tilde{R}(X,Y)Z=\tilde{\nabla}_X\tilde{\nabla}_X Z-\tilde{\nabla}_Y\tilde{\nabla}_X Z-\tilde{\nabla}_{[X,Y]}Z .$$

In view of equation (3.6) above equation reduces to

(3.9) $$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z+\left[\left(\nabla_X\eta\right)(Z)\phi Y-\left(\nabla_X\eta\right)(Z)\phi X\right]\\ &+\left[g(\phi X,Z)\left(\nabla_Y\xi\right)-g(\phi Y,Z)\left(\nabla_X\xi\right)\right]\\ &+\eta(Z)+\left[\left(\nabla_X\phi\right)(Y)-\left(\nabla_Y\phi\right)(X)\right]\\ &-\left[g\left(\left(\nabla_X\phi\right)(Y),Z\right)-g\left(\left(\nabla_Y\phi\right)(X),Z\right)\right]\xi ,\end{aligned}$$

where

(3.10) $$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$$

is the Riemannian curvature tensor of connection ∇. By virtue of equations (2.8), (2.9) and (2.10) equation (3.9) takes the form

(3.11) $$\tilde{R}(X,Y)Z = R(X,Y)Z + \left[g(X,Z)\phi Y - g(Y,Z)\phi X\right] + \varepsilon\left[g(\phi X,Z)Y - g(\phi Y,Z)X\right].$$

Which is the relation between curvature tensors of connections $\tilde{\nabla}$ and ∇. From equation (3.11), we have

(3.12) $$`\tilde{R}(X,Y,Z,U) = `R(X,Y,Z,U) + \left[g(X,Z)g(\phi Y,U) - g(Y,Z)g(\phi X,U)\right] + \varepsilon\left[g(\phi X,Z)g(Y,U) - g(\phi Y,Z)g(X,U)\right],$$

where

$$`R(X,Y,Z,U) = g\left(R(X,Y)Z,U\right)$$

and

$$`\tilde{R}(X,Y,Z,U) = g\left(\tilde{R}(X,Y)Z,U\right).$$

Putting $X = U = e_i$ in above equation and taking summation over i, $1 \le i \le n$, we get

(3.13) $$\tilde{S}(Y,Z) = S(Y,Z) + (1 - n\varepsilon)g(\phi Y,Z),$$

where $\tilde{S}$ and S are the Ricci tensors of the connections $\tilde{\nabla}$ and ∇ respectively.
Contracting equation (3.13), we obtain

(3.14) $$\tilde{r} = r,$$

where $\tilde{r}$ and r are the scalar curvatures of the connections $\tilde{\nabla}$ and ∇ respectively.
Writing two more equations by the cyclic permutations of X, Y and Z from equation (3.11), we get

(3.15) $$\tilde{R}(Y,Z)X = R(Y,Z)X + \left[g(Y,X)\phi Z - g(Z,X)\phi Y\right] + \varepsilon\left[g(\phi Y,X)Z - g(\phi Z,X)Y\right]$$

and

(3.16) $$\tilde{R}(Z,X)Y = R(Z,X)Y + \left[g(Z,Y)\phi X - g(X,Y)\phi Z\right] + \varepsilon\left[g(\phi Z,Y)X - g(\phi X,Y)Z\right].$$

Adding equations (3.11), (3.15) and (3.16) and using Bianchi's first identity, we get

(3.17) $$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = -2\varepsilon\left[g(\phi X,Y)Z + g(\phi Y,Z)X + g(\phi Z,X)Y\right]$$

Thus we can state as follows:

Theorem 3.1. *An (ε)-Kenmotsu manifold M^n with quarter-symmetric metric connection satisfies the equation (3.17).*

Now interchanging X and Y in equation (3.12), we get

(3.18) $$`\tilde{R}(Y,X,Z,U) = `R(Y,X,Z,U) + \left[g(Y,Z)g(\phi X,U) - g(X,Z)g(\phi Y,U)\right] + \varepsilon\left[g(\phi Y,Z)g(X,U) - g(\phi X,Z)g(Y,U)\right],$$

Adding equations (3.12) and (3.18) with the fact that $`\tilde{R}(X,Y,Z,U) + `\tilde{R}(Y,X,Z,U) = 0$, we get

(3.19) $$`\tilde{R}(X,Y,Z,U) + `\tilde{R}(Y,X,Z,U) = 0$$

Again interchanging Z and U in equation (3.12), we get

(3.20) $$`\tilde{R}(X,Y,U,Z) = `R(X,Y,U,Z) + \left[g(X,U)g(\phi Y,Z) - g(Y,U)g(\phi X,Z)\right]$$

$$+\varepsilon\left[g(\phi X,U)g(Y,Z)-g(\phi Y,U)g(X,Z)\right] ,$$

Adding equations (3.12) and (3.20) with the fact that ${}'\tilde{R}(X,Y,Z,U)$ $+{}'\tilde{R}(X,Y,U,Z)=0$, we get

(3.21) $${}'\tilde{R}(X,Y,Z,U)+{}'\tilde{R}(X,Y,U,Z)$$

$$=(1-\varepsilon)\left[g(X,U)g(\phi Y,Z)+g(X,Z)g(\phi Y,U)\right]$$

$$-\left[g(Y,U)g(\phi X,Z)+g(Y,Z)g(\phi X,U)\right].$$

Again interchanging pair of slots in equation (3.12), we get

(3.22) $${}'\tilde{R}(Z,U,X,Y)={}'R(Z,U,X,Y)$$

$$+\left[g(Z,X)g(\phi U,Y)-g(U,X)g(\phi Z,Y)\right]$$

$$+\varepsilon\left[g(\phi Z,X)g(U,Y)-g(\phi U,X)g(Z,Y)\right] ,$$

Now, subtracting equation (3.22) in equation (3.12) with the fact that ${}'R(X,Y,Z,U)-{}'R(Z,U,X,Y)=0$, we get

(3.23) $${}'\tilde{R}(Z,U,X,Y)-{}'R(Z,U,X,Y)$$

$$=2g(\phi Y,U)g(X,Z)+g(\phi Z,Y)g(U,X)$$

$$-g(\phi X,U)g(Y,Z)+\varepsilon\left[2g(\phi X,Z)g(Y,U)\right.$$

$$\left.-g(\phi Y,Z)g(X,U)+g(\phi U,X)g(Z,Y)\right].$$

Thus in view of equations (2.19), (2,21) and (2,23), we can state as follows:

Theorem 3.2. *The curvature tensor of type* $(0,4)$ *of a quarter-symmetric metric connection in an* (ε)*-Kenmotsu manifold is*

(i) Skew-symmetric in first two slots,
(ii) Not skew-symmetric in last two slots,
(iii) Not symmetric in pair of slots.

Now consider $\tilde{R}(X, Y)Z = 0$, which in view of equation (3.11) gives

$$R(X, Y)Z = g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + \varepsilon\left[g(X, \phi Z)Y - g(Y, \phi Z)X\right]. \tag{3.24}$$

Taking the inner product of above equation with ξ, we get

$$\eta R(X, Y)Z = \varepsilon\left[g(X, \phi Z)\eta(Y) - g(Y, \phi Z)\eta(X)\right], \tag{3.25}$$

which by virtue of equation (2.13) gives

$$\eta\left(R(X, Y)Z\right) = \eta\left(R(X, Y)\phi Z\right), \tag{3.26}$$

which gives

$$R(X, Y)Z = R(X, Y)\phi Z .$$

Thus, we can state as follow:

Theorem 3.3. *If the curvature tensor of a quarter-symmetric metric connection vanishes, then* $R(X, Y)Z = R(X, Y)\phi Z$.

4. Quasi-Projectively Flat (ε)-Kenmotsu Manifold with Respect to Quarter-Symmetric Metric Connection

The projective curvature tensor $\tilde{P}$ with respect to quarter-symmetric metric connection is defined by[6].

$$\tilde{P}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{n-1}\left[\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y\right]. \tag{4.1}$$

Definition 4.1. *An* (ε)*-Kenmotsu manifolds* M^n *is said to be quasi-projectively flat with respect to quarter-symmetric metric connection*[17]*, if*

$$g\left(\tilde{P}(\phi X, Y)Z, \phi W\right) = 0, \tag{4.2}$$

where $\tilde{P}$ *is the projective curvature tensor with respect to quarter-symmetric metric connection* $\tilde{\nabla}$.

In view of equation (4.1), we have

(4.3)
$$g\left(\tilde{P}(X, Y)Z, W\right) = g\left(\tilde{R}(X, Y)Z, W\right) - \frac{1}{(n-1)}\left[\tilde{S}(Y, Z)g(X, W) - \tilde{S}(X, Z)g(Y, W)\right].$$

Replacing X by ϕX and W by ϕW in above equation, we get

(4.4)
$$g\left(\tilde{P}(\phi X, Y)Z, \phi W\right) = g\left(\tilde{R}(\phi X, Y)Z, \phi W\right) - \frac{1}{(n-1)}\left[\tilde{S}(Y, Z)g(\phi X, \phi W) - \tilde{S}(\phi X, Z)g(Y, \phi W)\right]$$

Now w assuming that M^n is quasi-projectively flat with respect to quarter-symmetric metric connection. Then by virtue of equations (4.2) and (4.4), we have

(4.5)
$$g\left(\tilde{R}(\phi X, Y)Z, \phi W\right) = -\frac{1}{(n-1)}\left[\tilde{S}(Y, Z)g(\phi X, \phi W) - \tilde{S}(\phi X, Z)g(Y, \phi W)\right].$$

Using equations (3.11) and (3.17) in above equation, we get

(4.6)
$$g\left(R(\phi X, Y)Z, \phi W\right) = -\, g(\phi X, Z)\, g(\phi Y, \phi W) - g(Y, Z)g(X, \phi W)$$
$$-\frac{1}{(n-1)}\left[S(Y, Z)g(\phi X, \phi W) - S(\phi X, Z)g(Y, \phi W)\right]$$
$$+\left(\frac{1-\varepsilon}{n-1}\right)\left[g(X, Y)g(Y, \phi W) - n(X)n(Z)g(Y, \phi W)\right.$$
$$\left. + \, g(\phi Y, Z)g(\phi X, \phi W)\right].$$

Let $\{e_1, e_2, \ldots\ldots e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n. Then $\{\phi e_1, \phi e_2, \ldots\ldots \phi e_{n-1}, \xi\}$ is also a local orthonormal basis of M^n. Putting $X = W = e_i$ in equation (4.6) and taking summation over i, $1 \le i \le n-1$, we get

$$(4.7)\quad \sum_{i=1}^{n-1} g\left(R\left(\phi e_i, Y\right)Z, \phi e_i\right) = -\sum_{i=1}^{n-1}\left[g\left(\phi e_i, Z\right)g\left(\phi Y, \phi e_i\right) + g\left(Y, Z\right) g(\phi e_i, \phi e_i)\right]$$

$$+\frac{(1-\varepsilon)}{n-1}\sum_{i=1}^{n-1}\left[g\left(e_i, Z\right)g\left(Y, \phi e_i\right) - \eta\left(e_i\right)\eta(Z)g\left(Y, \phi e_i\right) + g\left(\phi Y, Z\right)\left(g(\phi e_i, \phi e_i)\right)\right]$$

$$+\frac{1}{(n-1)}\sum_{i=1}^{n-1}\left[S(Y, Z)g\left(\phi e_i, \phi e_i\right) - S\left(\phi e_i, Z\right)g\left(Y, \phi e_i\right)\right].$$

Also

$$(4.8)\qquad \sum_{i=1}^{n-1} g\left(R\left(\phi e_i, Y\right)Z, \phi e_i\right) = S(Y, Z) + g(Y, Z),$$

$$(4.9)\qquad \sum_{i=1}^{n-1} S\left(\phi e_i, Z\right)g\left(\phi e_i, Y\right) = S(Y, Z),$$

$$(4.10)\qquad \sum_{i=1}^{n-1} g\left(e_i, Z\right)g\left(Y, \phi e_i\right) = -\sum_{i=1}^{n-1} g\left(e_i, Z\right)g\left(\phi Y, e_i\right) = -g(\phi Y, Z),$$

$$(4.11)\qquad \sum_{i=1}^{n-1} g\left(\phi e_i, \phi e_i\right) = n-1,$$

$$(4.12)\qquad \sum_{i=1}^{n-1} \eta\left(e_i\right)\eta(Z)g\left(Y, \phi e_i\right) = -\eta(\phi Y)\eta(Z),$$

$$(4.13)\qquad \sum_{i=1}^{n-1} g\left(\phi e_i, Y\right)g\left(\phi e_i, Z\right) = g(Y, Z).$$

Hence by virtue of equations (4.8), (4.9), (4.10), (4.11), (4.12) and (4.13) equation (4.7) takes the form

(4.14) $$S(Y, Z) = -[(n-2)\varepsilon + 1] g(\phi Y, Z),$$

which leads to the following theorem-:

Theorem 4.1. *A quasi-projectively flat (ε)-Kenmotsu manifold with respect to a quarter- symmetric metric connection satisfies equation (4.14).*

5. ϕ-Projectively Flat (ε)-Kenmotsu Manifold With Respect to a Quarter-Symmetric Metric Connection

Definition 5.1. *An n-dimensional (ε)-Kenmotsu manifold with respect to a quarter- symmetric metric connection is said to be ϕ-projectively flat if*[17]

(5.1) $$\phi^2 \left(\tilde{P}(\phi X, \phi Y)\phi Z\right) = 0,$$

where $\tilde{P}$ is the projective curvature tensor of (ε)-Kenmotsu manifold with respect to quarter- symmetric metric connection.

Let M^n be ϕ-projectively flat (ε)-Kenmotsu manifold with respect to quarter- symmetric metric connection. It is easy to see that $\phi^2 \left(\tilde{P}(\phi X, \phi Y)\phi Z\right) = 0$ hold if and only if

(5.2) $$g\left(\tilde{P}(\phi X, \phi Y)\phi Z, \phi W\right) = 0,$$

for any $X, Y, Z, W \in TM^n$.
Putting $Y = \phi Y$ and $Z = \phi Z$ in equation (4.4), we obtain

(5.3) $$g\left(\tilde{P}(\phi X, \phi Y)\phi Z, \phi W\right) = g\left(\tilde{R}(\phi X, \phi Y)\phi Z, \phi W\right)$$

$$-\frac{1}{(n-1)}\left[\tilde{S}(\phi Y, \phi Z) g(\phi X, \phi W) - \tilde{S}(\phi X, \phi Z) g(\phi Y, \phi W)\right]$$

In view of equation (5.2), above equation takes the form

(5.4)
$$g\left(\tilde{R}(\phi X, \phi Y)\phi Z, \phi W\right) = \frac{1}{(n-1)}\Big[\tilde{S}(\phi Y, \phi Z)g(\phi X, \phi W) - \tilde{S}(\phi X, \phi Z)g(\phi Y, \phi W)\Big],$$

which on using equation (3.11) and (3.13) takes the form

(5.5)
$$\begin{aligned} g\left(R(\phi X, \phi Y)\phi Z, \phi W\right) &= g(\phi X, \phi Z)g(Y, \phi W) - g(\phi Y, \phi Z)g(X, \phi W) \\ &+\varepsilon\big[g(X, \phi Z)g(\phi Y, \phi W) - g(Y, \phi Z)g(\phi X, \phi W)\big] \\ &+\frac{1}{(n-1)}\Big[S(\phi Y, \phi Z)g(\phi X, \phi W) - S(\phi X, \phi Z)g(\phi Y, \phi W) \\ &-(1-n\varepsilon)\left\{g(Y, \phi Z)g(\phi X, \phi W) + g(X, \phi Z)g(\phi Y, \phi W)\right\}\Big] \end{aligned}$$

Let $\{e_1, e_2, \ldots\ldots e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n. We use the flat that $\{\phi e_1, \phi e_2, \ldots\ldots \phi e_{n-1}, \xi\}$ is also orthonormal basis. Putting $X = W = e_i$ in above equation and taking summation over i, $1 \le i \le n-1$, we get

(5.6)
$$\begin{aligned} \sum_{i=1}^{n-1} g\left(R(\phi e_i, \phi Y)\phi Z, \phi e_i\right) &= \sum_{i=1}^{n-1}\Big[g(\phi e_i, \phi Z)g(Y, \phi e_i) \\ &-g(\phi Y, \phi Z)g(e_i, \phi e_i)\Big] + \varepsilon\sum_{i=1}^{n-1}\Big[g(e_i, \phi Z)g(\phi Y, \phi e_i) \\ &-g(Y, \phi Z)g(\phi e_i, \phi e_i)\Big] + \frac{1}{(n-1)}\sum_{i=1}^{n-1}\Big[S(\phi Y, \phi Z)g(\phi e_i, \phi e_i) \\ &-S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) - (1-n\varepsilon)\left\{g(Y, \phi Z)g(\phi e_i, \phi e_i)\right. \\ &\left.+g(e_i, \phi Z)g(\phi Y, \phi e_i)\right\}\Big], \end{aligned}$$

which by virtue of equations (4.8), (4.9), (4.10), (4.11), (4.12) and (4.13) takes the form

(5.7) $$S(\phi Y, \phi Z) = \left(4n - n^2 - 5 + 2n\varepsilon\right) g(Y, \phi Z) - g(\phi Y, \phi Z) \quad ,$$

which in view of equations (2.5) and (2.16) takes the form

(5.8) $$S(Y, Z) = \left(4n - n^2 - 5 + 2n\varepsilon\right) g(Y, \phi Z) - g(Y, Z) + (2 - n\varepsilon)\eta(Y)\eta(Z).$$

Thus, we arise at

Theorem 5.1: *ϕ-Projectively flat (ε)-Kenmotsu manifold with quarter-symmetric metric connection satisfies equation (5.8).*

6. (ε)-Kenmotsu Manifold with Respect to a Quarter-Symmetric Metric Connection Satisfying $\tilde{P}.\tilde{S} = 0$

Consider (ε)-Kenmotsu manifold with respect to a quarter-symmetric metric connection satisfying

(6.1) $$\left(\tilde{P}(X, Y).\tilde{S}\right)(Z, U) = 0,$$

where $\tilde{S}$ is the Ricci tensor with respect to a quarter-symmetric metric connection. Then, we have

(6.2) $$\tilde{S}\left(\tilde{P}(X, Y)Z, U\right) + \tilde{S}\left(Z, \tilde{P}(X, Y)U\right) = 0.$$

Putting $X = \xi$ in above equation, we get

(6.3) $$\tilde{S}\left(\tilde{P}(\xi, Y)Z, U\right) + \tilde{S}\left(Z, \tilde{P}(\xi, Y)U\right) = 0$$

In view of equation (4.1), we have

(6.4) $$\tilde{P}(\xi, Y)Z = \tilde{R}(\xi, Y)Z - \frac{1}{(n-1)}\left[\tilde{S}(Y, Z)\xi - \tilde{S}(\xi, Z)Y\right].$$

By virtue of equations (3.11) and (3.13), we have

(6.5) $$\tilde{R}(\xi, Y)Z = R(\xi, Y)Z + g(\xi, Z)\phi Y - \varepsilon g(\phi Y, Z)\xi$$

and

(6.6) $$\tilde{S}(\xi, Z) = S(\xi, Z).$$

Using equations (6.5) and (6.6) in equation (6.4), we obtain

(6.7) $$\tilde{P}(\xi, Y)Z = 2\eta(Z)Y + \eta(Z)\phi Y - \varepsilon g(Y, Z)\xi - \left(\frac{1-\varepsilon}{n-1}\right) g(\phi Y, Z)\xi - \frac{1}{(n-1)}\tilde{S}(Y, Z)\xi .$$

Using equation (6.7) in equation (6.3), we get

(6.8) $$2\left[\eta(Z)\tilde{S}(Y, U) + \eta(U)\tilde{S}(Z, Y)\right] + \left[\eta(Z)\tilde{S}(\phi Y, U) + \eta(U)\tilde{S}(Z, \phi Y)\right] - \varepsilon\left[g(Y, Z)\tilde{S}(\xi, U) + g(Y, U)\tilde{S}(Z, \xi)\right] - \left(\frac{1-\varepsilon}{n-1}\right)\left[g(\phi Y, Z)\tilde{S}(\xi, U) + g(\phi Y, U)\tilde{S}(Z, \xi)\right] - \left(\frac{1}{n-1}\right)\left[S(Y, Z)\tilde{S}(\xi, U) + S(Y, U)\tilde{S}(Z, \xi)\right],$$

which on using equation (3.13) takes the form

(6.9) $$3\left[S(Y, Z)\eta(U) + S(Y, U)\eta(Z)\right] + \left[S(\phi Y, U)\eta(Z) + S(Z, \phi Y)\eta(U)\right] + \left[2(1-n\varepsilon) + (1-\varepsilon)\right] g(\phi Y, U)\eta(Z) + \left[2(1-n\varepsilon) + (1-\varepsilon)\right] \times g(\phi Z, U)\eta(U) + \left[(1-n\varepsilon) + (n-1)\varepsilon\right] g(Y, Z)\eta(U) - \left[(1-n\varepsilon) - (n-1)\varepsilon\right] g(Y, U)\eta(Z) + (1-n\varepsilon)(1-\varepsilon)\eta(Y)\eta(Z)\eta(U) = 0.$$

Putting $U = \xi$ in above equation and using equations (2.2) and (2.15), we have

(6.10)
$$S(Y,Z)+\left(\frac{1-\varepsilon}{3}\right)g(Y,Z)+\left(\frac{-2n+3-2\varepsilon+n\varepsilon}{3}\right)\eta(Y)\eta(Z)$$
$$=\frac{1}{3}S(\phi Z,Y)-\frac{2\varepsilon(n-1)}{3}g(\phi Z,Y).$$

Thus from the above discussions, we can state the following

Theorem (6.1): *An (ε)-Kenmotsu manifold M^n with quarter-symmetric metric connection satisfies* $\tilde{P}\,.\,\tilde{S}$ *= 0, is an η- Einstein manifold iff S(φZ, Y) = 2ε(n - 1) g(φZ, Y).*

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