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On the Degree Of Approximation of Conjugate Function Belonging to Lip $(\xi(t), p)$ Class of Conjugate Series of Fourier Series by Matrix Summability Method

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Abstract: we prove a new theorem on the degree of approximation of conjugate function belonging to the weighted Lip $(\xi(t), p)$ class of conjugate series of Fourier series by matrix summability method.

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1. Introduction

For the function $f \in Lip\alpha$ the degree of approximation by Cesâro means and Nörlund means of the Fourier series of f have been studied by Alexits[1], Sahney and Goel[2], Chandra[3], Qureshi[4,5], Qureshi and Neha[6], Leindler[7], Rhoads Lal and Nigam[8]. But till now nothing seems to have been done in the direction of present work . In this paper we established a new theorem on the degree of approximation of conjugate function belonging to the weighted Lip $(\xi(t), p)$ class of conjugate Fourier series by matrix summability method.

2. Preliminaries

Let $f(x)$ be periodic with period 2π and integrable in the sense of Labesgue. The Fourier series associated with f at a point x is defined as

(2.1) $$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

Partial sum of Fourier series usually written as

$$S_n(f,x)=\frac{a_0}{2}+\sum_{k=1}^{k}\left(a_k\cos kx+b_k\sin kx\right).$$

The conjugate series of Fourier series is given by

(2.2) $$\sum_{n=1}^{\infty}\left(b_n\cos nx-a_n\sin nx\right)=\sum_{n=1}^{\infty}B_n(x)=\bar{f}(x).$$

With partial sum $\bar{S}_n(f,x)$. We will use (2.2) as conjugate series of Fourier series.
Define,

(2.3) $$t_n(f,x)=\sum_{k=0}^{n}a_{n,k}\,S_k(f,x),$$

Where $a_{n,k}$ is a lower triangular matrix with non -negative entries such that

$$a_{n,-1}=0,\ A_{n,k}=\sum_{r=k}^{n}a_{n,k}\ ,\ A_{n,0}=1.$$

For all $n\geq 0$ The series (2.1) is said to T -summable to s If $t_n(f,x)\to s$ as $n\to\infty$. The T -operator reduces to the Nörlund N_p - operator, if

(2.4) $$a_{n,k}=\begin{cases}\dfrac{p_{n-k}}{P},\ 0\leq k\leq n\\ 0,\ k>n,\end{cases}$$

where $P_n=\sum_{k=0}^{n}p_k\neq 0$ and $p_{-1}=0=P_{-1}$. In this case. The transform $t_n(f,x)$ reduce to the Norlund transform $N_n(f,x)$. if $\lim_{n\to\infty}a_{n\,k}=0$ for each k, then T is reguar .

The L_p - norm is defined by

(2.5) $$\|f\|_p=\int_0^{2\pi}\left(\left|f(x)\right|^p dx\right)^{\frac{1}{p}},\ p\geq 1,\ \|f\|_\infty=\sup_{x\in[0,\,2\pi]}\left|f(x)\right|$$

and the degree of approximation $E_n(f)$ is given by

(2.6) $$E_n(f)=\min_n\left\|f(x)-T_n(x)\right\|_p,$$

Where $T_n(x)$ is a trigonometric polynomial of degree *n*. A function $f\in Lip\alpha$, if

(2.8) $$\left|f(x+t)-f(x)\right|=O\left(|t|^\alpha\right) \text{ for } 0<\alpha<1$$

And $f\in Lip(\alpha,p)$, for $0\leq x\leq 2\pi$, if

(2.9) $$W_p(t,f)=\left(\int_0^{2\pi}\left|f(x+t)-f(x)\right|^p dx\right)^{\frac{1}{p}}=O\left(|t|^\alpha\right),\ 0<\alpha\leq 1,\ p\geq 1.$$

Given a positive increasing function $\xi(t)$ and $p\geq 1$, $f(x)\in Lip(\xi(t),p)$ if

(2.10) $$W_p(t,f)=\left(\int_0^{2\pi}\left|f(x+t)-f(x)\right|^p dx\right)^{\frac{1}{p}}=O(\xi(t))$$

and $f(x)\in W\left(L_p,\xi(t)\right)$ if

$$W_p(t,f)=\left(\int_0^{2\pi}\left|f(x+t)-f(x)\right|^p\sin^\beta x dx\right)^{\frac{1}{p}}=O(\xi(t))\ .$$

If $\beta=0$, our newly defined class $W\left(L_p,\xi(t)\right)$ coincides with the class $Lip(\xi(t),p)$. We write

$$\psi(t)=\frac{1}{2}\left[f(x+t)-f(x-t)\right],$$

$$W_n=\left|\frac{1}{P_n}\right|\sum_{k=0}^{n}K\left|p_k-p_{k-1}\right|,$$

$$W_n(r)=\sum_{k=0}^{r}(k+1)\left|\Delta_k a_{n,\,n-k}\right|,\ 0\le r\le n,$$

$$M(n,t)=\sum_{k=0}^{n}a_{n,\,n-k}\cos\left(n-k+\frac{1}{2}\right)t,$$

$$\bar{f}(x)=\frac{1}{\pi}\int_0^{\pi}\psi(t)\cot\left(\frac{t}{2}\right)dt=\lim_{q\to 0}\int_q^{\pi}\psi(t)\cot\left(\frac{t}{2}\right)dt\ ,$$

$$A_{n,k}=\sum_{k=0}^{n}a_{n,r}\ ,$$

$$V_{n,k}=\frac{(n-k+1)a_{n,k}}{A_{n,k}}\ ,$$

$$\bar{K}(n,t)=\frac{1}{\sin\left(\frac{t}{2}\right)}\sum_{k=0}^{n}a_{n,\,n-k}\cos\left(n-k+\frac{1}{2}\right)t,$$

$$\bar{K}(n,t)=\frac{M(n,t)}{\sin\left(\frac{t}{2}\right)}.$$

$$\tau=\left[\frac{1}{t}\right],\ \text{integral part of }\frac{1}{t}$$

$$\Delta_k a_{n,n-k}=a_{n,k}-a_{n,k+1}.$$

3. Main theorem

Lemma[3] 3.1: *Let* $T=\left(a_{n,k}\right)$ *be an infinite triangular matrix satisfying (.Then*

(3.1) $R_{n,k}=O(1),\ 0\le k\le n.$

For $k=0,$

(3.2) $R_{n,0}=O(1),$

i.e.,

(3.3) $$(n+1)a_{n,0}=O(1).$$

Lemma[9] (3.2): *Let* $T=(a_{n,k})$ *be an infinite triangular matrix satisfying (2.4).Then*

(3.4) $$|M(n,t)|=O(A_{n,n-\tau})+O\left(\frac{1}{\tau}\right)\left(\sum_{k\tau}^{n-1}|\Delta_k a_{n,n-k}|+a_{n,0}\right).$$

Theorem 3.1: *Let* $T=(a_{n,k})$ *be an infinite triangular matrix with non – negative entries satisfying[9], then the degree of approximatiom of function* $\bar{f}(x)$*, conjugate to a* $2\pi-$*periodic function* $f(x)$ *belonging to class* $W(L_p,\xi(t)), p\geq 1$ *by using a matrix operator on its conjugate Fourier seires , is given by*

(3.5) $$\left\|\tilde{f}(x)-\tilde{t}_n(x)\right\|=O\left(n^{\phi+\frac{1}{p}}\xi(t)\right).$$

Provided that $\xi(t)$ *satisfies (2.4) and (2.5) uniformly in* $\xi(t)$*, in which* δ *is an arbitrary positive number with* $q(1-\delta)-1>0$*, where* $p^{-1}+q^{-1}=1,\ 1\leq p\leq\infty$ *and condition (2.8) holds*

Remark (3.1): in the case of the N_p- transform , condition[9], for $r=n$, reduce to (2.1) (ii) and thus theorem (3.1) extends theorem (2.6) to matrix summability for the waighted class function .

Remarks (3.2): Also for $\phi=0$, theoem (3.1) reduce to Corollary (4.1) , and thus generalizes the theorem of Lal and Nigam[2].

Proof: We know that

$$\bar{S}_{n-k}(x)-\bar{f}(x)=-\frac{1}{2\pi}\int_0^{\pi}\psi(x)\left[\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right]dt,$$

(3.6) $$\sum_{k=0}^{n}a_{n,n-k}\left\{\bar{f}(x)-\bar{S}_{n-k}(x)\right\}=\frac{1}{2\pi}\int_0^{\pi}\psi(x)\left[\sum_{k=0}^{n}a_{n,n-k}\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right]dt,$$

Therefore

$$\bar{f}(x)-\bar{t}_n(x)=\frac{1}{2\pi}\int_0^{\pi}\psi(x)\left[\sum_{k=0}^{n}a_{n,n-k}\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right]dt,$$

Hence

$$\left|\bar{f}(x)-\bar{t}_n(x)\right|\leq\frac{1}{2\pi}\int_0^{\pi}|\psi(x)|\left|\bar{K}(n,t)\right|dt$$

$$=\frac{1}{2\pi}\left[\int_0^{\frac{\pi}{n+1}}+\int_{\frac{\pi}{n+1}}^{\pi}\right]|\psi(x)|\left|\bar{K}(n,t)\right|dt=\frac{1}{2\pi}\left[I_1+I_2\right]|\psi(x)|,\text{ say}$$

Since $\bar{K}(n,t)=O\left(\frac{1}{t}\right)$, using Hölder's inequality , condition (2.4) and the fact that $(\sin t)^{-1}\leq\frac{\pi}{2t}$, for $0<t\leq\frac{\pi}{2}$, and the second mean value theorem for the integrals,

$$I_1=\int_0^{\frac{\pi}{n+1}}|\psi(x)|\left|\bar{K}(n,t)\right|dt$$

$$=O\left[\int_0^{\frac{\pi}{n+1}}\left\{\frac{t\psi(t)\sin^{\phi}t}{\xi(t)}\right\}^p dt\right]^{\frac{1}{p}}\left[\int_0^{\frac{\pi}{n+1}}\left\{\frac{t\psi(t)\bar{K}(n,t)}{t\sin^{\phi}t}\right\}^q dt\right]^{\frac{1}{q}}$$

(3.7) $$=O\left(\frac{1}{n+1}\right)\left[\int_h^{\frac{\pi}{n+1}}O\left\{\frac{\xi(t)}{t^2\sin^{\phi}t}\right\}^q dt\right]^{\frac{1}{q}}$$

$$=O\left(\frac{1}{n+1}\right)\left[\left(\frac{\frac{\pi}{n+1}}{\sin\frac{\pi}{n+1}}\right)^{\phi q}\int_h^{\frac{\pi}{n+1}}O\left\{\frac{\xi(t)}{t^{2+\phi}}\right\}dt\right]^{\frac{1}{q}}$$

$$=O\left(\frac{1}{n+1}\right)\left[\int_h^{\frac{\pi}{n+1}} O\left\{\frac{\xi(t)}{t^{2+\phi}}\right\}dt\right]^{\frac{1}{q}}$$

Since $\xi(t)$ is non- decreasing with t and also using condition (2.8),

$$I_1=O\left(\frac{1}{n+1}\right)O\left(\xi\left(\frac{\pi}{n+1}\right)\right)\left[\int_h^{\frac{\pi}{n+1}} t^{-(2+\phi)q}dt\right]^{\frac{1}{q}}$$

(3.8) $$=O\left((n+1)^{-1}\xi\left(\frac{1}{n+1}\right)\right)O\left(n^{2+\phi-\frac{1}{q}}\right)=O\left(n^{\phi+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right).$$

Using Hӧlder’s inequality, condition (2.5) , Lemma (3.2) , Minkowski ’s inequality , and condition (2.8).

$$I_1=\int_{\frac{\pi}{n+1}}^{\pi}|\psi(x)||\bar{K}(n,t)|dt$$

$$\leq\left[\int_{\frac{\pi}{n+1}}^{\pi}\left\{\frac{t^{-\delta}\psi(t)\sin^{\phi}t}{\xi(t)}\right\}^p dt\right]^{\frac{1}{p}}\left[\int_{\frac{\pi}{n+1}}^{\pi}\left\{\frac{\psi(t)\bar{K}(n,t)}{t^{-\delta}\sin^{\phi}t}\right\}^q dt\right]^{\frac{1}{q}}$$

$$\leq\left[\int_{\frac{\pi}{n+1}}^{\pi}\left\{\frac{t^{-\delta}|\psi(t)||\sin^{\phi}t|}{\xi(t)}\right\}^p dt\right]^{\frac{1}{p}}\left[\int_{\frac{\pi}{n+1}}^{\pi}\left\{\frac{\psi(t)M(n,t)}{t^{-\delta}\sin^{\phi}t\,Sin\frac{t}{2}}\right\}^q dt\right]^{\frac{1}{q}}$$

$$=O\left((n+1)^{\delta}\right)\left[\int_{\frac{\pi}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{-\delta+1+\phi}}\right\}^q O\left\{A_{n,n-\tau}+t^{-1}\left(a_{n,0}+\sum_{k=\tau}^{n-1}|\Delta_k a_{n,n-k}|\right)\right\}^q dt\right]^{\frac{1}{q}}$$

$$=O\left((n+1)^{\delta}\xi\left(\frac{\pi}{n+1}\right)\right)O\left[I_{2.1}+I_{2.2}+I_{2.3}\right],\text{ say}$$

Since A has non– negative entries and row sums one

$$I_{2.1} = \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left(t^{\delta-\phi} A_{n,\,n-\tau} \right)^q dt \right\}^{\frac{1}{q}}$$

(3.9)
$$= \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left(t^{\delta-\phi} \right)^q dt \right\}^{\frac{1}{q}}$$

$$= O\left(n^{\phi-\delta-\frac{1}{q}} \right).$$

Using Lemma (3.1)

$$I_{2.2} = \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left(t^{\delta-\phi-1} a_{n,\,0} \right)^q \right\}^{\frac{1}{q}}$$

(3.10)
$$= O(a_{n,\,0}) \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left(t^{\delta-\phi} - 1 \right)^q dt \right\}^{\frac{1}{q}}$$

$$= O\left((n+1)^{-1} \right) \left(n^{\phi-\delta+1-\frac{1}{q}} \right) = O\left(n^{\phi-\delta+1-\frac{1}{q}} \right).$$

From condition[9]

$$I_{2.3} = \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left(t^{\delta-1-\phi} \sum_{k=\tau}^{n-1} \left| a_{n,\,n-k} \right| \right)^q dt \right\}^{\frac{1}{q}}$$

$$= O\left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left(t^{\delta-\phi} (\tau+1) \sum_{k=\tau}^{n-1} \left| a_{n,\,n-k} \right| \right)^q dt \right\}^{\frac{1}{q}}$$

(3.11) $$=O\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left(t^{\delta-\phi}\sum_{k=\tau}^{n-1}(k+1)\left|a_{n,n-k}\right|\right)^{q}dt\right\}^{\frac{1}{q}}$$

$$=O\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left(t^{\delta-\phi}W_n(n+1)\right)^{q}dt\right\}^{\frac{1}{q}}=O\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left(t^{\delta-\phi}A_{n,0}\right)^{q}dt\right\}^{\frac{1}{q}}$$

$$=O\left(n^{\phi-\delta-\frac{1}{q}}\right).$$

Combining (3.9), (3.10) and (3.11)

(3.12) $$I_2=O\left(n^{\delta+1}\xi\left(\frac{1}{n}\right)\right)\left(n^{\phi-\delta-\frac{1}{q}}\right)=O\left(n^{\phi+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right).$$

Hence I_1 and I_2 yields

(3.13) $$\left|\bar{f}(x)-\bar{t}_n(x)\right|=O\left(n^{\phi+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right).$$

Using L_p- norm, we have

$$\left\|\bar{f}(x)-\bar{t}_n(x)\right\|_p=\left\{\int_0^{2\pi}\left|\bar{f}(x)-\bar{t}_n(x)\right|^p dx\right\}^{\frac{1}{p}}$$

(3.14) $$=O\left\{\int_0^{2\pi}\left(n^{\phi+\frac{1}{p}}\xi\left(\frac{1}{n}\right)\right)dx\right\}^{\frac{1}{p}}$$

$$=O\left\{n^{\phi+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\int_0^{2\pi}dx\right\}^{\frac{1}{p}}=O\left[n^{\phi+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right].$$

4. Particular cases

Corollary[13] 4.1: *If* $\xi(t)=t^{\alpha}$, $0<\alpha\le 1$, *then the waighted class* $W(L_p,\xi(t))$, $p\ge 1$, *reduce to the class* $Lip(\alpha, p)$ *and the degree of approximation of a function* $\bar{f}(x)$, *conjugate to* $2\pi-$ *periodic function* f *belonging to the class* $Lip(\alpha, p)$ *is given by*

(4.1) $$\left|\overline{t}(x)-\overline{f}(x)\right|=O\left(\frac{1}{n^{\alpha-\frac{1}{\phi}}}\right).$$

Proof: The result follows by setting $\phi=0$ in (3.1)

Corollary[12] 4.2: *If* $\xi(t)=t^{\alpha}$, *for* $p=\infty$, *in corollary (4.1) , then* $f\in Lip\,\alpha.$

Proof: In his case , using (4.1), one has theorem (3.1). for $p=\infty,$ we get

$$\left\|\overline{f}(x)-\overline{t}_n(x)\right\|_p=\sup_{0\le x\le 2\pi}\left|\overline{f}(x)-\overline{t}_n(x)\right|$$

(4.2) $$=O\left(\frac{1}{n^{\alpha}}\right)$$

$$=O\left(\frac{1}{P_n}\sum_{k=1}^{n}\frac{P_k}{K^{\alpha+1}}\right),\ for\ p_n\ge p_{n+1}.$$

i.e., $$\left|\overline{f}(x)-\overline{t}_n(x)\right|=O\left(\frac{1}{P_n}\sum_{k=1}^{n}\frac{P_k}{K^{\alpha+1}}\right).$$

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