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On the Approximation of Conjugate Lip α Class Function by $K^{\lambda}-$ Summability Means of its Fourier Series

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Abstract: A new theorem on the degree of approximation of conjugate lip α class function by $K^{\lambda}-$ summability means of its Fourier series is proved

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1. Introduction

Recently, the degree of approximation of function belonging to $K^{\lambda}-$summability of Fourier series using different summability method has been researched by various investigators[1-15] and references therein. Moreover we obtined the similar results for a class of more extensive function, we need to introduce some notations and definition. It is well known that the partial sums of the Foureir series usually be written as

$$S_n(f,x)=\frac{a_0}{2}+\sum_{k=0}^{n}\left(a_k\cos kx+b_k\sin kx\right),\ n\in N.$$

Where $f\in L^p=L^p[0,\,2\pi]$ is a $2\pi-$ periodic function and $p\geq 1$.

A function $f\in Lip\alpha,$ if

$$f(x+t)-f(x)=o\left(|t|^{\alpha}\right)\ \text{for}\ 0<\alpha\leq 1.$$

The degree of approximation $E_n(f)$ of a function $f \in L^p$ space by triginometric polynomial $T_n(x)$ of degree n is given by

$$E_n(f) = \min_{T_n} \left\| f(x) - T_n(x) \right\|_p .$$

Let us define, for $n = 0, 1, 2, 3.......$the number $\begin{bmatrix} n \\ m \end{bmatrix}$, $0 \le m \le n$ by

$$x(x+1)(x+2)(x+3)......(x+n-1) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m ,$$

i.e.,

$$\frac{\Gamma(x+n)}{\Gamma x} = \prod_{\gamma=0}^{n-1} (x+\gamma) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m .$$

The number $\begin{bmatrix} n \\ m \end{bmatrix} x^m$ are known as absolute value of stirling number of first kind. Let $\{S_n\}$ be the sequence of partial sums of an infinite series $\sum a_n$ and let us write,

$$S_n^\lambda = \frac{\Gamma \lambda}{\Gamma(\lambda+n)} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \lambda^m S_m .$$

To denote nth $K^\lambda -$means of order $\lambda > 0$. If $S_n^\lambda \to S$ as $n \to \infty$, where S is a fixed finite quantity, theb the sequence $\{S_n\}$ of the series $\sum a_n$ is said to be summable by Karamata method K^λ of order $\lambda > 0$ to sum S and we can write

$$S_n^\lambda = S(K^\lambda) \text{ as } n \to \infty.$$

We still need a few notations

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$K_n(t)=\frac{\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m \sin\left(m+\frac{1}{2}\right)t}{\Gamma(\lambda+n)\sin\frac{t}{2}}.$$

2. On the Approximation of Conjugate Lip α Class Function by $K^{\lambda}-$ Summability

Lemma 2.1: *(Vuckovic 1965) Let* $\lambda>0$ *and* $0<t<\frac{\pi}{2}$ *then imaginary part of* $\frac{\Gamma\left(\lambda e^{it}+n\right)}{\Gamma(\lambda\cos t+n)\sin\frac{t}{2}}=\frac{\left|\sin\{\lambda\log(n+1)\sin t\}\right|}{\sin\frac{t}{2}}+o(1)$ *as* $n\to\infty$, *uniformly in* t.

Theorem 2.1: *If* $f:R\to R$ *is* $2\pi-$ *periodic function and Lipschitz class function then the degree of approximation of function* f *by the* $K^{\lambda}-$*summability of Fourier series satisfies* $\left\|S_n^{\lambda}-f\right\|_{\infty}=o\left(\frac{\log(n+1)e}{(n+1)^{\alpha+1}}+\frac{1}{(n+1)^{\lambda}}\right)$ *for* $n=0,1,2,3,.....$

Proof: Titchmarsh (1939. p – 403) the n^th^ partial sum of Fourier series at $t=x$ is

$$S_m-f(x)=\frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\phi(t)\,dt.$$

Then

$$\frac{\Gamma\lambda}{\Gamma(\lambda+n)}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m\{S_m-f(x)\}=\frac{1}{2\pi}\frac{\Gamma\lambda}{\Gamma(\lambda+n)}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m\int_0^{\pi}\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\phi(t)\,dt.$$

$$S_n^{\lambda}-\frac{\Gamma\lambda}{\Gamma(\lambda+n)}\frac{\Gamma(\lambda+n)}{\Gamma\lambda}f(x)=\frac{1}{2\pi}\int_0^{\pi}K_n(t)\phi(t)\,dt$$

$$S_n^\lambda - f(x) = \frac{\Gamma\lambda}{2\pi}\int_0^\pi K_n(t)\phi(t)\,dt = o\left[\int_0^\pi K_n(t)\phi(t)\,dt\right]$$

(2.1)
$$= o\left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\left|K_n(t)\right|\left|\phi(t)\right|dt$$

$$= o\left[I_1\right] + o\left[I_2\right].$$

To evaluate I_1

$$K_n(t) = \frac{\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m e^{i\left(m+\frac{1}{2}\right)t}}{\Gamma(\lambda+n)\sin\frac{t}{2}},$$

$$K_n(t) = \mathrm{Im}aginary\ part\,o\ \frac{\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m e^{i\left(m+\frac{1}{2}\right)t}}{\Gamma(\lambda+n)\sin\frac{t}{2}},$$

$$= I_p\,\frac{\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m e^{imt} e^{i\frac{1}{2}t}}{\Gamma(\lambda+n)\sin\frac{t}{2}} = I_p\,\frac{\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda e^{it} e^{\frac{it}{2}}}{\Gamma(\lambda+n)\sin\frac{t}{2}} = I_p\,\frac{\left\{\frac{\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda e^{it}\right)} e^{\frac{it}{2}}\right\}}{\Gamma(\lambda+n)\sin\frac{t}{2}}$$

$$= I_p\left\{\cos\frac{t}{2} + \sin\frac{t}{2}\right\}\frac{\left\{\frac{\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda e^{it}\right)}\right\}}{\Gamma(\lambda+n)\sin\frac{t}{2}}$$

$$= \frac{\cos\frac{t}{2} I_p\left\{\frac{\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda e^{it}\right)}\right\} + \sin\frac{t}{2}\,real\ part\,of\ \left\{\frac{\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda e^{it}\right)}\right\}}{\Gamma(\lambda+n)\sin\frac{t}{2}}$$

$$=\frac{\cos\frac{t}{2}I_p\left\{\frac{\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda e^{it}\right)}\right\}}{\Gamma(\lambda+n)\sin\frac{t}{2}}+\sin\frac{t}{2}\,real\ part\ of\ \frac{\left\{\frac{\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda e^{it}\right)}\right\}}{\Gamma(\lambda+n)\sin\frac{t}{2}}$$

$$=o\left[\frac{I_p\left\{\frac{\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda e^{it}\right)}\right\}}{\Gamma(\lambda+n)\sin\frac{t}{2}}\right]+o\left[\frac{real\ part\ of\left\{\frac{\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda e^{it}\right)}\right\}}{\Gamma(\lambda+n)\sin\frac{t}{2}}\right]$$

$$K_n(t)=o\left[\frac{\left\{\frac{\Gamma(\lambda\cos t+n)I_p\Gamma\left(\lambda e^{it}+n\right)}{\Gamma(\lambda+n)}\right\}}{\Gamma(\lambda\cos t+n)\sin\frac{t}{2}}\right]+o\left[\frac{\Gamma(\lambda\cos t+n)}{\Gamma(\lambda+n)}\right].$$

For $0<t<\frac{1}{n+1}$

$$\frac{\Gamma(\lambda\cos t+n)}{\Gamma(\lambda+n)}=o\left[n^{-\lambda(1-\cos t)}\right]=o\left[e^{-\frac{\lambda}{2}t^2\log n}\right].$$

Since for $0<t<\frac{1}{n+1}$, $0<1-\cos t<\frac{t^2}{2}$

$$K_n(t)=o\left[\frac{e^{-\frac{\lambda}{2}t^2\log n}I_p\Gamma\left(\lambda e^{it}+n\right)}{\Gamma(\lambda\cos t+n)\sin\frac{t}{2}}\right]+o\left[e^{-\frac{\lambda}{2}t^2\log n}\right]\ \text{for}\ 0<t<\frac{1}{n+1}$$

and

$$K_n(t)=o\left\{\frac{1}{(n+1)^{\lambda}t}\right\}\ \text{for}\ \frac{1}{n+1}<t<\pi$$

$$I_1=\int_0^{\frac{1}{n+1}}|\phi(t)||K_n(t)|dt$$

$$=o\left\{\int_0^{\frac{1}{n+1}}\frac{e^{-\frac{\lambda}{2}t^2\log n}I_p\Gamma\left(\lambda e^{it}+n\right)}{\Gamma(\lambda\cos t+n)\sin\frac{t}{2}}|\phi(t)|dt\right\}+o\left[\int_0^{\frac{1}{n+1}}e^{-\frac{\lambda}{2}t^2\log n}|\phi(t)|dt\right].$$

Applying lemma, we have

$$I_1=o\left[\int_0^{\frac{1}{n+1}}\left[\left[e^{-\frac{\lambda}{2}t^2\log n}\left[\frac{|\{\sin[\lambda\log(n+1)\sin t]\}|}{\sin\frac{t}{2}}\right]+I(1)\right]|\phi(t)|dt\right]+o\left[\int_0^{\frac{1}{n+1}}e^{-\frac{\lambda}{2}t^2\log n}|\phi(t)|dt\right]$$

$$=o\left[\int_0^{\frac{1}{n+1}}\left[\frac{e^{-\frac{\lambda}{2}t^2\log n}}{\sin\frac{t}{2}}\{\sin[\lambda\log(n+1)\sin t]\}\right]|\phi(t)|dt\right]+o\left[\int_0^{\frac{1}{n+1}}e^{-\frac{\lambda}{2}t^2\log n}|\phi(t)|dt\right]$$

$$=o\left[\int_0^{\frac{1}{n+1}}\left[\frac{e^{-\frac{\lambda}{2}t^2\log n}}{\sin\frac{t}{2}}\{\sin[\lambda\log(n+1)\sin t]\}\right]|\phi(t)|dt\right]+o(1)\left[\int_0^{\frac{1}{n+1}}|\phi(t)|dt\right].$$

$$I_1=o\left[\lambda(n+1)\int_0^{\frac{1}{n+1}}|\phi(t)|dt\right]+o(1)\left[\int_0^{\frac{1}{n+1}}|\phi(t)|dt\right].$$

Since $f(x+t)-f(x)=o\left(|t|^{\alpha}\right)$ i.e. $f\in Lip\alpha$. We have

$$\begin{aligned}\phi(t)&=f(x+t)+f(x-t)-2f(x)\\&=|f(x+t)-f(x)|+|f(x-t)-f(x)|\\&=o\left(|t|^{\alpha}\right)+o\left(|t|^{\alpha}\right)\\&=o\left(|t|^{\alpha}\right)\end{aligned}$$

thus $\phi \in Lip\alpha$.
Hence

$$I_1 = o\left[\lambda \log(n+1)\int_0^{\frac{1}{n+1}} o\left(\left|t^\alpha\right|\right)dt\right] + o(1)\left[\int_0^{\frac{1}{n+1}} o\left(\left|t^\alpha\right|\right)dt\right],$$

$$I_1 = o\left[\lambda \log(n+1)\right]\int_0^{\frac{1}{n+1}} \left|t^\alpha\right|dt + o(1)\left[\int_0^{\frac{1}{n+1}} \left|t^\alpha\right|dt\right]$$

$$= o\left[\lambda \log(n+1)\right]\left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\frac{1}{n+1}} + o(1)\left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\frac{1}{n+1}}$$

$$= \left[\frac{\lambda \log(n+1)}{(\alpha+1)(n+1)^{\alpha+1}}\right] + o(1)\left[\frac{1}{(\alpha+1)(n+1)^{\alpha+1}}\right]$$

$$= o\left[\frac{\lambda \log(n+1)+1}{(n+1)^{\alpha+1}}\right] = \left[\frac{\log(n+1)e}{(n+1)^{\alpha+1}}\right]$$

Now I_2, for $\frac{1}{n+1} < t < \pi$, $\phi(t)$ is bounded.

$$I_2 = \int_{\frac{1}{n+1}}^{\pi} \left|\phi(t)\right|\left|K_n(t)\right|dt$$

$$= o\left[\int_{\frac{1}{n+1}}^{\pi} \frac{1}{(n+1)^\lambda}\left|\phi(t)\right| dt\right]$$

$$= o\left(\frac{1}{(n+1)^\lambda}\right)\int_{\frac{1}{n+1}}^{\pi} \frac{\phi(t)}{t}dt$$

$$=o\left(\frac{1}{(n+1)^{\lambda}}\right)\int_{\frac{1}{n+1}}^{\pi}\frac{t^{\alpha}}{t}\,dt$$

$$=o\left(\frac{1}{(n+1)^{\lambda}}\right)\left[\frac{t^{\alpha}}{\alpha}\right]_{\frac{1}{n+1}}^{\pi}$$

$$=o\left(\frac{1}{(n+1)^{\lambda}}\right)\left[\frac{\pi^{\alpha}}{\alpha}-\frac{1}{\alpha(n+1)^{\alpha}}\right].$$

$$I_2=o\left(\frac{1}{(n+1)^{\lambda}}\right)$$

$$o(I_1)+o(I_2)=o\left[\frac{\log(n+1)e}{(n+1)^{\alpha+1}}\right]+o\left[\frac{1}{(n+1)^{\lambda}}\right]$$

Thus

$$\left|S^{\lambda}-f(x)\right|=o\left(\frac{\log(n+1)e}{(n+1)^{\alpha+1}}+\frac{1}{(n+1)^{\lambda}}\right).$$

Hence

$$\left\|S_n^{\lambda}-f\right\|_{\infty}=o\left(\frac{\log(n+1)e}{(n+1)^{\alpha+1}}+\frac{1}{(n+1)^{\lambda}}\right).$$

This proof the theorem.

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