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# Generalized Sasakian-Space-Form with $W_4$ Curvature Tensor

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**Abstract:** In this paper, we study generalized Sasakian-space-form with $W_4$ curvature tensor. We find some relations between differentiable functions $f_1$, $f_2$ and $f_3$ and we also find Ricci-tensor, Ricci-operator and scalar curvature in a $W_4$ flat generalized Sasakian-space-form.

**Keywords:** Ricci-tensor, Ricci-operator and scalar curvature, Sasakian-space-form, $W_4$ curvature tensor.

**2010 AMS Classification Number:** 53C25, 53D15.

## 1. Introduction

Alegre et al.[1] introduced and studied the notion of generalized Sasakian-space-form. A generalized Sasakian-space-form is an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ whose curvature tensor is given by

$$
\begin{aligned}
(1.1) \qquad R(X,Y)Z = {} & f_1[g(Y,Z)\,X - g(X,Z)\,Y] \otimes \\
& + f_2[g(X,\varphi Z)\,\varphi Y - g(Y,\varphi Z)\,\varphi X + 2g(X,\varphi Y)\,\varphi Z] \\
& + f_3[\eta(X)\,\eta(Z)\,Y - \eta(Y)\,\eta(Z)\,X \\
& + g(X,Z)\,\eta(Y)\,\xi - g(Y,Z)\,\eta(X)\,\xi],
\end{aligned}
$$

where $f_1, f_2$ and $f_3$ are differentiable functions on $M$ and $X, Y, Z$ are vector fields on $M$. In such case we shall write generalized Sasakian-space-form as $M(f_1, f_2, f_3)$. This type of manifold appears as a natural generalization of the well known Sasakian-space-form $M(c)$, which can be obtain as a particular case of generalized Sasakian-space-form by taking $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$, where $c$ denotes the constant $\varphi -$ sectional curvature. Moreover, cosympletic space form, Kenmotsu space forms are also particular cases of generalized Sasakian-space-form $M(f_1, f_2, f_3)$. Alegre and Carriazo[2] also studied contact metric and trans-Sasakian generalized Sasakian-space-forms. Kim[3] in his paper studied conformally flat and locally symmetric generalized Sasakian-space-form.

In this present paper generalized Sasakian-space-form with $W_4$ curvature tensor has been studied. In a $W_4$ flat generalized Sasakian-space-form we also find Ricci-tensor, Ricci-operator and scalar curvature. The notion of $W_4$ curvature tensor was introduced by G. P. Pokhariyal[4]. A $(2n+1)-$dimensional Riemannian $M$ is $W_4$ flat if $W_4 = 0$, where $W_4$ curvature tensor is defined as

(1.2) $$W_4(X, Y)Z = R(X, Y)Z + \frac{1}{2n}[g(X, Z)QY - g(X, Y)QZ],$$

Where $Q$ is the field of symmetric endomorphism corresponding to the Ricci tensor $S$ i.e. $g(QX, Y) = S(X, Y)$.

If a Riemannian manifold satisfies $R(X, Y)W_4 = 0$, where $W_4$ is a $W_4$ curvature tensor, then the manifold is said to be $W_4$ semi-symmetric manifold.

## 2. Preliminaries

In this section, we recall some general definitions and basic formulas which will use later. For this, we recommend the reference[5]. A $(2n+1)-$dimensional Riemannian manifold $(M, g)$ is said to be an almost contact metric manifold if there exist a $(1,1)$ tensor field $\varphi$, a unique global non-vanishing structural vector field $\xi$ (called the vector field) and a $1-$form $\eta$ such that

(2.1) $$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\xi) = 1,$$

(2.2) $$d\eta(X,\xi) = 0, \qquad g(X,\xi) = \eta(X),$$

(2.3) $$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$

(2.4) $$d\eta(X,Y) = g(X,\varphi Y), \qquad \eta o \varphi = 0.$$

Such a manifold is called contact manifold if $\eta \wedge (d\eta)^n \neq 0,$ where $n$ is $n^{\text{th}}$ exterior power. For contact manifold we also have $d\eta = \Phi$, where $\Phi(X,Y) = g(\varphi X, Y)$ is called fundamental $2-$form on $M$. If $\xi$ is killing vector field, then $M$ is said to be $K-$contact manifold. The almost contact metric structure $(\varphi, \xi, \eta, g)$ on $M$ is said to be normal if

(2.5) $$[\varphi,\varphi](X,Y) + 2d\eta\,(X,Y)\xi = 0,$$

for all vector field $X, Y$ on $M$, where $[\varphi,\varphi]$ denotes the Nijenhuis tensor of $\varphi$ given by

(2.6) $$[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

An almost contact metric manifold $M$ is said to be $\eta-$Einstein if its Ricci-tensor $S$ is of the form

(2.7) $$S(X,Y) = cg(X,Y) + d\eta(X)\,\eta(Y),$$

Where $c$ and $d$ are smooth functions on $M$. A $\eta-$Einstein manifold becomes Einstein if $d = 0$.

If $\{e_1, e_2, ......, e_{2n}, \xi\}$ is a local orthonormal basis of vector fields in an almost contact metric manifold $M$ of dimension $(2n+1)$, then $\{\varphi e_1, \varphi e_2, ......, \varphi e_{2n}, \xi\}$ is also a local orthonormal basis. It is easy to verify that

(2.8) $$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\varphi e_i, \varphi e_i) = 2n,$$

(2.9) $$\sum_{i=1}^{2n} g(e_i, Y)\, S(X, e_i) = \sum_{i=1}^{2n} g(\varphi e_i, Y)\, S(X, \varphi e_i)$$
$$= S(X, Y) - S(X, \xi)\, \eta(Y),$$

for all $X, Y \in T(M)$. In view of (2.4) and (2.9) and we have

(2.10) $$\sum_{i=1}^{2n} g(e_i, \varphi Y)\, S(\varphi X, e_i) = \sum_{i=1}^{2n} g(\varphi e_i, \varphi Y)\, S(\varphi X, \varphi e_i)$$
$$= S(\varphi X, \varphi Y).$$

## 3. Some Results on Generalized Sasakian-Space-Form

For a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ of dimension $(2n+1)$, we have

(3.1) $$R(X, Y)\xi = (f_1 - f_3)\,[\eta(Y)\, X - \eta(X)\, Y],$$

(3.2) $$S(X, Y) = (2nf_1 + 3f_2 - f_3)\, g(X, Y)$$
$$- (3f_2 + (2n-1)\, f_3)\, \eta(X)\, \eta(Y).$$

From (3.1), we have

(3.3) $$R(X, \xi)\, \xi = (f_1 - f_3)[X - \eta(X)\, \xi],$$

(3.4) $$R(X, \xi)\, Y = (f_1 - f_3)(\eta(Y)\, X - g(X, Y)\, \xi),$$

(3.5) $$Q(X) = (2nf_1 + 3f_2 - f_3)\, X - (3f_2 + (2n-1)\, f_3)\, \eta(X)\, \xi.$$

(3.6) $$r = 2n(2n+1)\, f_1 + 6nf_2 - 4nf_3,$$

Where $Q$ is the Ricci operator and $r$ is the scalar curvature of $M(f_1, f_2, f_3)$. Now from (3.2) and (3.5), we have

(3.7) $$S(X,\xi) = 2n(f_1 - f_3)\,\eta(X),$$

and

(3.8) $$Q\xi = 2n(f_1 - f_3)\,\xi.$$

from (3.7), we get

(3.9) $$\sum_{i=1}^{2n} S(e_i, e_i) = \sum_{i=1}^{2n} S(\varphi e_i, \varphi e_i) = r - 2n(f_1 - f_3),$$

Where $r$ is scalar curvature. In a generalized Sasakian-space-form $M\,(f_1, f_2, f_3)$, we also have

(3.10) $$R(X,\xi,\xi,Y) = R(\xi, X, Y, \xi) = (f_1 - f_3)\,g(\varphi X, \varphi Y),$$

and

(3.10) $$\sum_{i=1}^{2n} R(e_i, X, Y, e_i) = \sum_{i=1}^{2n} R(\varphi e_i, X, Y, \varphi e_i) = S(X,Y) - (f_1 - f_3)\,g(\varphi X, \varphi Y),$$

for all $X, Y \in T(M)$.

## 4. $W_4$ - Flat Generalized Sasakian-Space-Form

Let $M\,(f_1, f_2, f_3)$, be a $(2n+1)-$dimensional generalized Sasakian-space form. The Riemannian curvature tensor $R$, the Ricci-tensor $S$ and the Ricci-operator $Q$ of $M$ are given by equations (1.1), (3.2) and (3.5) respectively. Putting the value of $R(X,Y)\,Z$, $S(X,Y)$ and $QX$ in the equation (1.2), we get

(4.1) $$W_4(X,Y)Z = f_1[g(Y,Z)\,X - g(X,Z)\,Y] + f_2[g(X,\varphi Z)\,\varphi Y - g(Y,\varphi Z)\,\varphi X + 2g(X,\varphi Y)\varphi Z] + f_3[\eta(X)\eta(Z)\,Y - \eta(Y)\,\eta(Z)\,X$$

$$+g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi]$$
$$+\frac{1}{2n}[g(X,Z)\{(2nf_1+3f_2-f_3)Y$$
$$-(3f_2+(2n-1)f_3)\eta(Y)\xi\}$$
$$-g(X,Y)\{(2nf_1+3f_2-f_3)Z$$
$$-(3f_2+(2n-1)f_3)\eta(Z)\xi\}].$$

On simplifying above equation, we get

(4.2)
$$W_4(X,Y)Z = \frac{1}{2n}(3f_2-f_3)g(X,Z)Y + f_1(g(Y,Z)X)$$
$$+f_2[g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X$$
$$+2g(X,\varphi Y)\varphi Z] + f_3[\eta(X)\eta(Z)Y$$
$$-\eta(Y)\eta(Z)X - g(Y,Z)\eta(X)\xi]$$
$$+\frac{1}{2n}(f_3-3f_2)g(X,Z)\eta(Y)\xi$$
$$-\frac{1}{2n}(2nf_1+3f_2-f_3)g(X,Y)Z$$
$$+\frac{1}{2n}(3f_2+(2n-1)f_3)g(X,Y)\eta(Z)\xi.$$

If $M(f_1, f_2, f_3)$ is $W_4$ flat, then we have $W_4(X,Y)Z=0$. If we put $X=\varphi Y$ in the above equation, we get

$$\frac{1}{2n}(3f_2-f_3)g(\varphi Y,Z)Y + f_1(g(Y,Z)\varphi Y) + f_2[g(Y,Z)\varphi Y$$
$$-\eta(Y)\eta(Z)\varphi Y + g(Y,\varphi Z)Y - g(Y,\varphi Z)\eta(Y)\xi + 2g(Y,Y)\varphi Z$$
$$-2\eta(Y)\eta(Y)\varphi Z] + f_3[-\eta(Y)\eta(Y)\varphi Y$$

(4.3)
$$+\frac{1}{2n}(f_3-3f_2)g(\varphi Y,Z)\eta(Y)\xi$$
$$-\frac{1}{2n}(2nf_1+3f_2-f_3)g(\varphi Y,Y)Z$$
$$+\frac{1}{2n}(3f_2+(2n-1)f_3)g(\varphi Y,Y)\eta(Z)\xi = 0.$$

If we choose a unit vector field $U$ such that $\eta(U)=O$ and putting $Y=U$ in the equation (4.3), then we get

(4.4)
$$\frac{1}{2n}(3f_2-f_3)\,g(\varphi U,Z)U+f_1(g(U,Z)\varphi U)+f_2[g(U,Z)\,\varphi U$$
$$-g(\varphi U,Z)\,U+2\varphi Z]-\frac{1}{2n}(2nf_1+3f_2-f_3)\,g(\varphi U,U)\,Z$$
$$+\frac{1}{2n}(3f_2+(2n-1)\,f_3)\,g(\varphi U,U)\,\eta(Z)\,\xi=0.$$

Again taking $Z=U$ in the equation (4.4), we get

(4.5) $$(f_1+3\,f_2)\varphi U=0.$$

In view of equation (4.5), we have the following theorem:

**Theorem 4.1:** *In a $W_4$ flat generalized Sasakian-space-form* $(f_1+3\,f_2)=0.$

Now under the consideration of $W_4$ flat manifold equation (1.2) reduces to

(4.6) $$R(X,Y,Z,U)=\frac{1}{2n}[g(X,Z)\,S(Z,U)-g(X,Z)\,S(Y,U)],$$

where $R(X,Y,Z,U)=g(R(X,Y)\,Z,U)$. Putting $Z=\xi$ in the equation (4.6) and using equations (2.2), (3.1) and (3.7), we get

(4.7)
$$(f_1-f_3)[\eta(Y)\,g(X,U)-\eta(X)\,g(Y,U)]$$
$$=\frac{1}{2n}[2n(f_1-f_3)\,\eta(U)\,g(X,Y)-\eta(X)\,S(Y,U)].$$

Now putting $X=\xi$ and using equations (2.1) and (2.2), we get

(4.8) $$S(Y,U)=2n(f_1-f_3)\,g(Y,U).$$

Putting $U=\xi$ in the equation (4.8), we get

(4.9) $$QY=2n(f_1-f_3)\,Y.$$

If $\{e_1, e_2, ......, e_{2n}, e_{2n+1}\xi\}$ is a local orthonormal basis of vector fields in $M\,(f_1,\ f_2,\ f_3),$ then from equation (4.8), we get

$$\sum_{i=1}^{2n+1} S(e_i, e_i) = 2n(f_1 - f_3)\sum_{i=1}^{2n+1} g(e_i, e_i).$$

Using equation (2.8), we get

(4.10) $$r = 2n(2n+1)\,(f_1 - f_3).$$

**Theorem 4.2:** *In a* $W_4$ *flat generalized Sasakian-space-form, the Ricci-tensor* $S$, *the Ricci-operator* $Q$ *and the scalar curvature* $r$ *are given by the equations (4.8), (4.9) and (4.10) respectively.*

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