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Almost Kenmotsu Manifold Admitting Semi-Symmetric Metric Connection

Shweta Naik and H. G. Nagaraja

Department of Mathematics
Bangalore University, Bengaluru-560056, India.
Email: naikshwetamaths@gmail.com, hgnraj@yahoo.com

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Abstract: We study almost Kenmotsu manifold admitting semi-symmetric metric connection. We proved the conditions for this manifold to be of constant curvature. Further we verify our results by giving an example.

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1. Introduction

A normal manifold with closed 1-form η and $d\varphi = 2\eta\Lambda\varphi$ called almost Kenmotsu manifold was studied by Dileo and Pastore[1]. They also investigated locally symmetric almost Kenmotsu manifolds. Wang and Liu[2-4] Dey and Majhi5 proved some interesting theorems in almost Kenmotsu manifolds with nullity distributions.

The notion semi-symmetric linear connection was initially studied by Friedmann and Schouten[6] and the study was continued by Hayden[7]. Further Yano systematically studied semi-symmetric metric connection on Riemannian manifolds[8], and the study extended to almost contact metric manifolds by several others.

Here we study almost Kenmotsu manifold M admitting semi-symmetric metric connection $\hat{\nabla}$. We give preliminaries and basic results in section 2. In Section 3, we obtain conditions for M with $\hat{\nabla}$ to be of constant curvature provided it satisfies certain semi-symmetry, Ricci-semi symmetry conditions with respect to $\hat{\nabla}$ and flatness like curvature conditions with respect to conformal, concircular and projective curvature tensors. We constructed an example in section 4 to verify our results.

2. Preliminaries

Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be an almost contact metric manifold, where φ, ξ, η and g are respectively a $(1, 1)$ tensor field, characteristic vector field and a 1-form on M satisfying

(2.1) $$\phi^2 Z = -Z + \eta(Z)\xi, \qquad \eta(\xi) = 1.$$

From (2.1) we have $\operatorname{rank}(\phi) = 2n$ and

(2.2) $$\eta \cdot \phi = 0, \quad \phi\xi = 0,$$

(2.3) $$g(\phi Y, \phi Z) = g(Y, Z) - \eta(Y)\eta(Z).$$

Now, we denote by $l = R(\cdot, \xi)\xi$ and $h = \frac{1}{2} L_\xi \phi$, two symmetric (1, 1)-type tensors on M. The tensors l and h satisfy:

$$h\xi = 0, \quad \operatorname{tr} h = 0, \quad \operatorname{tr}(h\varphi) = 0, \quad h\varphi + \varphi h = 0.$$

(2.4) $$\nabla_X \xi = -\varphi^2 X + h\varphi X,$$

(2.5) $$(\nabla_Y \eta) Z = g(Y, Z) - \eta(Y)\eta(Z) + g(h\varphi Y, Z),$$

(2.6) $$l - \varphi l \varphi = -2(h^2 - \varphi^2),$$

(2.7) $$(\nabla_X \varphi) Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X,$$

(2.8) $$R(X, Y)\xi = \eta(X)(Y - \varphi h Y) - \eta(Y)(X - \varphi h X) - (\nabla_X \varphi h)Y + (\nabla_Y \varphi h)X,$$

(2.9) $$\nabla_\xi h = -\varphi h^2 - \varphi - 2h - \varphi l,$$

(2.10) $$S(X,\xi) = -2n\eta(X) + g(div(\varphi h), X),$$

(2.11) $$\mathrm{tr}(l) = -2n + \mathrm{tr}\,h^2 = S(\xi,\xi),$$

where $X, Y \in TM$, S is Ricci tensor, ∇ is Levi-civita connection in M respectively. Also $\nabla_\xi \varphi = 0$.

Throughout this paper the quantities with cap are with respect to semi - symmetric metric connection $\hat{\nabla}$ and the quantity without cap are with respect to Levi-civita connection ∇. The connections $\hat{\nabla}$ and ∇ are related by[4]

(2.12) $$\hat{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi.$$

Taking $Y = \xi$ in (2.12) we have,

(2.13) $$\hat{\nabla}_X \xi = -2\varphi^2 X - \varphi h X.$$

Using the definition of R and (2.12), we have

(2.14) $$\begin{aligned}\hat{R}(V,U)Z = R(V,U)Z &+ 3\left[g(V,Z)U - g(U,Z)V\right] \\ &+2\eta(Z)\left[\eta(U)V - \eta(V)U\right] \\ &+2\left[g(U,Z)\eta(V) - g(V,Z)\eta(U)\right]\xi \\ &+g(\varphi hU, Z)V - g(\varphi hV, Z)U \\ &-g(V,Z)\varphi hU + g(U,Z)\varphi hV.\end{aligned}$$

Contracting (2.14) on V, we have

(2.15) $$\begin{aligned}\hat{S}(U,Z) = S(U,Z) &- (6n-2)g(U,Z) \\ &+(4n-2)\eta(U)\eta(Z) + (2n-1)g(\varphi hU, Z).\end{aligned}$$

Again contracting (2.15), we get

(2.16) $$\hat{r} = r - 2n(6n-1).$$

From (2.16), we have the following:

$$\hat{R}(V, Z)\xi = 2\eta(V)(Z - \varphi hZ) - 2\eta(Z)(V - \varphi hV) - (\nabla_V \varphi h)Z + (\nabla_Z \varphi h)V . \tag{2.17}$$

$$\hat{R}(V, \xi)Z = 2g(V, Z) - g(\varphi hV, Z)\xi - \eta(Z) \times\left[2V - 3\varphi V + h^2 V - \varphi\left(\nabla_\xi h\right)V\right]. \tag{2.18}$$

From (2.15), we derive

$$\hat{S}(\xi, U) = -4n\eta(U) + g\left(div(\varphi h), U\right), \tag{2.19}$$

$$\hat{S}(\xi, \xi) = -4n + \mathrm{tr}\left(h^2\right). \tag{2.20}$$

Definition 2.1: *A Riemannian manifold M is said to be*
(i) Projectively flat if the projective curvature tensor H given by

$$H(V, Z)U = R(V, Z)U - \frac{1}{2n}\left[S(Z, U)V - S(V, U)Z\right] \tag{2.21}$$

vanishes identically on M.
(ii) Concircularly flat if a $(0, 4)$ tensor $C(V, U, W, Z)$ invariant under concircular transformation called the concircular curvature tensor[9] *given by*

$$C(X, U, W, Z) = R(X, U, W, Z) - \frac{r}{2n(2n-1)} \times\left[g(U, W)g(X, Z) - g(X, W)g(U, Z)\right] \tag{2.22}$$

vanishes. i.e. $C(V, U, W, Z) = 0$.
(iii) Conformally flat if the conformal curvature tensor K given by

$$K(V, U)Z = R(V, U)Z + \frac{r}{2n(2n-1)}\left[g(U, Z)V - g(V, Z)U\right] \tag{2.23}$$

$$-\frac{1}{2n-1}\left[S(U,Z)V-S(V,Z)U+g(U,Z)QV-g(V,Z)QU\right]$$

vanishes.

3. Results and Analysis

Throughout this section M^* denote $(2n+1)$ dimensional almost Kenmotsu manifold admitting semi-symmetric metric connection $\hat{\nabla}$ and $\{e_i : i=1,2,......2n+1\}$ denote an orthonormal basis of the tangent space at each point of M .

Lemma 3.1: *If M^* is Ricci-semi-symmetric with respect to $\hat{\nabla}$ then* $\operatorname{tr} h^2 = 0$.

Proof: Suppose in M^*, $\left(\hat{R}(U,V)\cdot\hat{S}\right)(Z,W)=0$ holds.

(3.1) $$\hat{S}\left(\hat{R}(U,V)W, Z\right)+\hat{S}\left(W, \hat{R}(V,U)Z\right)=0.$$

Using (2.14), (2.15) in (3.1), and taking $U=V=\xi$, we get

(3.2) $$\begin{aligned} &S\left(R(V,\xi)W,\xi\right)+S\left(W,R(V,\xi)\xi\right)=-(4n-2)\\ &\times\left[g(V,W)-\eta(V)\eta(W)\right]-(2n-1)g\left(\varphi hW, R(V,\xi)\xi\right)\\ &-g(V,W)\left[-4n+tr\,h^2\right]+\eta(W)\left[g\left(div(\varphi h),V\right)\right]\\ &+g(\varphi hV,W)\left[-4n+tr\,h^2\right]-\eta(V)\left[g\left(div(\varphi h),W\right)\right]\\ &+\left[S(W,V)-(6n-2)g(W,V)+(4n-2)\eta(V)\eta(W)\right.\\ &\left.+(2n-1)g(\varphi hV,W)\right]-\left[S(W,\varphi hV)-(6n-2)\right.\\ &\left.\times g(W,\varphi hV)+(2n-1)g(hV,hW)\right]. \end{aligned}$$

Now, we take $W=\xi$ in (3.2), and simplify by using (2.8) and (2.10) to get,

$$\operatorname{tr} h^2 = 0.$$

Lemma 3.2: *If* M^* *is*
(i) Concircularlly flat with respect to $\hat{\nabla}$ *, then*

$$r = tr\left(h^2\right) + 2n\left(2n-3\right).$$

(ii) Projectivelly flat with respect to $\hat{\nabla}$ *, then*

$$r = \left(4n+1\right) tr\, h^2 + 2n\left(2n-3\right).$$

(iii) φ*-projectively semi-symmetric with respect to* $\hat{\nabla}$ *, then*

$$r = 2n\left(2n-3\right) + \left(4n+1\right) tr\, h^2.$$

Proof: Case (i): Suppose M^* is concircularly flat with respect to $\hat{\nabla}$. Then from (2.22), we have

$$g\left(\hat{R}\left(U,V\right)W,\, X\right) = \frac{\hat{r}}{2n\left(2n+1\right)}\left[g\left(U,W\right)g\left(V,X\right) - g\left(V,W\right)g\left(U,X\right)\right]. \tag{3.3}$$

Setting $V = X = \xi$, and using (2.16), (2.22) we get,

$$-2g\left(\varphi U,\, \varphi W\right) + 3g\left(\varphi hU,\, W\right) - g\left(h^2 U,\, W\right) + g\left(\varphi\left(\nabla_\xi h\right)U,\, W\right) = \frac{r - 2n\left(6n-1\right)}{2n\left(2n+1\right)}\left[g\left(U,W\right) - \eta\left(U\right)\eta\left(W\right)\right]. \tag{3.4}$$

Setting $U = W = e_i$, in (3.4) and then summing over $i = 1, 2, ...2n+1$, we obtain

$$r = \left(2n+1\right) tr\left(\varphi\left(\nabla_\xi h\right)\right) - \left(2n+1\right) tr\left(h^2\right) + 2n\left(2n-3\right). \tag{3.5}$$

Now from (2.9), (2.11) and $tr\left(h\varphi\right) = 0$, it follows that $tr\left(\varphi\left(\nabla_\xi h\right)\right) = 2tr\left(h^2\right)$. Substituting this in (3.5), we get

$$r = tr\left(h^2\right) + 2n\left(2n-3\right). \tag{3.6}$$

Case (ii): If M^* is projectively flat, then (2.21), becomes

(3.7) $$\hat{R}(U,V)Z=\frac{1}{2n}\left[\hat{S}(V,Z)U-\hat{S}(U,Z)V\right].$$

Taking inner product with W and setting $V=W=\xi$, and by using (2.14) and (2.20), we get

(3.8) $$S(V,Z)=\left(2n-2+tr\,h^2\right)g(V,Z)-(4n-2)\eta(V)\eta(Z)$$
$$-(4n+1)g(\varphi hV,Z).$$

Setting $V=Z=e_i$ in (3.8) and summing over i from $i=1,2,......2n+1$, we get

(3.9) $$r=(4n+1)tr\,h^2+2n(2n-3).$$

Case (iii): Suppose M^* is φ-projectively semi-symmetric with respect to $\hat{\nabla}$ i.e., $\hat{H}o\varphi=0$. Then

(3.10) $$\hat{H}(U,V)\varphi W-\varphi\hat{H}(U,V)W=0,$$

which implies

(3.11) $$\hat{R}(U,V)\varphi W-\varphi\hat{R}(U,V)W-\frac{1}{2n}\Big[\hat{S}(V,\varphi W)U$$
$$-\hat{S}(U,\varphi W)V+\hat{S}(V,W)\varphi U-\hat{S}(U,W)\varphi V\Big]=0.$$

Taking $V=\xi$ in (3.11) we have,

(3.12) $$\hat{R}(U,\xi)\varphi W-\varphi\hat{R}(U,\xi)W-\frac{1}{2n}\Big[\hat{S}(\xi,\varphi W)U$$
$$-\hat{S}(U,\varphi W)\xi+\hat{S}(\xi,W)\varphi U\Big]=0.$$

By using (2.14) and (2.20) and taking inner product with ξ, we get

(3.13) $$S(U,\varphi W)=(2n-2+tr\,h^2)g(U,\varphi W)+g(hU,W).$$

Replacing W by φW in (3.13), we have

(3.14) $$S(U,W)=2\left(n-1+tr\,h^2\right)g(U,W)-\left(4n-2+tr\,h^2\right)\times\eta(U)\eta(W)+g(\varphi hU,W).$$

Setting $U=W=e_i$, in (3.14), summing over $i=1,2,......2n+1$, we get

(3.15) $$r=2n(2n-3)+(4n+1)tr\,h^2.$$

Hence lemma (3.1) follows from case (i)-(iii). Now from Lemma 3.1 and Lemma 3.2), we state the following

Theorem 3.1: *If M^* is either concircularlly flat or projectivelly flat or φ-projectively semi-symmetric with respect to $\hat{\nabla}$, then M is of constant scalar curvature.*

Theorem 3.2: *The manifold M^* is conformally flat (or φ-conformally semi- symmetric) with respect to $\hat{\nabla}$ if and only if* $\mathrm{tr}\,h^2=0$.

Proof: Case (i): Suppose M^* is conformally flat with respect to $\hat{\nabla}$. Then from (2.23), we have

(3.16) $$\hat{R}(U,V)W=\frac{1}{2n-1}\left[\hat{S}(V,W)U-\hat{S}(U,W)V+g(V,W)\hat{Q}U-g(U,W)\hat{Q}V\right]-\frac{\hat{r}}{2n(2n-1)}\left[g(V,W)U-g(U,W)V\right].$$

Contracting (3.16) with ξ and using (2.14), (2.15) and (2.20), we have

(3.17) $$S(U,W)=\left(tr\,h^2+1+\frac{r}{2n}\right)g(U,W)-\left(2n+1+\frac{r}{2n}\right)\eta(U)\eta(W).$$

Setting $U=W=e_i$ in (3.17) taking summation over $i=1,2,.....,(2n+1)$ we obtain

(3.18) $$tr\,h^2=0.$$

Conversely, from (3.17) it follows that $\hat{K}(U,V)W=0$.

Case (ii): Suppose M^* is φ-conformally semi-symmetric with respect to $\hat{\nabla}$. Then $\hat{K}\cdot\varphi = 0$, i.e.

(3.19) $$\hat{K}(U,V)\varphi W - \varphi\hat{K}(U,V)W = 0.$$

From equation (2.23) we have

(3.20) $$\hat{R}(U,V)\varphi W - \varphi\hat{R}(U,V)W - \frac{1}{2n-1}\Big[\hat{S}(V,\varphi W)U - \hat{S}(U,\varphi W)V$$
$$+g(V,W)\hat{Q}U - g(U,W)\hat{S}(V,W)\varphi U - \hat{S}(U,W)\varphi V$$
$$+g(V,W)\varphi\hat{Q}U - g(U,W)\varphi\hat{Q}V\Big] + \frac{\hat{r}}{2n(2n-1)}$$
$$\Big[g(V,\varphi W)g(U,\varphi W)V - g(V,W)\varphi U + g(U,W)\varphi V\Big] = 0.$$

Contraction of (3.20) with ξ, we get

(3.21) $$\eta\Big(\hat{R}(U,V)\varphi W\Big) - \frac{1}{2n-1}\Big[\hat{S}(V,\varphi W)\eta(U) - \hat{S}(U,\varphi W)\eta(V)$$
$$+g(V,W)\eta\big(\hat{Q}U\big) - g(U,W)\eta\big(\hat{Q}V\big)\Big] + \frac{\hat{r}}{2n(2n-1)}$$
$$\Big[g(V,\varphi W)\eta(U) - g(U,\varphi W)\eta(V)\Big] = 0.$$

Taking $V = \xi$ in (3.21) and using (2.16) and (2.21) we have,

(3.22) $$S(U,\varphi W) = \left(1 + \frac{r}{2n}\right)g(U,\varphi W).$$

Replacing W by φW in (3.22), we have

(3.23) $$S(U,W) = \left(1 + \frac{r}{2n}\right)g(U,W) - \left(1 + \frac{r}{2n} + 2n - tr\,h^2\right)\eta(U)\eta(W).$$

Putting $U = W = e_i$ in (3.23) and summation over i gives

(3.24) $$tr\, h^2 = 0\,.$$

Conversely, from (3.23) it follows that $\hat{K}\cdot\varphi = 0$. Therefore the Theorem follows from (3.18) and (3.24).

Theorem 3.3: *If in M^*, φh is of Codazzi type then M^* is semi-symmetric.*

Proof: If φh is of Codazzi type, i.e.,

(3.25) $$g\left((\nabla_U \varphi h)V,\, Z\right) - g\left((\nabla_V \varphi h)U,\, Z\right) = 0\,,$$

then, it follows from (2.8) that

(3.26) $$R(U,\, Z)\xi = \eta(U)(Z - \varphi h Z) - \eta(Z)(U - \varphi h U)\,.$$

And (2.14) becomes

(3.27) $$\hat{R}(U,\, Z)V = 4\left[g(U, V)Z - g(Z, V)U\right] + 2\eta(V)$$
$$\left[\eta(Z)U - \eta(U)Z\right] + 2\left[g(Z, V)\eta(U) - g(U, V)\eta(Z)\right]\xi$$
$$+2\left[g(\varphi h Z, V)U - g(\varphi h U, V)Z\right] - \left[g(U, V)\varphi h Z - g(Z, V)\varphi h U\right].$$

Consider $\hat{R}(U,\, Z)\cdot\hat{R} = 0$ i.e.,

(3.28) $$\left(\hat{R}(U,\, Z)\right)\hat{R}(V,\, Y)W = \hat{R}(U,\, Z)\hat{R}(V,\, Y)W - \hat{R}\left(\hat{R}(U,Z)V, Y\right)W$$
$$-\hat{R}\left(V,\, \hat{R}(U,\, Z)Y\right)W - \hat{R}(V,\, Y)\hat{R}(U,\, Z)W\,.$$

Taking $Z = W = \xi$ in (3.28) we have

(3.29) $$\left(\hat{R}(U,\, \xi)\hat{R}\right)(V,\, Y)\xi = \hat{R}(U,\, \xi)\hat{R}(V,\, Y)\xi - \hat{R}\left(\hat{R}(U,\, \xi)V,\, Y\right)\xi$$
$$-\hat{R}\left(V,\, \hat{R}(U,\, \xi)Y\right)\xi - \hat{R}(V,\, Y)\hat{R}(U,\, \xi)\xi\,.$$

Computing each of four terms of RHS of (3.29) separately and after simplification, we get

(3.30) $$\hat{R}(U,\xi)\cdot\hat{R}=0.$$

Hence the proof.

Example: Let (x, y, z) are the standard coordinates in R. We consider the 3-dimensional manifold $M=\{(x, y, z)\in R^3\}$. The vector fields $E_1=\frac{\partial}{\partial x}$, $E_2=\frac{\partial}{\partial y}$, $\xi=E_3=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{\partial}{\partial z}$ form a basis for T_pM at each p in M. We define the Riemannian metric g by,

$$g_{ij}=\begin{cases}1 \text{ for } i=j\\ 0 \text{ for } i\neq j\end{cases}.$$

Let $\eta(Z)=g(Z, E_3)$ for any $Z\in\chi(M)$. Then $\eta(E_3)=1$. Let φ be defined by $\varphi(E_1)=E_2$, $\varphi(E_2)=E_1$, $\varphi(E_3)=0$. We see from the definition of φ and g that $\varphi^2Z=-Z+\eta(Z)E_3$, $g(\varphi Z,\varphi W)=g(Z,W)-\eta(Z)\eta(W)$, for any vector fields Z and W on M. Thus the structure (φ, ξ, η, g) is an almost contact structure. We also derive that $[E_1, E_3]=E_1$, $[E_2, E_3]=E_2$, $[E_1, E_2]=0$. By Koszul's formula the connection ∇ of g is given by

$$2g(\nabla_V U, W)=Vg(U,W)+Ug(W,V)-Wg(V,U)$$
$$-g(V,[U,W])-g(U,[V,W])+g(W,[V,U]).$$

By the use of the above formula, we obtain

$$\nabla_{E_1}E_3=E_1,\ \ \nabla_{E_2}E_3=E_2,\ \ \nabla_{E_1}E_1=-E_3,\ \ \nabla_{E_2}E_2=0,$$

$$\nabla_{E_3}E_3=0,\ \ \nabla_{E_2}E_1=0,\ \ \nabla_{E_3}E_1=0,\ \ \nabla_{E_1}E_2=0, \nabla_{E_3}E_2=0.$$

From $h=\frac{1}{2}L_{\varepsilon}\varphi$ and the above equations we see that

(3.31) $$hE_1=hE_2=hE_3=0.$$

Also we get $\nabla_X \xi = -\varphi^2 X + h\varphi X$ for any $X \in \chi(M)$. Therefore, M is an almost Kenmotsu manifold.

By the above results, we obtain the components of the curvature tensor R as follows:

$$R(E_1, E_2)E_3 = 0, \quad R(E_2, E_3)E_3 = -E_3, \quad R(E_1, E_3)E_3 = -E_1,$$

$$R(E_1, E_2)E_2 = -E_1, \quad R(E_3, E_2)E_2 = E_3, \quad R(E_1, E_3)E_2 = 0,$$

$$R(E_1, E_2)E_1 = E_2, \quad R(E_2, E_3)E_1 = 0, \quad R(E_1, E_3)E_1 = E_3.$$

Now the semi-symmetric metric connection on M is given by

$$\hat{\nabla}_{E_1} E_3 = 2E_1, \quad \hat{\nabla}_{E_2} E_3 = 2E_2, \quad \hat{\nabla}_{E_1} E_1 = -2E_3, \quad \hat{\nabla}_{E_2} E_2 = 0,$$

$$\hat{\nabla}_{E_3} E_3 = 0, \quad \hat{\nabla}_{E_2} E_1 = 0, \quad \hat{\nabla}_{E_3} E_1 = 0, \quad \hat{\nabla}_{E_1} E_2 = 0, \quad \hat{\nabla}_{E_3} E_2 = 0.$$

In view of above relations M is a 3-dimensional almost Kenmotsu manifold which admits semi-symmetric metric connection and

$$\hat{R}(E_1, E_2)E_3 = 0, \quad \hat{R}(E_2, E_3)E_3 = -2E_3, \quad \hat{R}(E_1, E_3)E_3 = -2E_1,$$

$$\hat{R}(E_1, E_2)E_2 = -4E_1, \quad \hat{R}(E_3, E_2)E_2 = -2E_3, \quad \hat{R}(E_1, E_3)E_2 = 0,$$

$$\hat{R}(E_1, E_2)E_1 = 4E_2, \quad \hat{R}(E_2, E_3)E_1 = 0, \quad \hat{R}(E_1, E_3)E_1 = 2E_3.$$

Making use of the above results we obtain the Ricci tensor as follows:

$$S(E_1, E_1) = g\big(R(E_1, E_2)E_2, E_1\big) + g\big(R(E_1, E_3)E_3, E_1\big) = -2.$$

Similarly we have

$$S(E_2, E_2) = S(E_3, E_3) = -2, \quad \hat{S}(E_1, E_1) = -6 \text{ and } \hat{S}(E_3, E_3) = -4.$$

$$r = \sum_{i=1}^{3} S\left(E_i, E_i\right) = -6 \text{ and } \hat{r} = \sum_{i=1}^{3} \hat{S}\left(E_i, E_i\right) = -16 .$$

Further we have from the above equations $\hat{R}\cdot\hat{S} = 0$. For instance,

$$\left(\hat{R}\left(E_1, E_3\right)\cdot\hat{S}\right)\left(E_1, E_1\right) = 0, \quad \left(\hat{R}\left(E_1, E_2\right)\cdot\hat{S}\right)\left(E_1, E_1\right) = 0,$$

$$\left(\hat{R}\left(E_1, E_1\right)\cdot\hat{S}\right)\left(E_1, E_1\right) = 0 .$$

This is true for other components also. From equation (4.1) we get

$$tr\, h^2 = \sum_{i=1}^{3} g\left(h^2 E_i, E_i\right) = 0 .$$

Thus Lemma 3.1 is verified.

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