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Non Invariant Hypersurfaces of Quasi Sasakian Manifolds

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Abstract: The aim of the present paper is to study noninvariant hypersurfaces of a quasi sasakian manifold with (f, g, u, v, λ) structure. We have broken the paper into some parts. The first part contains the properties of this structure. In the second part the second fundamental form of noninvariant hypersurfaces of quasi sasakian manifolds with (f, g, u, v, λ) structure have been calculated provided that f is parallel . Third part is mainly based on non invariant hypersurface with (f, g, u, v, λ) structure and totally geodesic submanifold of quasi sasakian manifolds. Finaly we conclude the paper by obtaining the necessary condition for totally geodesic or totally umbilical noninvariant hypersurfaces with (f, g, u, v, λ) structure of a quasi sasakian manifold.

Keywords: Remannian metric, tensor field, noninvariant hypersurfaces, quasi sasakian manifold.

1. Introduction

The transform of a tangent vector field of a hypersurface by the (1,1) structure tensor field φ defining an almost contact structure could not be tangent to the hypersurface was noticed by Goldberg and Yano[1]. Yano and Okumura[2] defined (f, g, u, v, λ) structure and named it as a noninvariant hypersurface of an almost contact metric manifold. In that paper they established that there always exists a (f, g, u, v, λ) structure and stated the results that there does not have any invariant hypersurface of a contact manifold. As a result, invariant hypersurface of almost cosymplectic structure is almost Kahelerian. In the present paper we investigate noninvariant hypersurfaces of quasi sasakian manifolds .

2. Preliminaries

Let $\tilde{M}$ be a $2n+1$ dimensional connected differentiable manifold endowed with an almost contact structure (φ, η, ξ, g) where φ is a tensor field of type $(1,1)$, ξ is a vector field, η is a $1-$form and g is the Riemannian metric on $\tilde{M}$ such tha[3, 4].

(2.1) $$\varphi^2(X)=-X+\eta(X)\xi, \qquad \eta(\xi)=1$$

(2.2) $$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y), \quad X,Y\in T\tilde{M}$$

Then also,

(2.3) $$\varphi\xi=0,\ \eta(\varphi X)=0,\ \eta(X)=g(X,\xi).$$

Let φ be a fundamental $2-$form defined by

$$\varphi(X,Y)=g(X,\varphi Y), \quad X,Y\in T\tilde{M}\ .$$

$\tilde{M}$ said to be quasi sasakian if the almost contact structure (φ,η,ξ,g) is normal and the fundamental $2-$form φ is closed $(d\varphi=0)$ which was first introduced by Blair[4]. The normality condition gives that the induced almost contact structure on $\tilde{M}\times R$ is integrable or equivalently, the tensor field. $N[\varphi,\varphi]-2\xi\times\eta$ vanishes identically on $\tilde{M}$. The rank of a quasi sasakian structure is always odd[4]. It is equal to 1, If the structure is cosymplectic and it is equal to $2n+1$, if the structure is sasakian. An almost contact metric manifold M of dimension three is quasi sasakian manifold if and only if[5].

(2.4) $$\tilde{\nabla}_X\xi=-\beta\varphi X, \qquad X\in T\tilde{M},$$

for a certain function β on M such that $\xi\beta=0$, $\tilde{\nabla}$ being the operator of the covariant differentiation with respect to the Levi-Civita connection of $\tilde{M}$. Clearly such a quasi sasakian manifold is cosymplectic if and only if $\beta=0$. As a consequence of (2.4) we have[5].

(2.5) $$(\tilde{\nabla}_X\varphi)Y=\beta(g(X,Y))\xi-\eta(Y)X, \quad X,Y\in T\tilde{M}$$

(2.6) $$(\tilde{\nabla}_X\eta)Y=g(\tilde{\nabla}_X\xi,Y)=-\beta g(\varphi X,Y).$$

The Gauss and Weingarten formulae are given by

(2.7) $\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)\tilde{N}.$

(2.8) $\tilde{\nabla}_X \tilde{N} = -\tilde{A}_N X,$ for all $X, Y \in TM$.

$\tilde{\nabla}$ and ∇ are the Riemannian and induced Riemannian connection $\tilde{M}$, M respectively, and $\tilde{N}$ is the unit normal vector in the normal bundle $T^{\perp}M$ In this formula σ is the second fundamental form on M related to $\tilde{A}_{\tilde{N}}$ by

(2.9) $\sigma(X, Y) = g\left(\tilde{A}_{\tilde{N}} X, Y\right).$

Let M be the hypersurface of an almost contact metric structure manifold then we define the following

(2.10) $\varphi X = f X + u(X)\tilde{N}.$

(2.11) $\varphi \tilde{N} = -U.$

(2.12) $\xi = V + \lambda \tilde{N}.$

(2.13) $\lambda = \eta(\tilde{N}).$

(2.14) $\eta(X) = v(X),$ for all $X \in TM.$

We get an induced (f, g, u, v, λ) structure on the invariant hypersurface satisfying

(2.15) $f^2 = -I + u \times U + v \times V.$

(2.16) $fU = -\lambda V,$

(2.17) $fV = \lambda U,$

(2.18) $u * f = \lambda v,$

(2.19) $v * f = -\lambda u,$

(2.20) $u(U) = 1 - \lambda^2,$

(2.21) $u(V)=v(U)=0,$

(2.22) $v(V)=1-\lambda^2,$

(2.23) $g(fX, fY)=g(X,Y)-u(X)u(y),$

(2.24) $g(X, fY)=-g(fX,Y),$

(2.25) $g(X,U)=u(X),$

(2.26) $g(X,V)=v(X),$ for all $X, Y \in TM$.

(2.27) $\lambda=\eta(\tilde{N}).$

we see that every transversal hypersurface of an almost contact Riemannian manifold also admits a (f, g, u, v, λ) structure .

3. Noninvariant Hypersurfaces with (F, G, U, V, Λ) Structure

The transformation of a tangent vector of the hypersurface due to linear transformation, field φ of type $(1,1)$ working in each tangent space $T_P M$ of M , $p \in M$ in almost contact structure , never be a tangent to the noninvariant hypersurface. If X be a tangent vector implies that φX is not tangent to the hypersurface defined by (2.10).

In an almost contact structure a hypersurface does not hold almost complex structure . Embedding a manifold to an invariant hypersurface of contact structure is impossible, Goldberg and Yano[6] showed that. However hypersurface of real codimension 1 is noninvariant with almost complex structure permits an almost contact structure.

Lemma 3.1*:* *Let* M *be a noninvariant hypersurface with* (f,g,u,v,λ) *structure of a quasi sasakian manifold* $\tilde{M}$ *. Then*

(a) $(\tilde{\nabla}_X \varphi)Y=(\nabla_X f)Y+(\nabla_X u)Y+\sigma(X, fY)\tilde{N}+\sigma(X,Y)U-u(Y)\tilde{A}_{\tilde{N}} X$

(b) $(\tilde{\nabla}_X \eta)Y=\nabla_X(v(Y))-v(\nabla_X Y)-\lambda\sigma(X,Y).$

(c) $\tilde{\nabla}_X \xi = \nabla_X V - \lambda \tilde{A}_{\tilde{N}} X + (X\lambda)\tilde{N} + \sigma(X,V)\tilde{N}.$

Proof:

(a)
$$\begin{aligned}(\tilde{\nabla}_X \varphi)Y &= \tilde{\nabla}_X \varphi Y - \varphi \tilde{\nabla}_X Y \\ &= \tilde{\nabla}_X (fY + u(Y)\tilde{N}) - \varphi(\nabla_X Y + \sigma(X,Y)\tilde{N}) \\ &= \nabla_X fY + \sigma(X, fY)\tilde{N} + u(Y)\tilde{\nabla}_X \tilde{N} \\ &+ (\tilde{\nabla}_X u(Y))\tilde{N} - \varphi(\nabla_X Y) - \sigma(X,Y)\varphi\tilde{N}\end{aligned}$$

Using (2.7), (2.8) and (2.11), we get

(3.1) $$(\tilde{\nabla}_X \varphi)Y = (\nabla_X f)Y + (\nabla_X u)Y + (X, fY)\tilde{N} + \sigma(X,Y)U - u(Y)\tilde{A}_{\tilde{N}} X$$

(b) Now ,

$$(\tilde{\nabla}_X \eta)Y = \tilde{\nabla}_X \eta(Y) - \eta(\tilde{\nabla}_X Y).$$

Using (2.7), (2.13) and (2.14), we have from above

(3.2) $$(\tilde{\nabla}_X \eta)Y = \nabla_X(v(Y)) - v(\nabla_X Y) - \lambda\sigma(X,Y).$$

Proving (b)

(c) Finally from (2.7), we have

$$\tilde{\nabla}_X \xi = \nabla_X \xi + \sigma(X,\xi)\tilde{N}.$$

Using (2.8) and (2.12), we have

(3.3) $$\tilde{\nabla}_X \xi = \nabla_X V - \lambda \tilde{A}_{\tilde{N}} X + (X\lambda)\tilde{N} + \sigma(X,V)\tilde{N}.$$

Proposition 3.1: *Let M be a noninvariant hypersurface with (f,g,u,v,λ) structure of a quasi sasakian manifold $\tilde{M}$. Then*

(3.4) $$\sigma(X,V) = -X\lambda - \beta u(X).$$

(3.5) $$\nabla_X V = -\beta f X + \lambda \tilde{A}_{\tilde{N}} X.$$

Proof: Considering

$$\left(\nabla_X V-\lambda \tilde{A}_{\tilde{N}} X\right)+\left(\sigma(X,V)+X\lambda\right)\tilde{N}$$
$$=\tilde{\nabla}_X \xi-(X\lambda)\tilde{N}-\sigma(X,V)\tilde{N}+\left(\sigma(X,V)+X\lambda\right)\tilde{N}$$
$$=\tilde{\nabla}_X \xi=-\beta\varphi X=-\beta\left(f X+u(X)\tilde{N}\right).$$

Now equating tangential and normal part we get,

$$\nabla_X V-\lambda \tilde{A}_{\tilde{N}} X=-\beta f X \quad \text{and} \quad \sigma(X,V)+X\lambda=-\beta u(X).$$

Hence the result follows .

Theorem 3.1*: Let M be a totally umbilical noninvariant hypersurface with (f,g,u,v,λ) structure of a quasi sasakian manifold. Then it is totally geodesic if relation $\beta u(X)+X\lambda=0$. Also if the quasi sasakian manifold admits contact structure then $\frac{1}{\lambda}\beta u+d(\log\lambda)=0.$*

Proof: Consider

$$\tilde{\nabla}_X \xi=\nabla_X \xi+\sigma(X,\xi)\tilde{N}$$
$$=\nabla_X V+\nabla_X \lambda\tilde{N}+\sigma(X,V)\tilde{N}$$
$$=\nabla_X V+\lambda\nabla_X \tilde{N}+(X\lambda)\tilde{N}+\sigma(X,V)\tilde{N}\,.$$

Then we have

$$\tilde{\nabla}_X \xi=\left(\nabla_X V-\lambda \tilde{A}_{\tilde{N}} X\right)+(X\lambda)\tilde{N}+\left(\sigma(X,V)+X\lambda\right)\tilde{N}$$

And ultimately from above,

$$\sigma(X,V)=-X\lambda-\beta u(X).$$

If M is totally umbilical then $\tilde{A}_{\tilde{N}}=\zeta I$, where ζ is Kahelerian metric[7] and we know theof σ on M related to $\tilde{A}_{\tilde{N}}$ by

$$\sigma(X,Y)=g\left(\tilde{A}_{\tilde{N}} X,Y\right)=g(\zeta X,Y)=\zeta g(X,Y)$$

Therefore $\sigma(X,V)=\zeta g(X,V)=\zeta v(X)$ and then

$$\sigma(X,V)=\zeta v(X)=-X\lambda-\beta u(X)$$

Hence $\zeta v(X)+X\lambda+\beta u(X)=0$.

If M is totally geodesic then $\zeta=0$. And above gives $X\lambda+\beta u(X)=0$. If quasi sasakian structure is contact structure then

$$\frac{1}{\lambda}\beta u(X)+\frac{X\lambda}{\lambda}=0 \quad [X\lambda=\nabla_X \lambda]$$

$$\frac{\beta}{\lambda}u(X)+d(\log\lambda)=0.$$

Theorem 3.2: *Let M be a totally umbilical noninvariant hypersurface with (f,g,u,v,λ) structure of a quasi sasakian manifold . If f is parallel then one has*

(a) $$\sigma(X,Y)=\frac{u(Y)u(\tilde{A}_{\tilde{N}} X)}{1-\lambda^2}-\frac{\beta u(X)v(Y)}{1-\lambda^2}.$$

(b) $$u(\tilde{A}_{\tilde{N}} X)=\frac{\sigma(X,X)(1-\lambda^2)}{u(X,X)}+\beta v(X).$$

(c) $$u(\tilde{A}_{\tilde{N}} U)=\mu.$$

Provided that $\sigma(U,U)=\mu$.

(d) $$\sigma(U,Y)=-\beta v(Y)+\frac{\mu}{1-\lambda^2}u(Y).$$

Proof: We have

$$(\tilde{\nabla}_X \varphi)Y=(\nabla_X f)Y+(\sigma(X,fY)+(\nabla_X u)Y)\tilde{N}+\sigma(X,Y)U-u(Y)\tilde{A}_{\tilde{N}} X$$

$$(\nabla_X f)Y=(\tilde{\nabla}_X \varphi)Y-(\sigma(X,fY)+(\nabla_X u)Y)\tilde{N}-\sigma(X,Y)U+u(Y)\tilde{A}_{\tilde{N}} X$$

Since $(\tilde{\nabla}_X \varphi)Y=\beta(g(X,Y))\xi-\eta(Y)X$.

Again parallelity of f leads to

$$(\nabla_X f)Y=0,$$

$$\beta\left(g(X,Y)\xi-\eta(Y)X\right)-\left(\sigma(X,fY)+(\nabla_X u)Y\right)\tilde{N}-\sigma(X,Y)U+u(Y)\tilde{A}_{\tilde{N}}X.$$

Equating tangential part we have , from above

$$\sigma(X,Y)U=\beta\left(g(X,Y)V-V(Y)X\right)+u(Y)\tilde{A}_{\tilde{N}}X,\ [\text{as }\xi=V+\lambda\tilde{N}].$$

Applying u on both side we get

$$\sigma(X,Y)\left(1-\lambda^2\right)=u(Y)u\left(\tilde{A}_{\tilde{N}}X\right)-\beta u(X)v(Y).$$

Hence

$$\sigma(X,Y)=\frac{u(Y)u\left(\tilde{A}_{\tilde{N}}X\right)}{1-\lambda^2}-\frac{\beta u(X)v(Y)}{1-\lambda^2}.$$

Taking $\sigma(U,U)=\mu,$ we have

$$\sigma(U,U)\left(1-\lambda^2\right)=u(U)u\left(\tilde{A}_{\tilde{N}}U\right)-\beta u(U)v(U).$$

Which gives

$$u\left(\tilde{A}_{\tilde{N}}U\right)=\mu.$$

Again , we have from above

$$u\left(\tilde{A}_{\tilde{N}}X\right)=\frac{\sigma(X,X)\left(1-\lambda^2\right)}{u(X,X)}+\beta v(X),$$

and

$$\sigma(U,Y)=-\beta v(Y)+\frac{\mu}{1-\lambda^2}u(Y).$$

Theorem 3.3: *Let M be a totally umbilical noninvariant hypersurface with (f,g,u,v,λ) structure of a quasi sasakian manifold $\tilde{M}$. If a vector field U is parallel then $\beta\lambda X+f\left(\tilde{A}_{\tilde{N}}X\right)=0,$ for a nonzero vector field X, $\tilde{M}$ never be totally geodesic.*

Proof: consider

$$\left(\tilde{\nabla}_X\varphi\right)\tilde{N}=\tilde{\nabla}_X\varphi\tilde{N}-\varphi\left(\tilde{\nabla}_X\tilde{N}\right)$$

$$=-\tilde{\nabla}_X U-\varphi\left(\tilde{A}_{\tilde{N}} X\right)$$

$$=-\tilde{\nabla}_X U-\sigma(X,U)-f\left(\tilde{A}_{\tilde{N}} X\right)+\mu\left(\tilde{A}_{\tilde{N}} X\right)\tilde{N}.$$

Hence equating tangential and normal component we have ,

$$\left(\tilde{\nabla}_X \varphi\right)\tilde{N}=-\tilde{\nabla}_X U+f\left(\tilde{A}_{\tilde{N}} X\right).$$

Now

$$\left(\tilde{\nabla}_X \varphi\right)Y=\beta\left(g(X,Y)\xi-\eta(Y)X\right)$$

$$=\beta\left\{\left(g(X,Y)\left(V+\lambda\tilde{N}\right)-V(Y)X\right)\right\}$$

$$=\beta\left\{\left(g(X,Y)V-V(Y)X\right)\right\}+\beta g(X,Y)\lambda\tilde{N}.$$

$$\left(\tilde{\nabla}_X \varphi\right)\tilde{N}=\beta\left\{\left(g(X,N)V-\eta(N)X\right)\right\}+\beta g\left(X,\tilde{N}\right)\lambda\tilde{N}$$

$$=-\beta\lambda X \qquad [\text{ as } X\in TM \text{ , hence } g\left(X,\tilde{N}\right)=0.]$$

Hence

$$\nabla_X U=\beta\lambda X+f\left(\tilde{A}_{\tilde{N}} X\right).$$

If U is parallel then $\nabla_X U=0.$ Now suppose that $\tilde{M}$ is totally geodesic , then $\zeta=0,$ implies that $\tilde{A}_{\tilde{N}}=0.$ This leads to

$$\beta\lambda X=0, \quad \text{as } \lambda\neq 0,\ \beta\neq 0.$$

Shows $X=0.$ Which is impossible as X is nonzero vector field so $\tilde{M}$ never be totally geodesic .

Theorem 3.4: *If V is parallel vector field in noninvariant hypersurface $\tilde{M}$ with (f,g,u,v,λ) structure of a cosymplectic manifold $\tilde{M}$ follows the contact structure, then $\sigma(X,Y)=0$; i.e. $\tilde{M}$ is totally geodesic .*

Proof: We know for a cosymplectic manifold $\beta=0,$ implies $\tilde{\nabla}_X\xi=0$ and from

$$\tilde{\nabla}_X \xi=\left(\nabla_X V-\lambda\tilde{A}_{\tilde{N}} X\right)+\left(\sigma(X,V)+X\lambda\right)\tilde{N}$$

We have,

$$\left(\nabla_X V-\lambda\tilde{A}_{\tilde{N}} X\right)+\left(\sigma(X,V)+X\lambda\right)\tilde{N}=0.$$

So

$$\left(\nabla_X V - \lambda \tilde{A}_{\tilde{N}} X\right) = 0,$$

implies that

$$\nabla_X V = \lambda \tilde{A}_{\tilde{N}} X.$$

If V is parallel then

$$\nabla_X V = 0 = \lambda \tilde{A}_{\tilde{N}} X \text{ , as } \lambda \neq 0,$$

leads to

$$\tilde{A}_{\tilde{N}} X = 0 \text{, for all } X \in TM.$$

Gives

$$g\left(\tilde{A}_{\tilde{N}} X, Y\right) = \sigma\left(X, Y\right),$$

So

$$\sigma\left(X, Y\right) = g\left(0, Y\right) = 0,$$

implies that $\tilde{M}$ is totally geodesic for all $X \in T\tilde{M}$.

References

1. S. P. Yadav and S. Kishore, Noninvariant Hypersurfaces of a Nearly Trans-sasakian Manifolds , *Journal of Mathematics*, Article ID 657690, (2014), 1-5 .
2. S. I. Goldberg and K. Yano, Noninvariant Hypersurfaces of Almost Contact Manifolds, *Journal of the Mathematical Society of Japan*, **22** (1970), 25-34 .
3. D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. No 509 . Springer Verlag, 1976.
4. D. E. Blair, The Theory of Quasi-Sasakian Structure, *J . Differential Geom.*, **1** (1967), 331-345 .
5. Z. Olszak, Normal Almost Contact Metric Manifolds of Dimension Three, *Ann. Polon. Math.*, **47** (1986), 41-50 .
6. S. I. Goldberg , Conformal Transformations of Kaehler Manifolds, *Bulletin of the American Mathematical society*, **66** (1960), 54-58 .
7. K. Yano and Okumura , On (f, g, u, v, λ) -Structures, *Kodai Mathematical Seminar Reports*, **22** (1970), 401-423 .
8. R. Prasad, On Noninvariant Hypersurfaces of Trans – Sasakian Manifolds, *Bulletin of Calcutta Mathematical Society*, **99** (2007), 501-510 .
9. U. C. De and A. Sarkar, On Three Dimensional Locally φ -Recurrent Quasi– Sasakian Manifolds, *Demostratio Mathematica*, **42** (2007), 677-684 .
10. D. E. Blair, The Theory of Quasi Sasakian Structure , *J . Differential Geom.*, **1** (1967), 331-345 .