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On a Quasi-Einstein Kähler Manifold and its Application to General Relativity

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Abstract: In the present paper, a quasi-Einstein Kähler manifold is studied. Certain results related to pseudo-projective curvature tensor and M-projective curvature tensor of a quasi-Einstein Kähler manifold $(QEK)_4$ have been studied. A $(QEK)_4$ spacetime with space-matter tensor is also studied and a result on $(QEK)_4$ spacetime is obtained.

Keywords: Quasi-Einstein Kähler manifold, pseudo-projective curvature tensor, energy momentum tensor, Einstein's fields equation, space-matter tensor.

2010 Mathematics Subject Classification: 53C25, 53C50.

1. Introduction

In 2000, Chaki and Maity[1] introduced the notion of an $n-$dimensional Quasi-Einstein manifold and denoted it by $(QE)_n$. The study of quasi-Einstein manifolds was further enriched by Chaki[2], Guha[3], De and Ghosh[4,5], Debnath and Konar[6], Bejan[7] and many others.

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetime is a quasi-Einstein manifold. Also, in general relativity[8], a general quasi-Einstein manifold can be taken as a model of the perfect fluid spacetime. So quasi-Einstein manifolds have some importance in the general theory of relativity.

In this paper, we study quasi-Einstein Kähler manifolds. The paper is organized as follows: Section 1 is introductory. In Section 2, some geometric properties of quasi-Einstein Kähler manifolds have been studied. Section 3 is devoted to obtain a certain result on quasi-Einstein Kähler manifold in the case of pseudo-projective curvature tensor and in Section 4 in the case of $M-$projective curvature tensor. In Section 5, we discuss $(QEK)_4$ spacetime with space-matter tensor and obtain some results on $(QEK)_4$ spacetime.

2. Preliminaries

Let (M_{2n}, g) be a $2n-$dimensional Kähler manifold with respect to the Levi-Civita connection ∇, then we have

(2.1) $$\bar{\bar{X}} + X = 0 \text{ and } \bar{X} = FX,$$

(2.2) $$(\nabla_X F) = 0,$$

(2.3) $$g(\bar{X}, \bar{Y}) = g(X, Y),$$

(2.4) $$g(\bar{X}, Y) + g(X, \bar{Y}) = 0,$$

where X and Y are arbitrary vector fields. The curvature tensor R on the Kähler manifold is defined as

(2.5) $$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

and it satisfies

(2.6) $$\begin{aligned}\tilde{R}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) &= \tilde{R}(X, Y, Z, W) = \tilde{R}(X, Y, \bar{Z}, \bar{W}) \\ &= \tilde{R}(\bar{X}, \bar{Y}, Z, W).\end{aligned}$$

The Ricci tensor satisfies

(2.7) $$S(\bar{X}, \bar{Y}) = S(X, Y).$$

The pseudo-projective curvature tensor P is defined as

(2.8) $$\begin{aligned}P(X,Y)Z &= aR(X,Y)Z + b\left[S(Y,Z)X - S(X,Z)Y\right] \\ &\quad - \frac{r}{n}\left(\frac{a}{n-1} + b\right)\left[g(Y,Z)X - g(X,Z)Y\right],\end{aligned}$$

while $M-$projective curvature tensor denoted by $W,$ is defined as

(2.9) $$W(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY].$$

A non-flat Riemannian or a semi-Riemannian manifold (M_n, g) is called a quasi-Einstein manifold[1] if its Ricci tensor S is non-vanishing and satisfies the condition

(2.10) $$S(X,Y) = ag(X,Y) + bA(X)A(Y),$$

or equivalently, its Ricci operator Q satisfies

(2.11) $$QX = aX + bA(X)U,$$

where a and b are scalars, $b \neq 0$ and A is a non-zero 1-form such that

(2.12) $$g(X,U) = A(X),$$

for all vector fields X and a unit vector field U. Contracting X and Y in equation (2.10), we get

(2.13) $$r = na + b.$$

In 1972, Chen and Yano[9] introduced the notion of a manifold of quasi-constant curvature. A non-flat Riemannian or a semi-Riemannian manifold is said to be a manifold of quasi-constant curvature if its curvature tensor R of type (0, 4) satisfies the condition

(2.14) $$\begin{aligned} R(X,Y,Z,W) = {} & a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & + b[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\ & + g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)], \end{aligned}$$

where a and b are scalars of which $b \neq 0$ and A is a non-zero 1-form, such that $g(X,U) = A(X),$ for all vector fields X and a unit vector field $U.$ Scalars a and b appears in (2.14), are called the associated scalars. A is called associated $1-$form and U is called generator of the manifold. Such an n-dimensional manifold is denoted by $(QC)_n$.

3. Pseudo-Projective Curvature Tensor

Theorem 3.1: *On Quasi-Einstein Kähler manifold* $(QEK)_n$, *the following conditions are equivalent for pseudo-projective curvature tensor:*

(3.1) $$P^*(\bar{Y},\bar{Z}) = P^*(Y,Z)$$

and

(3.2) $$A(\bar{Y})A(Z) + A(Y)A(\bar{Z}) = 0$$

provided $(n-2)a + (n-1)b \neq 0.$

Proof: From equation (2.8), the pseudo-projective curvature tensor of type (0,4) may be written as

(3.3) $$\begin{aligned} P(X,Y,Z,W) &= aR(X,Y,Z,W) + b[S(Y,Z)g(X,W) \\ &\quad - S(X,Z)g(Y,W)] - \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)g(X,W) \\ &\quad - g(X,Z)g(Y,W)]. \end{aligned}$$

Using equations (2.10), (2.13) and (2.14) in equation (3.3), we get

(3.4) $$\begin{aligned} P(X,Y,Z,W) &= \left\{a^2 + ab - \frac{an+b}{n}\left(\frac{a}{(n-1)} + b\right)\right\}[g(Y,Z)g(X,W) \\ &\quad - g(X,Z)g(Y,W)] + b(a+b)[A(Y)A(Z)g(X,W) \\ &\quad - A(X)A(Z)g(Y,W)] + ab[g(Y,Z)A(X)A(W) \\ &\quad - g(X,Z)A(Y)A(W)]. \end{aligned}$$

Contracting equation (3.4) over X and W, we get

(3.5) $$\begin{aligned} P^*(Y,Z) &= \left\{a^2(n-1) + abn - \frac{an+b}{n}(a-b+bn)\right\}g(Y,Z) \\ &\quad + \left\{ab(n-2) + b^2(n-1)\right\}A(Y)A(Z) \end{aligned}$$

Replacing Y and Z and by $\bar{Y}$ and $\bar{Z}$, and using (2.3) we have

(3.6) $$\begin{aligned} P^*(\bar{Y},\bar{Z}) &= \left\{a^2(n-1) + abn - \frac{an+b}{n}(a-b+bn)\right\}g(Y,Z) \\ &\quad + \left\{ab(n-2) + b^2(n-1)\right\}A(\bar{Y})A(\bar{Z}), \end{aligned}$$

From equations (3.5) and (3.6), we see that (3.1) holds if and only if

$$A(\bar{Y})A(\bar{Z}) = A(Y)A(Z), \tag{3.7}$$

provided $(n-2)a+(n-1)b \neq 0$. Replacing Z by $\bar{Z}$ and using (2.1) we get (3.2). Also replacing Z by $\bar{Z}$ in (3.2), we get (3.7). Thus (3.2) and (3.7) are equivalent.

Theorem 3.2: *On Quasi-Einstein Kähler manifold $(QEK)_n$, pseudo-projective curvature tensor satisfies*

$$P(\bar{X},\bar{Y})\bar{Z} = P(\bar{X},Y)Z + P(X,\bar{Y})Z + P(X,Y)\bar{Z} \tag{3.8}$$

if

$$\begin{aligned} A(\bar{Y})A(\bar{Z})\bar{X} &= A(Y)A(Z)\bar{X} + A(\bar{Y})A(Z)X \\ &\quad + A(Y)A(\bar{Z})X. \end{aligned} \tag{3.9}$$

***Proof**:* From equation (3.4), we have

$$\begin{aligned} P(X,Y)Z = \left\{a^2 + ab - \frac{an+b}{n}\left(\frac{a}{(n-1)} + b\right)\right\}[g(Y,Z)X \\ -g(X,Z)Y] + b(a+b)[A(Y)A(Z)X - A(X)A(Z)Y] \\ +ab[g(Y,Z)A(X)U - g(X,Z)A(Y)U]. \end{aligned} \tag{3.10}$$

Replacing X,Y,Z by $\bar{X},\bar{Y},\bar{Z}$ in equation (3.10), we get

$$\begin{aligned} P(\bar{X},\bar{Y})\bar{Z} = \left\{a^2 + ab - \frac{an+b}{n}\left(\frac{a}{(n-1)} + b\right)\right\}[g(\bar{Y},\bar{Z})\bar{X} \\ -g(\bar{X},\bar{Z})\bar{Y}] + b(a+b)[A(\bar{Y})A(\bar{Z})\bar{X} - A(\bar{X})A(\bar{Z})\bar{Y}] \\ +ab[g(\bar{Y},\bar{Z})A(\bar{X})U - g(\bar{X},\bar{Z})A(\bar{Y})U]. \end{aligned} \tag{3.11}$$

Using equation (2.3), equation (3.11) may be written as

$$\begin{aligned} P(\bar{X},\bar{Y})\bar{Z} = \left\{a^2 + ab - \frac{an+b}{n}\left(\frac{a}{(n-1)} + b\right)\right\}[g(Y,Z)\bar{X} \\ -g(X,Z)\bar{Y}] + b(a+b)[A(\bar{Y})A(\bar{Z})\bar{X} - A(\bar{X})A(\bar{Z})\bar{Y}] \\ +ab[g(Y,Z)A(\bar{X})U - g(X,Z)A(\bar{Y})U]. \end{aligned} \tag{3.12}$$

Replacing X by $\bar{X}$ in equation (3.10), we get

$$P(\bar{X},Y)Z=\left\{a^2+ab-\frac{an+b}{n}\left(\frac{a}{(n-1)}+b\right)\right\}[g(Y,Z)\bar{X} \tag{3.13}$$
$$-g(\bar{X},Z)Y]+b(a+b)[A(Y)A(Z)\bar{X}-A(\bar{X})A(Z)Y]$$
$$+ab[g(Y,Z)A(\bar{X})U-g(\bar{X},Z)A(Y)U].$$

Replacing Y by $\bar{Y}$ in equation (3.10), we get

$$P(X,\bar{Y})Z=\left\{a^2+ab-\frac{an+b}{n}\left(\frac{a}{(n-1)}+b\right)\right\}[g(\bar{Y},Z)X \tag{3.14}$$
$$-g(X,Z)\bar{Y}]+b(a+b)[A(\bar{Y})A(Z)X-A(X)A(Z)\bar{Y}]$$
$$+ab[g(\bar{Y},Z)A(X)U-g(X,Z)A(\bar{Y})U].$$

Replacing Z by $\bar{Z}$ in equation (3.10), we get

$$P(X,Y)\bar{Z}=\left\{a^2+ab-\frac{an+b}{n}\left(\frac{a}{(n-1)}+b\right)\right\}[g(Y,\bar{Z})X \tag{3.15}$$
$$-g(X,\bar{Z})Y]+b(a+b)[A(Y)A(\bar{Z})X-A(X)A(\bar{Z})Y]$$
$$+ab[g(Y,\bar{Z})A(X)U-g(X,\bar{Z})A(Y)U].$$

Adding (3.13), (3.14), (3.15), we get

$$P(\bar{X},Y)Z+P(X,\bar{Y})Z+P(X,Y)\bar{Z}=\{a^2+ab- \tag{3.16}$$
$$\frac{an+b}{n}\left(\frac{a}{(n-1)}+b\right)\}[g(Y,Z)\bar{X}-g(X,Z)\bar{Y}]$$
$$+b(a+b)[A(Y)A(Z)\bar{X}+A(\bar{Y})A(Z)X+A(Y)A(\bar{Z})X$$
$$-A(\bar{X})A(Z)Y-A(X)A(Z)\bar{Y}-A(X)A(\bar{Z})Y]$$
$$+ab[g(Y,Z)A(\bar{X})U-g(X,Z)A(\bar{Y})U].$$

Using equation (3.12) in (3.16) we get

$$P(\bar{X},Y)Z+P(X,\bar{Y})Z+P(X,Y)\bar{Z}=P(\bar{X},\bar{Y})\bar{Z}+b(a+b)[A(\bar{X})A(\bar{Z})\bar{Y} \tag{3.17}$$
$$-A(\bar{Y})A(\bar{Z})\bar{X}+A(Y)A(Z)\bar{X}+A(\bar{Y})A(Z)X+A(Y)A(\bar{Z})X$$
$$-A(\bar{X})A(Z)Y-A(X)A(Z)\bar{Y}-A(X)A(\bar{Z})Y].$$

Suppose (3.9) holds. Interchanging X and Y in (3.9), we get

(3.18) $$A(\bar{X})A(\bar{Z})\bar{Y} = A(Y)A(Z)\bar{X} + A(\bar{X})A(Z)Y + A(X)A(\bar{Z})Y.$$

Subtracting (3.18) from (3.9), we get

(3.19) $$\begin{aligned} A(\bar{X})A(\bar{Z})\bar{Y} - A(\bar{Y})A(\bar{Z})\bar{X} = A(Y)A(Z)\bar{X} + A(\bar{Y})A(Z)X \\ +A(Y)A(\bar{Z})X - A(\bar{X})A(Z)Y \\ -A(X)A(Z)\bar{Y} - A(X)A(\bar{Z})Y \end{aligned}$$

Using (3.19) in (3.17), we get (3.8).

4. $M-$ Projective Curvature Tensor

Theorem 4.1: *On Quasi-Einstein Kähler manifold* $(QEK)_n$, *the following conditions are equivalent for* $M-$*projective curvature tensor*

(4.1) $$W^*(\bar{Y},\bar{Z}) = W^*(Y,Z).$$

and

(4.2) $$A(\bar{Y})A(\bar{Z}) = A(Y)A(Z)$$

for $n>2$.

Proof: From equation (2.9), the $M-$projective curvature tensor of type (0,4) may be written as

(4.3) $$W(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{2(n-1)}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)g(QX,W) - g(X,Z)g(QY,W)].$$

Using equations (2.10), (2.11) and (2.14) in equation (4.3), we get

(4.4) $$\begin{aligned} W(X,Y,Z,W) = \left\{a - \frac{a}{(n-1)}\right\}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ + \left\{b - \frac{b}{2(n-1)}\right\}[A(Y)A(Z)g(X,W) - A(X)A(Z)g(Y,W)] \\ + \left\{b - \frac{b}{2(n-1)}\right\}[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)]. \end{aligned}$$

Contracting equation (4.4) over X and W, we get

(4.5) $$W^*(Y,Z)=\left\{\frac{2an^2-2n(3a-b)+4a-3b}{2(n-1)}\right\}g(Y,Z)+\left\{\frac{2n^2b-7nb+6b}{2(n-1)}\right\}A(Y)A(Z).$$

Replacing Y and Z and by $\bar{Y}$ and $\bar{Z}$, and using (2.3), we have

(4.6) $$W^*(\bar{Y},\bar{Z})=\left\{\frac{2an^2-2n(3a-b)+4a-3b}{2(n-1)}\right\}g(Y,Z)+\left\{\frac{2n^2b-7nb+6b}{2(n-1)}\right\}A(\bar{Y})A(\bar{Z}).$$

From equations (4.5) and (4.6), we have

(4.7) $$W^*(\bar{Y},\bar{Z})=W^*(Y,Z)+\left\{\frac{(2n-3)(n-2)}{2(n-1)}\right\}b[A(\bar{Y})A(\bar{Z})-A(Y)A(Z)].$$

This shows that (4.1) holds if and only if (4.2) holds for $n>2$.

Theorem 4.2: *On Quasi-Einstein Kähler manifold* $(QEK)_n$, *M -projective curvature tensor satisfies*

(4.8) $$W(\bar{X},\bar{Y})\bar{Z}=W(\bar{X},Y)Z+W(X,\bar{Y})Z+W(X,Y)\bar{Z}$$

if

(4.9) $$A(\bar{Y})A(\bar{Z})\bar{X}=A(Y)A(Z)\bar{X}+A(\bar{Y})A(Z)X+A(Y)A(\bar{Z})X.$$

Proof: From equation (4.4), we get

(4.10) $$W(X,Y)Z=\left\{a-\frac{a}{(n-1)}\right\}[g(Y,Z)X-g(X,Z)Y]+\left\{b-\frac{b}{2(n-1)}\right\}[A(Y)A(Z)X-A(X)A(Z)Y]+\left\{b-\frac{b}{2(n-1)}\right\}[g(Y,Z)A(X)U-g(X,Z)A(Y)U].$$

Replacing X,Y,Z by $\bar{X},\bar{Y},\bar{Z}$ in equation (4.10), we get

$$\begin{aligned}(4.11)\qquad W(\bar{X},\bar{Y})\bar{Z} &= \left\{a-\frac{a}{(n-1)}\right\}[g(\bar{Y},\bar{Z})\bar{X}-g(\bar{X},\bar{Z})\bar{Y}] \\ &+\left\{b-\frac{b}{2(n-1)}\right\}[A(\bar{Y})A(\bar{Z})\bar{X}-A(\bar{X})A(\bar{Z})\bar{Y}] \\ &+\left\{b-\frac{b}{2(n-1)}\right\}[g(\bar{Y},\bar{Z})A(\bar{X})U-g(\bar{X},\bar{Z})A(\bar{Y})U].\end{aligned}$$

Using (2.3) in (4.11), we get

$$\begin{aligned}(4.12)\qquad W(\bar{X},\bar{Y})\bar{Z} &= \left\{a-\frac{a}{(n-1)}\right\}[g(Y,Z)\bar{X}-g(X,Z)\bar{Y}] \\ &+\left\{b-\frac{b}{2(n-1)}\right\}[A(\bar{Y})A(\bar{Z})\bar{X}-A(\bar{X})A(\bar{Z})\bar{Y}] \\ &+\left\{b-\frac{b}{2(n-1)}\right\}[g(Y,Z)A(\bar{X})U-g(X,Z)A(\bar{Y})U].\end{aligned}$$

Replacing X by $\bar{X}$ in equation (4.10), we get

$$\begin{aligned}(4.13)\qquad W(\bar{X},Y)Z &= \left\{a-\frac{a}{(n-1)}\right\}[g(Y,Z)\bar{X}-g(\bar{X},Z)Y] \\ &+\left\{b-\frac{b}{2(n-1)}\right\}[A(Y)A(Z)\bar{X}-A(\bar{X})A(Z)Y] \\ &+\left\{b-\frac{b}{2(n-1)}\right\}[g(Y,Z)A(\bar{X})U-g(\bar{X},Z)A(Y)U].\end{aligned}$$

Replacing Y by $\bar{Y}$ in equation (4.10), we get

$$\begin{aligned}(4.14)\qquad W(X,\bar{Y})Z &= \left\{a-\frac{a}{(n-1)}\right\}[g(\bar{Y},Z)X-g(X,Z)\bar{Y}] \\ &+\left\{b-\frac{b}{2(n-1)}\right\}[A(\bar{Y})A(Z)X-A(X)A(Z)\bar{Y}] \\ &+\left\{b-\frac{b}{2(n-1)}\right\}[g(\bar{Y},Z)A(X)U-g(X,Z)A(\bar{Y})U].\end{aligned}$$

Replacing Z by $\bar{Z}$ in equation (4.10), we get

$$W(X,Y)\bar{Z}=\left\{a-\frac{a}{(n-1)}\right\}[g(Y,\bar{Z})X-g(X,\bar{Z})Y]$$
$$+\left\{b-\frac{b}{2(n-1)}\right\}[A(Y)A(\bar{Z})X-A(X)A(\bar{Z})Y]$$
$$+\left\{b-\frac{b}{2(n-1)}\right\}[g(Y,\bar{Z})A(X)U-g(X,\bar{Z})A(Y)U]. \tag{4.15}$$

Adding (4.13), (4.14), (4.15), we get

$$W(\bar{X},Y)Z+W(X,\bar{Y})Z+W(X,Y)\bar{Z}=\left\{a-\frac{a}{(n-1)}\right\}[g(Y,Z)\bar{X}$$
$$-g(X,Z)\bar{Y}]+\left\{b-\frac{b}{2(n-1)}\right\}[A(Y)A(Z)\bar{X}+A(\bar{Y})A(Z)X$$
$$+A(Y)A(\bar{Z})X-A(\bar{X})A(Z)Y-A(X)A(Z)\bar{Y}-A(X)A(\bar{Z})Y]$$
$$+\left\{b-\frac{b}{2(n-1)}\right\}[g(Y,Z)A(\bar{X})U-g(X,Z)A(\bar{Y})U]. \tag{4.16}$$

Using equation (4.12) in (4.16), we get

$$P(\bar{X},Y)Z+P(X,\bar{Y})Z+P(X,Y)\bar{Z}=P(\bar{X},\bar{Y})\bar{Z}$$
$$+\left\{b-\frac{b}{2(n-1)}\right\}[A(\bar{X})A(\bar{Z})\bar{Y}-A(\bar{Y})A(\bar{Z})\bar{X}$$
$$+A(Y)A(Z)\bar{X}+A(\bar{Y})A(Z)X+A(Y)A(\bar{Z})X$$
$$-A(\bar{X})A(Z)Y-A(X)A(Z)\bar{Y}-A(X)A(\bar{Z})Y]. \tag{4.17}$$

Suppose (4.9) holds. Interchanging X and Y in (4.9), we get

$$A(\bar{X})A(\bar{Z})\bar{Y}=A(Y)A(Z)\bar{X}+A(\bar{X})A(Z)Y+A(X)A(\bar{Z})Y. \tag{4.18}$$

Subtracting (4.18) from (4.9), we get

$$A(\bar{X})A(\bar{Z})\bar{Y}-A(\bar{Y})A(\bar{Z})\bar{X}=A(Y)A(Z)\bar{X}+A(\bar{Y})A(Z)X$$
$$+A(Y)A(\bar{Z})X-A(\bar{X})A(Z)Y-A(X)A(Z)\bar{Y}-A(X)A(\bar{Z})Y \tag{4.19}$$

Using (4.19) in (4.17), we get (4.8).

5. $(QEK)_4$ Spacetime

In this section we study space matter-tensor of $(QEK)_4$ spacetime. In a smooth manifold Petrov[10] defined a tensor $\tilde{P}$ of type $(0,4)$ as

(5.1) $$\tilde{P} = R + \frac{k}{2} g \wedge T - \sigma G,$$

where R is the curvature tensor of type $(0,4)$, T is the energy momentum tensor of type $(0,2)$, k is the gravitational constant, σ is the energy density and G is a tensor of type $(0,4)$ given by

(5.2) $$G(X,Y,Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W),$$

for all $X,Y,Z,W \in TM$. Kulkarni-Nomizu product $g \wedge T$ is given by

(5.3) $$(g \wedge T)(X,Y,Z,W) = g(Y,Z)T(X,W) + g(X,W)T(Y,Z) - g(X,Z)T(Y,W) - g(Y,W)T(X,Z),$$

where $X,Y,Z,W \in TM$. The tensor $\tilde{P}$ is known as the space-matter tensor of type $(0,4)$ of the manifold M. The space-matter tensor has been studied by Ahasan and Siddiqui[11,12] and many others.

Theorem 5.1: *The space-matter tensor in a Quasi-Einstein Kähler spacetime $(QEK)_4$ obeying Einstein field equation satisfies*

(5.4) $$\tilde{P}^*(\bar{Y},\bar{Z}) = \tilde{P}^*(Y,Z)$$

if and only if

(5.5) $$A(\bar{Y})A(\bar{Z}) = A(Y)A(Z)$$

for $b \neq 0$.

Proof: Equation (5.1) can also be written as

(5.6) $$\tilde{P}(X,Y,Z,W) = R(X,Y,Z,W) + \frac{k}{2}[g(Y,Z)T(X,W) + g(X,W)T(Y,Z) - g(X,Z)T(Y,W) - g(Y,W)T(X,Z)] - \sigma\left[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\right].$$

The Einstein's field equation without cosmological constant is given by

(5.7) $$S(X,Y)-\frac{r}{2}g(X,Y)=kT(X,Y),$$

where k is the gravitational constant and r is the scalar curvature of the spacetime.

Using equations (2.10), (2.14) and (5.7) in equation (5.6), we get

(5.8) $$\begin{aligned}\tilde{P}(X,Y,Z,W)=&\left\{a+\sigma+\frac{a(2-n)+b}{2}\right\}[g(Y,Z)g(X,W)\\&-g(X,Z)g(Y,W)]+\frac{3b}{2}[A(Y)A(Z)g(X,W)\\&-A(X)A(Z)g(Y,W)]+\frac{b}{2}[g(Y,Z)A(X)A(W)\\&-g(X,Z)A(Y)A(W)].\end{aligned}$$

Contracting equation (5.8) over X and W, we get

(5.9) $$\tilde{P}^*(Y,Z)=\left\{\frac{12a+6\sigma-3an+4b}{2}\right\}g(Y,Z)+4bA(Y)A(Z).$$

Replacing Y and Z and by $\bar{Y}$ and $\bar{Z}$ in (5.9), we have

(5.10) $$\tilde{P}^*(\bar{Y},\bar{Z})=\left\{\frac{12a+6\sigma-3an+4b}{2}\right\}g(Y,Z)+4bA(\bar{Y})A(\bar{Z}),$$

for $g(\bar{Y},\bar{Z})=g(Y,Z)$. From equations (5.9) and (5.10), we see that (5.4) holds if and only if (5.5) holds for $b\neq 0$.

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