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State of Distinct Exceptional Solutions in the Dimension Generalization Spaces

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Abstract: Once the solutions are obtained, sophisticated conditions are needed to prove their uniqueness, particularly in the Dimension Generalization spaces. To prove the uniqueness of the final solutions, this work first establishes the equivalence of Cauchy sequences that encapsulate the presence of the solution in the Dimensions' Generalization.

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1. Introduction

Research on the existence of solutions to differential, integral, or boundary equations in a variety of sciences is a rich area where scientists regularly publish their findings and frequently employ fixed-point theorems to demonstrate the existence of solutions[1-9]. This does not always include proving the solution's uniqueness.

Because of their breadth and difficulty, authors tend to steer clear of the study of the uniqueness of solutions in the generalization of dimension spaces.

Finding a point in big spaces and demonstrating its uniqueness is challenging. But there is one way to prove that, and this way what the authors will introduce by authors in this paper, which is proving the equivalence of the sequences to prove the uniqueness of the solution. The dimension space generalization was established by H. Gunawan[10] by the definition of m-normed space and recently Raj[11] showed some properties in the dimension space generalizations. Current studies that improve the dimension space generalizations and their properties, we refer to the references[12-18].

This paper's primary goal is to investigate the circumstances that allow solutions in dimension space generalizations to be unique.

2. Main Results

Definition 2.1. m-Cauchy sequences $\{x_{1_c}\},\{x_{2_c}\},\dots,\{x_{m_c}\}$ in an linear m–normed spaces $(X,\|\cdot,\dots,\cdot\|)$ are said to be equivalent, denoted by $\{x_{1_c}\}\approxeq\{x_{2_c}\}\approxeq,\dots,\approxeq\{x_{m_c}\}$ if for every neighborhood $\tau$of 0 there is an integer $N(\tau)$ such that $1_c,\dots,m_c\geq N(\tau)$ implies that:

(2.1) $x_{1_c}-x_{2_c}-\cdots-x_{m_c}\in\tau\Rightarrow\left\|x_{1_c}-x_{2_c}-\cdots-x_{m_c},y_2,\dots,y_m\right\|\in\tau,$

with respect to the independent set $\{y_2,\dots,y_m\}$ in X.

Theorem 2.1. *For every* $y_2,\dots,y_m\in X$, $\{x_{1_c}\}\approxeq\{x_{2_c}\}\approxeq,\dots,\approxeq\{x_{m_c}\}\in(X,\|\cdot,\dots,\cdot\|)$ *if and only if* $\lim\limits_{c\to\infty}\left\|x_{1_c}-x_{2_c}-\cdots-x_{m_c},y_2,\dots,y_m\right\|=0.$

Proof. Let $\{x_{1_c}\}\approxeq\{x_{2_c}\}\approxeq,\dots,\approxeq\{x_{m_c}\}$ then for τ of 0 there is an integer $N(\tau)$ such that $1_c,\dots,m_c\geq N(\tau)$ implies that (2.1) is fulfilled for every $y_2,\dots,y_m\in X$, getting $\lim\limits_{k\to\infty}\left\|x_{1_c}-x_{2_c}-\cdots-x_{m_c},y_2,\dots,y_m\right\|=0$ using Definition 2.1, which is proof the part if. Let $\lim\limits_{k\to\infty}\left\|x_{1_c}-x_{2_c}-\cdots-x_{m_c},y_2,\dots,y_m\right\|=0$, concluding $x_{1_c}\approxeq x_{2_c}\approxeq,\dots,\approxeq x_{m_c}$ are m-Cauchy sequences in $(X,\|\cdot,\dots,\cdot\|)$, then $\left\|x_{1_c}-x_{2_c}-\cdots-x_{m_c},y_2,\dots,y_m\right\|\in\tau$ for every $y_2,\dots,y_m\in X$, such that τ a neighborhood of 0 and $x_{1_c}-x_{2_c}-\cdots-x_{m_c}\in\tau$, when there exist an integer $N(\tau)$. Hence, $\{x_{1_c}\}\approxeq\{x_{2_c}\}\approxeq,\dots,\approxeq\{x_{m_c}\}$ which proves the part only if.

Theorem 2.2. If $\{x_{1_c}\}$ *is equivalent to* $\{\vartheta_{1_c}\}$, $\{x_{2_c}\}$ *is equivalent to* $\{\vartheta_{2_c}\}$ *and* $\{x_{m_c}\}$ *is equivalent to* $\{\vartheta_{m_c}\}$ *in* $(X,\|\cdot,\dots,\cdot\|)$ *then for all* $c\in\mathbb{N}$, $\omega\in\mathbb{R}$ and $y_2,\dots,y_m\in X$.

i) $\{x_{1_c}+x_{2_c}+\cdots+x_{m_c}\}$ is equivalent to $\{\vartheta_{1_c}+\vartheta_{2_c}+\cdots+\vartheta_{m_c}\}$.

ii) $\{\omega x_{1_c}\}$ is equivalent to $\{\omega\vartheta_{1_c}\}$,…, $\{\omega x_{m_c}\}$ is equivalent to $\{\omega\vartheta_{m_c}\}$.

Proof. $\left\|\left(x_{1_c}+x_{2_c}+\cdots+x_{m_c}\right)-\left(\vartheta_{1_c}+\vartheta_{2_c}+\cdots+\vartheta_{m_c}\right),y_2,\ldots,y_m\right\|$

$$=\left\|\left(x_{1_c}-\vartheta_{1_c}\right)+\left(x_{2_c}-\vartheta_{2_c}\right)+\cdots+\left(x_{m_c}-\vartheta_{m_c}\right),y_2,\ldots,y_m\right\|$$
$$\leq\left\|\left(x_{1_c}-\vartheta_{1_c}\right),y_2,\ldots,y_m\right\|+\left\|\left(x_{2_c}-\vartheta_{2_c}\right),y_2,\ldots,y_m\right\|+$$
$$\ldots+\left\|\left(x_{m_c}-\vartheta_{m_c}\right),y_2,\ldots,y_m\right\|$$

when $c\to\infty$. Using Theorem 3.1 to getting $\left\{x_{1_c}+x_{2_c}+\cdots+x_{m_c}\right\}\simeq\left\{\vartheta_{1_c}+\vartheta_{2_c}+\cdots+\vartheta_{m_c}\right\}$. Then (i) is proved.
To prove (ii), taking $c\to\infty$,

$$\left\|\left(\omega x_{1_c}-\omega\vartheta_{1_c}\right)\left(\omega x_{2_c}-\omega\vartheta_{2_c}\right)\ldots\left(\omega x_{m_c}-\omega\vartheta_{m_c}\right),y_2,\ldots,y_m\right\|$$
$$=|\omega|\left\|\left(x_{1_c}-\vartheta_{1_c}\right)\left(x_{2_c}-\vartheta_{2_c}\right)\ldots\left(x_{m_c}-\vartheta_{m_c}\right),y_2,\ldots,y_m\right\|$$
$$=|\omega|.0$$

Then, $\left\{\omega x_{1_c}\right\}\approxeq\left\{\omega\vartheta_{1_c}\right\}$, $\left\{\omega x_{2_c}\right\}\approxeq\left\{\omega\vartheta_{2_c}\right\}$ and $\left\{\omega x_{m_c}\right\}\approxeq\left\{\omega\vartheta_{m_c}\right\}$.

Theorem 2.3. *On the set of m-Cauchy sequences on X the relation $\approxeq$ is an equivalent relation in* $(X,\|\cdot,\ldots,\cdot\|)$.

*Proof***.** i) Since $\left\{x_{1_c}\right\}\approxeq\left\{x_{1_c}\right\}\approxeq\cdots\approxeq\left\{x_{1_c}\right\}$, then the reflexivity property is satisfied.
ii) For any permutation $\ell_1,\ell_2,\ldots,\ell_m$, in $\left\{x_{1_\ell}\right\}\approxeq\left\{x_{2_\ell}\right\}\approxeq,\ldots,\approxeq\left\{x_{m_\ell}\right\}$ we get that $\left\{x_{1_c}\right\}\approxeq\left\{x_{2_c}\right\}\approxeq,\ldots,\approxeq\left\{x_{m_c}\right\}$ then the symmetry and transitive properties have been fulfilled.
Hence, $\approxeq$ is the equivalent relation on$(X,\|\cdot,\ldots,\cdot\|)$.

Example. Let $\sigma_m\leq m$ the set of all real polynomials on [0,1]. Considering usual addition and scalar multiplication, σ_m is a linear vector space over $\mathbb{R}$. Define the following m-normed on σ_m
$\lambda(\mu,\Im)=\sum_{\daleth}^{m}\|\mu(x_\daleth)-\Im(x_\daleth),y_2,\ldots,y_m\|$ were μ and $\Im$ are linearly independent and $y_2,\ldots,y_m\in\sigma_m$, the $(\lambda(\mu,\Im),y_2,\ldots,y_m\in\sigma_m)$ be a complete m-normed space. Using Banach contraction principle and apply our result to prove that any contraction self-mapping g on $(\lambda(\mu,\Im),y_2,\ldots,y_m\in\sigma_m)$ has a unique fixed point.

Remark. It is crucial to establish the equivalence of Cauchy sequences to demonstrate that the fixed points are distinct using the Banach contraction principle in the areas spaces (2-normed spaces).

3. Conclusion

To prove that the fixed points are unique using the Banach contraction principle in the dimension space generalizations, it is crucial to establish the equivalence of Cauchy sequences.

Paper significance: This paper presents a study that allows us to prove the uniqueness of the solutions in the dimension space generalizations without requiring complex extra conditionals.

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