DOI: 10.61294/jiaps2024.2835 ISSN 0974 - 9373

Vol. 28 No. 3 (2024) **J. Int. Acad. Phys. Sci.** pp. 223-240

Comparative Fuzzy and Intuitionistic Fuzzy Study of a Retrial Queue with Negative Arrivals, Breakdown and Delayed Repairs

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(Received August 10, 2024, Accepted August 30, 2024)

Abstract: In this paper, comparative study of the steady state retrial system investigated by Han et al.[1] under fuzzy and Intuitionistic fuzzy number has been made by us. The retrial system has negative arrivals, and breakdown with delayed repair. Studies have been made by constructing a tele-medicine consultation model with numerical parameters satisfying the stability condition for the steady state system as proposed in the original model. Triangular fuzzy and triangular intuitionistic fuzzy mathematics are our tools. To validate the stability condition, a ranking method (for fuzzy number) and a magnitude method (for intuitionistic fuzzy number) have been used. Various system probabilities together with multitude of performance measures, under both the environment, have been evaluated. We have supported the important effectiveness measures by relevant graphs.

Keywords: Intuitionistic triangular fuzzy number, triangular fuzzy number, unreliable retrial system, negative customers, delayed repairs.

1. Introduction

Here, we have considered a retrial queue with negative arrivals (customers), breakdown with delayed server repair under the umbrella of fuzzy and intuitionistic fuzzy numbers. Retrial queues are those queues where a customer makes repeated attempts in case it does not get immediate service upon its arrival at a service facility. The most realized practical instance of retrial concept is that of our phone calling to other person in busy-tone condition where we try dialing again and again at random intervals till, we succeed. In fact, investigations on retrial queues started in an attempt to comprehend the redialing behavior phone users, see, e.g., Kosten[2] and

Cohen[3]. Books by Falin and Templeton[4] and by Artalezo and Gomez-Corral[5] are specifically written for retrial queues. Bibliography on retrial queues for the period 2000-2009 has been produced by Artalejo[6].

Unreliability (or breakdown) of a server is a very common phenomenon in any service system. Queue theorists have taken unreliability of a server into their consideration to model various real queue systems since the paper by White and Christie[7] with the notion of unreliability of a server. Kulkarni and Choi[8] incorporated the idea of unreliability into retrial queueing system. For recent investigations on retrial queueing with unreliability of a server coupled with other specifications one may see, e.g., Ruiling Tian *et al*.[9], Upadhyay *et al*.[10], Poongothai *et al*.[11] and Gupta[12] etc.

A negative customer, in most simple case, forces a common customer (positive customer) in the system to be removed. Concept of negative customers was introduced by Gelenbe[13]. Shin[14] has dealt with a retrial queue with two types of negative customers - one for removing customers in orbit and other for removing customers in service arena. Wang and Zhang[15] studied a negative-customer-discrete-time retrial queue. Dimitriou[16] has presented cases of removal of ordinary positive customer by a negative customer. Malik *et al*.[17] have presented a review paper on retrial G-queues. Bharathi *et al*.[18] investigated a bulk arrival retrial queue with negative customers and reneging.

In some cases, breakdown of a server gets immediate starting for repairing process by inbuilt repairing mechanism (or through other means) and in other cases there is a delay in repairing. Queues where server waits for its repair is called queue with delayed repair (may be due to non-detection of failure). For retrial queues with delayed repairs one can see, e.g., Jain and Bhagat[19], Choudhary and Ke[20-21], Gao et al.[22], Singh *et al*.[23], Liu *et al*.[24], Bharathi and Nandhini[25] etc. Han *et al*.[1] investigated a retrial queue with negative customers and delayed repair when the server breaks down during idle period (so breakdown goes unnoticed until a customer arrives). They obtained various steady state system probabilities as well as various performance measures, optimization of operating cost and also the optimal strategy analysis.

All the above literature has used deterministic exact values for their numerical illustration. But exact deterministic values of system parameters are often difficult to obtain and so we have to content ourselves with values of parameters which are spread over a small range, i.e., the system parameters have fuzziness in their values. To overcome these difficulties fuzzy numbers and its theory came into being. Zadeh[26] is credited with introduction of fuzzy sets and its theory. Fuzzy theory with multitude of its application can be found in Zimmerman[27]. There are various ways under fuzzy theory and fuzzy numbers in which queueing system can be studied, each having its own merit and demerit - alpha cuts with parametric nonlinear programming, fuzzy L-R method, flexible alpha cuts method, use of triangular fuzzy numbers, use of trapezoidal numbers etc. [see, e.g., Ke *et al.*[28], Mukeba[29], Kanyinda[30], Kannadasan and Padmavathi[31-32]].

In fuzzy theory, we use a membership function for a crisp value to assess its fuzziness to belong to a set of discourse and complement (with respect to unity) of this membership function gives fuzziness in non-belonging the set of discourse. But real system data are not only inadequate but ambiguous too and so the condition 'membership function plus non-membership function is equal to one' may break (see Dymova and Sevastjanov[33]). This defect from unity is indeterminate part of the data or information. This situation forced scholars to think some other ways to deal with impreciseness of data or information. One of these other ways is the intuitionistic fuzzy theory, introduced by Atanassove[34]. In intuitionistic fuzzy theory, we specify non-membership function together with membership function. These two functions are almost independent - the only constraint being that their sum must lie in the interval [0,1] (see Dubey and Mehra[35]). Aarathi[36], Aarathi and Shanmugasundari[37-38], Chandrasekaran and Bindu Kumari[39] are some of the relevant recent papers that incorporate intuitionistic fuzzy numbers.

In this paper, we intend to make a comparative study of the performance measures of the retrial system described by Han *et al.*[1] under fuzzy and intuitionistic fuzzy environments. Above literature review shows that no one has made such a comparative study attempted for a retrial queue under fuzzy and intuitionistic fuzzy environments with more than five parameters as our novelty.

The organization of the paper follows as: introduction is given in section 1. Description of notations and symbols are given in section 2. Model is described in section 3. Fuzzy performance measures are given in section 4. Intuitionistic fuzzy performance measures and its comparison with fuzzy performance measures are given in 5. Section 6 opines on results obtained. Lastly, section 7 winds up the paper with conclusion, references as well as an appendix that describes intuitionistic fuzzy and fuzzy mathematics that are used in calculations.

2. Notations and Symbols

$\lambda, \lambda^{f}, \lambda^{if}$ = Idle time customer arrival rate in crisp model, fuzzy model, intuitionistic fuzzy model,

ν, ν^f, ν^{if} = Retrial rate in crisp model, fuzzy model, intuitionistic fuzzy model,
$\beta, \beta^f, \beta^{if}$ = Repair rate during active breakdown in crisp model, fuzzy model, intuitionistic fuzzy model,
μ, μ^f, μ^{if} = Service rate in crisp model, fuzzy model, intuitionistic fuzzy model,
η, η^f, η^{if} = Passive breakdown rate in crisp model, fuzzy model, intuitionistic fuzzy model,
$\theta, \theta^f, \theta^{if}$ = Repair rate of passive breakdown in crisp model, fuzzy model, intuitionistic fuzzy model,
$\delta, \delta^f, \delta^{if}$ = Delay time in crisp model, fuzzy model, intuitionistic fuzzy model,
$\varphi, \varphi^f, \varphi^{if}$ = Arrival rate of negative customers in crisp model, fuzzy model, intuitionistic fuzzy model,
q = Probability of joining the orbit.

Various rates, system probabilities and efficacy measures in triangular fuzzy number (tfn) and triangular intuitionistic fuzzy number (tifn) are given by, e.g., (see appendix for definitions and other useful details)--
$\lambda^f = (a, b, c)$, $\lambda^{fp} = (m, \alpha, \beta)$, $\lambda^{if} = (a, b, c; a', b, c')$, $\lambda^{ifp} = (m, \alpha, \beta; m, \alpha', \beta')$, $m = b, \alpha = b - a, \beta = c - b, \alpha' = b - a', \beta' = c' - b$ etc., where fp and ifp in superscript signifies fuzzy parametric form of fuzzy and intuitionistic fuzzy numbers respectively.

3. Model Description

The model [see Han *et al.*[1]] talks about a non-reliable retrial queue that incorporates active breakdown, passive breakdown, negative customers and delayed repairs. Poisson arrival rate of the customers is λ, get their service with exponential rate of μ. If the arriving customer observes the system idle, he immediately gets the service. In the contrary case (unavailability of the server), he enters the orbit with probability q or balk with probability 1-q. Exponential retrial rate is ν. Server breaks down during idle period (passive breakdown) with Poisson rate η. Passive breakdown is not repaired immediately and repair time and delayed time during this breakdown is exponential with parameter θ and δ respectively. During repair of passive breakdown customers do not flee without getting service. Negative customers arrive when server is busy (active breakdown) and oust the service receiving customer. Arrival of negative customers is Poisson with rate φ. Exponential rate of repair of active breakdown is β. All the times such as inter-arrival time, repair time, service time etc. are independent. State of the system, at any time t, is described by the pair (N(t)= number of customers at time t, I(t)= state of the system at time t), where I(t)= 0, 1, 2, 3, 4, respectively for idle server, busy server, repairing of server due to passive breakdown, repairing of server due to negative customers, server in the delayed repair condition. Figure 1 shows state transition.

In order to investigate the model under fuzzy and intuitionistic fuzzy umbrella, we propose following practical problem based on the aforementioned model. Consider an unreliable tele-medicine consultation system. A telephone operator is given the responsibility of establishing conversation between the doctor (the server) and the patients (the customers). The calls are coming with Poisson fashion with rate around 3. If the doctor is engaged with a patient, then the operator notes down the caller's detail (orbit) and directs him to call after sometime later (retrial) and this caller either waits for retrial with approximate probability 0.6 or balks with approximate complementary probability 0.4 (=1-0.6). Retrial is exponential with rate nearly 3. On the other hand, if the doctor is idle and patient call comes in, the operator immediately makes the contact between the doctor and the patient and exponential service rate by the doctor is nearly 8. The unreliable system may fail during idle period (passive breakdown) with Poisson rate of nearly 1.1 and this failure undergoes unnoticed till the next call comes in. This causes delay in repair- the delay time being exponential with rate around 4. The delayed repair rate is exponential with parameter around 6. Poisson arrival

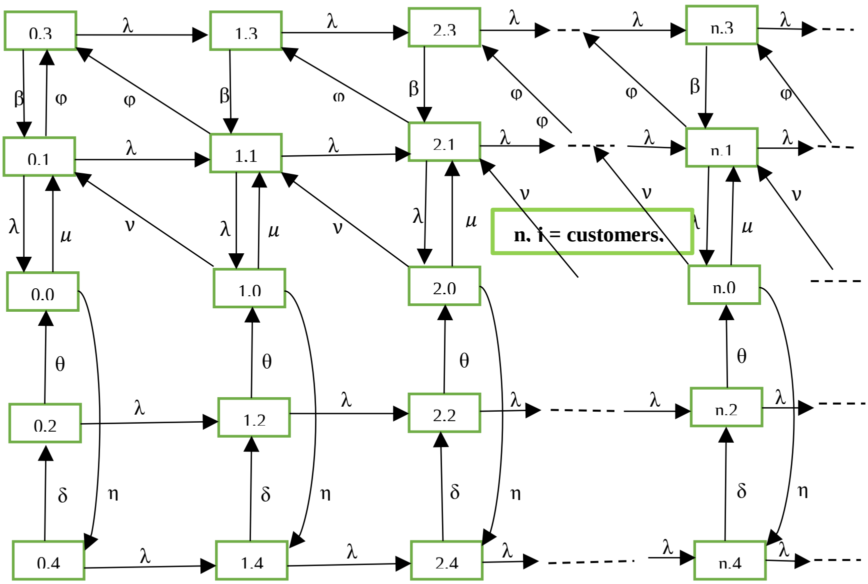

Figure 1. State-transition

of negative customers during ongoing consultation (active breakdown) with rate nearly 1.5 not only affects the system badly but also oust the patient undergoing the consultation. Exponential repair rate of active breakdown is approximately 1.4. Discuss the different performance measures (all the quantities are in appropriate units).

4. Fuzzy Performance Measures

Retrial system probabilities are as under:

- Probability of idle server $\wp_0^f$--

$$\wp_0^f = \frac{\delta^f \mu^f \theta^f \nu^f k_1^f \{(\lambda q)^f + \theta^f\}\{(\lambda q)^f + \delta^f\} + \delta^f \beta^f \mu^f \varphi^f \theta^f k_2^f}{A} \wp_{0,1}^f$$

- Probability that the server is busy $\wp_1^f$--

$$\wp_1^f = \frac{\mu^f k_3^f \beta^f \nu^f \{(\lambda q)^f + \theta^f\}\{(\lambda q)^f + \delta^f\} + k_2^f \delta^f \beta^f \varphi^f \theta^f (\lambda^f + \nu^f)}{A} \wp_{0,1}^f$$

- Repaired state server probability because of passive breakdown $\wp_2^f$ --

$$\wp_2^f = \frac{\nu^f k_1^f \delta^f \mu^f \eta^f \{(\lambda q)^f + \theta^f\}\{(\lambda q)^f + \delta^f\} + k_2^f \varphi^f \delta^f \beta^f \mu^f \eta^f}{A} \wp_{0,1}^f$$

- Repaired state server probability because of active breakdown $\wp_3^f$--

$$\wp_3^f = \frac{\mu^f k_3^f \lambda^f \varphi^f \nu^F \{(\lambda q)^f + \theta^f\}\{(\lambda q)^f + \delta^f\} + k_2^f \theta^f (\varphi^f)^2 \delta^f (\lambda^f + \nu^f)}{A} \wp_{0,1}^f$$

- Delayed repaired state server probability because of delayed repair $\wp_4^f$--

$$\wp_4^f = \frac{\nu^f \theta^f k_1^f \mu^f \eta^f \{(\lambda q)^f + \theta^f\}\{(\lambda q)^f + \delta^f\} + k_2^f \varphi^f \theta^f \beta^f \mu^f \eta^f}{A} \wp_{0,1}^f$$

where $A, A_1, A_2, k_1^f, k_2^f, k_3^f$ and $\wp_{0,1}^f$ are given by

$$A = k_2^f \left[\theta^f k_1^f \delta^f (\lambda^f + \nu^f) - \mu^f \beta^f \lambda^f k_3\right]$$

$$A_1 = \nu^f k_1^f \{(\lambda q)^f + \theta^f\}\{(\lambda q)^f + \delta^f\} + k_2^f \varphi^f$$

$$A_2 = \mu^f \nu^f \lambda^f k_3^f \{(\lambda q)^f + \theta^f\}\{(\lambda q)^f + \delta^f\} + k_2^f \varphi^f \theta^f \delta^f (\lambda^f + \nu^f)$$

$$k_1^f = \beta^f \varphi^f (\varphi^f + \mu^f) - (\lambda q)^f (\beta^f + \varphi^f)$$

$$k_2^f = \lambda^f [(\lambda q)^f \{((\lambda q)^f + (\eta q)^f + \theta^f + \delta^f) + (\eta q)^f (\theta^f + \delta^f) + \theta^f \delta^f\}]$$

$$k_3^f = (\eta q)^f (\theta^f + \delta^f) + \theta^f \delta^f$$

$$\wp_{0,1}^f = \frac{A}{\mu^f [(\eta q)^f (\theta^f + \delta^f) + \theta^f \delta^f] A_1 + (\beta^f + \varphi^f) A_2}$$

Steady-state constraint (stability condition) is

$\mu^f\beta^f\lambda^f k_3 < \theta^f k_1^f \delta^f(\lambda^f+\nu^f) \Leftrightarrow Rank[\mu^f\beta^f\lambda^f k_3] < Rank[\theta^f k_1^f \delta^f(\lambda^f+\nu^f)]$

Various performance measures of retrial system are as follows--

- Busy period orbital mean queue length $\mathbb{N}_{orbB}^f$

$$\mathbb{N}_{orbB}^f = \left[\frac{\omega_1^f(\lambda^f+\nu^f)}{\delta^f\mu^f\theta^f A_{11}} + \frac{\lambda^f\beta^f\mu^f - k_1^f(\lambda^f+\nu^f)}{\beta^f(\mu^f)^2}\right]\wp_0^f + \left[\frac{\alpha_1^f\varphi^f\theta^f\beta^f\delta^f(\lambda q)^f(\lambda^f+\nu^f)}{\mu^f k_2^f A_{11}} + \frac{\beta^f\varphi^f k_2^f - k_1^f\nu^f\{(\lambda q)^f+\theta^f\}\{(\lambda q)^f+\delta^f\}}{\beta^f\mu^f k_2^f}\right]\wp_{0,1}^f,$$

where

$$A_{11} = k_1^f\left[\theta^f k_1^f\delta^f(\lambda^f+\nu^f) - \mu^f\beta^f\lambda^f k_3\right],$$

$$\alpha_1^f = \mu^f + \beta^f + \varphi^f - (\lambda q)^f$$

$$\omega_1^f = (\lambda q)^f\delta^f\eta^f\theta^f\left(k_1^f\right)^2\left(\theta^f+\delta^f\right) - \lambda^f(\lambda q)^f\delta^f k_1^f\theta^f\mu^f\left[k_3^f + \beta^f\left((\eta q)^f+\theta^f+\delta^f\right)\right] + \lambda^f(\lambda q)^f\beta^f\mu^f k_3^f\left[k_1^f\left(\theta^f+\delta^f\right)+\theta^f\delta^f\alpha_1^f\right]$$

- Idle period orbital mean queue length $\mathbb{N}_{orbI}^f$

$$\mathbb{N}_{orbI}^f = \frac{\omega_1^f}{\theta^f\delta^f A_{11}}\wp_0^f + \frac{\alpha_1^f\varphi^f\theta^f\beta^f\delta^f(\lambda q)^f\mu^f}{A_{11}}\wp_{0,1}^f$$

- Active breakdown period orbital mean queue length $\mathbb{N}_{orbA}^f$

$$\mathbb{N}_{orbA}^f = \left[\frac{\omega_1^f\varphi^f(\lambda^f+\nu^f)}{\delta^f\beta^f\theta^f\mu^f A_{11}} + \frac{\lambda^f\beta^f\varphi^f\mu^f - \varphi^f k_1^f(\lambda^f+\nu^f)}{(\beta^f)^2(\mu^f)^2}\right]\wp_0^f + \left[\frac{\omega_2^f\mu^f + \alpha_1^f(\varphi^f)^2\theta^f\beta^f\delta^f(\lambda q)^f(\lambda^f+\nu^f)}{\beta^f\mu^f k_2^f A_{11}} + \frac{\beta^f\mu^f\varphi^f k_2^f(\varphi^f+\mu^f) - \varphi^f\mu^f k_1^f\nu^f\{(\lambda q)^f+\theta^f\}\{(\lambda q)^f+\delta^f\}}{(\beta^f)^2(\mu^f)^2 k_2}\right]\wp_{0,1}^f,$$

where

$$\omega_2^f = \lambda^f\varphi^f\mu^f k_3^f\nu^f\{(\lambda q)^f+\theta^f\}\{(\lambda q)^f+\delta^f\}\{(\lambda q)^f-\beta^f\} + k_2^f(\varphi^f)^2\theta^f\delta^f\{(\lambda q)^f-\beta^f\}(\lambda^f+\nu^f)$$

- Passive breakdown period orbital mean queue length $\mathbb{N}_{orbP}^f$

$$\mathbb{N}_{orbP}^f = \left[\frac{\eta^f\omega_1^f}{(\theta^f)^2\delta^f A_{11}} + \frac{(\lambda q)^f(\theta^f+\delta^f)\eta^f}{(\theta^f)^2\delta^f}\right]\wp_0^f + \frac{\alpha_1^f\varphi^f\theta^f\beta^f\delta^f(\lambda q)^f\mu^f}{A_{11}}\wp_{0,1}^f$$

- Average orbital size during delayed time $\mathbb{N}_{orbD}^f$

$$\mathbb{N}_{orbD}^f = \left[\frac{\eta^f\omega_1^f}{(\theta^f)^2\delta^f A_{11}} + \frac{(\lambda q)^f\eta^f}{(\delta^f)^2}\right]\wp_0^f + \frac{\alpha_1^f\varphi^f\theta^f\beta^f\delta^f(\lambda q)^f\mu^f}{A_{11}}\wp_{0,1}^f$$

- Mean number of customers in the orbit $\mathbb{N}_{ORB}^f$

$$\mathbb{N}_{ORB}^f = \mathbb{N}_{orbB}^f + \mathbb{N}_{orbI}^f + \mathbb{N}_{orbP}^f + \mathbb{N}_{orbA}^f + \mathbb{N}_{orbD}^f$$

- Mean number of system customers $\mathbb{N}_{Sys}^{f}$
$$\mathbb{N}_{Sys}^{f} = \mathbb{N}_{ORB}^{f} + \wp_1^f + \wp_2^f + \wp_4^f$$
- Mean orbital waiting time of a tagged customer who finds the server unavailable $₩_{orb}^{F}$
$$₩_{orb}^{f} = \frac{\mathbb{N}_{ORB}^{f}}{(\lambda q)^f\left[\wp_1^f+\wp_2^f+\wp_3^f+\wp_4^f\right]},$$
- Average busy cycle length $\mathcal{E}(T)$
$$\mathcal{E}(T) = \frac{\mu^f[(\eta q)^f(\theta^f+\delta^f)+\theta^f\delta^f]A_1+(\beta^f+\varphi^f)A_2}{\lambda^f\mu^f\{(\lambda q)^f+\theta^f\}\{(\lambda q)^f+\delta^f\}\left[\theta^f k_1^f\delta^f(\lambda^f+\nu^f)-\mu^f\beta^f\lambda^f k_3\right]}$$
- Expected span of idle period during the busy cycle $\mathcal{E}(T_0)$
$$\mathcal{E}(T_0) = \mathcal{E}(T)\wp_0^f$$
- Expected span of busy period during the busy cycle $\mathcal{E}(T_1)$
$$\mathcal{E}(T_1) = \mathcal{E}(T)\wp_1^f$$
- Expected span of repair period due to passive breakdown during the busy cycle $\mathcal{E}(T_2)$
$$\mathcal{E}(T_2) = \mathcal{E}(T)\wp_2^f$$
- Expected span of delayed period due to active breakdown during the busy cycle $\mathcal{E}(T_3)$
$$\mathcal{E}(T_3) = \mathcal{E}(T)\wp_3^f$$
- Expected span of delayed period during the busy cycle $\mathcal{E}(T_4)$
$$\mathcal{E}(T_4) = \mathcal{E}(T)\wp_4^f$$
- Breakdown probability of the server $\mathbb{P}_R = \wp_2^f + \wp_3^f + \wp_4^f$
- Probability that the server is in working state $\mathbb{P}_W = \wp_0^f + \wp_1^f$

Now suppose $\lambda^f = (2,3,4) = \nu^f$, $\beta^f = (0.4, 1.4, 2.4)$, $\varphi^f = (0.5, 1.5, 2.5)$, $\delta^F = (3,4,5)$, $\eta^f = (0.5, 1.1, 1.7)$, $q^f = (0.5, 0.6, 0.7)$, $\theta^F = (5,6,7)$, $\mu^f = (7,8,9)$.

The parametric form of these fuzzy numbers is (superscript *fp* for fuzzy parametric)
$\lambda^{fp} = (3,1,1) = \nu^{fp}$, $\beta^{fp} = (1.4,1,1)$ $\varphi^{fp} = (1.5,1,1)$, $\delta^{fp} = (4,1,1)$, $\eta^{fp} = (1.1, 0.6, 0.6)$, $q^{fp} = (0.6, 0.1, 0.1)$, $\theta^{fp} = (6,1,1)$, $\mu^{fp} = (8,1,1)$.

These parametric forms will be used for fuzzy arithmetic. Using the triangular fuzzy arithmetic given in the appendix we get
$\theta^f k_1^f \delta^f(\lambda^f + \nu^f) \approx (1162.52, 1163.52, 1164.52)$ and $\mu^f\beta^f\lambda^f k_3 \approx (1027.16, 1028.16, 1029.16)$.
$$Rank\left[\theta^f k_1^f \delta^f(\lambda^f + \nu^f)\right] = 1163.52 > 1028.16 = Rank[\mu^f\beta^f\lambda^f k_3]$$

Hence stability condition is fulfilled. Triangular fuzzy system probabilities are--

- $\wp_0^f \approx (-0.66237, 0.33763, 1.33763)$
- $\wp_1^f \approx (-0.80917, 0.19083, 1.19083)$
- $\wp_2^f \approx (-0.9381, 0.06190, 1.06190)$
- $\wp_3^f \approx (-0.73744, 0.26256, 1.26256)$
- $\wp_4^f \approx (-0.90014, 0.09986, 1.09986)$

Different effectiveness measures of retrial system are as under--

- Busy period orbital mean queue length
$$\mathbb{N}_{orbB}^f = (0.91571, 1.91571, 2.91571)$$
- Idle period orbital mean queue length
$$\mathbb{N}_{orbI}^f = (2.11087, 3.11087, 4.11087)$$
- Active breakdown period orbital mean queue length
$$\mathbb{N}_{orbA}^f = (4.5488, 5.5488, 6.5488)$$
- Passive breakdown period orbital mean queue length
$$\mathbb{N}_{orbP}^f = (1.35298, 2.35298, 3.35298)$$
- Average orbital size during delayed time
$$\mathbb{N}_{orbD}^f = (1.91596, 2.91596, 3.91596)$$
- Mean number of customers in the orbit
$$\mathbb{N}_{ORB}^f = (14.84432, 15.84432, 16.84456)$$
- Mean number of system customers
$$\mathbb{N}_{Sys}^f = (15.19691, 16.19691, 17.19691)$$
- Mean orbital waiting time of a tagged customer who finds the server unavailable $₩_{orb}^f$
$$₩_{orb}^f = (13.30912, 14.30912, 15.30912)$$
- Average busy cycle length
$$\mathcal{E}^f(T) = (14.31896, 15.31896, 16.31896)$$
- Expected span of idle period during the busy cycle
$$\mathcal{E}^f(T_0) = (4.17224, 5.17224, 6.17224)$$
- Expected span of busy period during the busy cycle
$$\mathcal{E}^f(T_1) = (1.92332, 2.92332, 3.92332)$$
- Expected span of repair period due to passive breakdown during the busy cycle $\mathcal{E}^f(T_2)$
$$\mathcal{E}^f(T_2) = (-0.05176, 0.94824, 1.94824)$$
- Expected span of delayed period due to active breakdown during the busy cycle $\mathcal{E}^f(T_3)$

$$\mathcal{E}^f(T_3) = (3.02215, 4.02215, 5.02215)$$

- ❖ Expected span of delayed period during the busy cycle $\mathcal{E}^f(T_4)$

$$\mathcal{E}^f(T_4) = (0.52975. 1.52975, 2.52975)$$

- ❖ Breakdown probability of the server

$$\mathbb{P}^f{}_R = (-0.57298, 0.42702, 1.42702)$$

- ❖ Probability that the server is in working state

$$\mathbb{P}^f{}_W = (-0.47154, 0.52846, 1.52846\ )$$

5. Intuitionistic Performance Measures

All the expressions for different measures will remain the same as in the case of fuzzy environment except that the parameters are now become intuitionistic fuzzy. Now in the intuitionistic fuzzy case, we take (superscript 'if' signifies intuitionistic fuzzy)

$$\lambda^{if} = (2.5, 3, 3.5; 2, 3, 4) = \nu^{if}, \qquad \beta^{if} = (0.9, 1.4, 1.9; 0.4, 1.4, 2.4),$$
$$\varphi^{if} = (1, 1.5, 2; 0.5, 1.5, 2.5), \qquad \delta^{iF} = (3.5, 4, 4.5; 3, 4, 5),$$
$$\eta^{if} = (0.6, 1.1, 1.6; 0.5, 1.1, 1.7), \quad q^{if} = (0.55, 0.6, 0.65; 0.5, 0.6, 0.7),$$
$$\theta^{if} = (5.5, 6, 6.5; 5, 6, 7), \qquad \mu^{if} = (7.5, 8, 8.5; 7, 8, 9).$$

The parametric form of these intuitionistic fuzzy numbers is (superscript i*fp* for intuitionistic fuzzy parametric)

$$\lambda^{ifp} = (\ 3, 0.5, 0{,}5; 3, 1, 1) = \nu^{fp}, \qquad \beta^{ifp} = (\ 1.4, 0.5, 0.51.4, 1, 1),$$
$$\varphi^{ifp} = (1{,}5, 0.5, 0.5; 1.5, 1, 1), \qquad \delta^{ifp} = (4, 0.5, 0.5; 4, 1, 1),$$
$$\eta^{ifp} = (1.1, 0.5, 0.5; 1.1, 0.6, 0.6), q^{ifp} = (0.6, 0.05, 0.05; 0.6, 0.1, 0.1),$$
$$\theta^{ifp} = (6, 0.5, 0.5; 6, 1, 1), \qquad \mu^{ifp} = (8, 0.5, 0.5; 8, 1, 1).$$

These parametric forms will be used for intuitionistic fuzzy arithmetic. Using the triangular intuitionistic fuzzy arithmetic given in the appendix, we get

$$\theta^{if} k_1^{if} \delta^{if}\left(\lambda^{if} + \nu^{if}\right) \approx (1163.02, 1163.52, 1164.02; 1162.52, 1163.52, 1164.52),$$
$$\mu^{if} \beta^{if} \lambda^{if} k_3 \approx (1027.66, 1028.16, 1028.66; 1027.16, 1028.16,\ 1029.16),$$
$$Mag\left[\theta^{if} k_1^{if} \delta^{if}\left(\lambda^{if} + \nu^{if}\right)\right] = 1163.52 > 1028.16 = Mag\left[\mu^{if} \beta^{if} \lambda^{if} k_3\right].$$

Hence stability condition is fulfilled. Triangular intuitionistic fuzzy system probabilities are--

- ➢ $\wp_0^{if} \approx (-0.16237, 0.33763, 0.83763; -0.66237, 0.33763, 1.33763)$
- ➢ $\wp_1^{if} \approx (-0.30917, 0.19083, 0.69083 - 0.80917, 0.19083, 1.19083)$
- ➢ $\wp_2^{if} \approx (-0.4381, 0.06190, 0.5619; -0.9381, 0.06190, 1.06190)$
- ➢ $\wp_3^{if} \approx (-0.23744, 0.26256, 0.76256; -0.73744, 0.26256, 1.26256)$
- ➢ $\wp_4^{if} \approx (-0.40014, 0.09986, 0.59986; -0.90014, 0.09986, 1.09986)$

Different effectiveness measures of retrial system are as under--

- Busy period orbital mean queue length

$$\mathbb{N}_{orbB}^{if} = (1.41571, 1.91571, 2.41571; 0.91571, 1.91571, 2.91571)$$

- Idle period orbital mean queue length

$$\mathbb{N}_{orbI}^{if} = (2.61087, 3.11087, 3.61087; 2.11087, 3.11087, 4.11087)$$

- Active breakdown period orbital mean queue length

$$\mathbb{N}_{orbA}^{if} = (5.0488, 5.5488, 6.0488; 4.5488, 5.5488, 6.5488)$$

- Passive breakdown period orbital mean queue length

$$\mathbb{N}_{orbP}^{if} = (1.85298, 2.35298, 2.85298; 1.35298, 2.35298, 3.35298)$$

- Average orbital size during delayed time

$$\mathbb{N}\,_{orbD}^{if} = (2.41596, 2.91596, 3.41596; 1.91596, 2.91596, 3.91596)$$

- Mean number of customers in the orbit

$$\mathbb{N}\,_{ORB}^{if} = (15.34432, 15.84432, 16.34456; 114.84432, 15.84432, 16.84456\,)$$

- Mean number of system customers

$$\mathbb{N}_{Sys}^{if} = (15.69691, 16.19691, 16.69691; 15.19691, 16.19691, 17.19691)$$

- Mean orbital waiting time of a tagged customer who finds the server unavailable $₩_{orb}^{if}$

$$₩_{orb}^{if} \ = (13.80912, 14.30912, 14.80912; 13.30912, 14.30912, 15.30912)$$

- Average busy cycle length

$$\mathcal{E}^{if}(T) = (14.81896, 15.31896, 15.81896; 14.31896, 15.31896, 16.31896)$$

- Expected span of idle period during the busy cycle

$$\mathcal{E}^{f}(T_0) = (4.67214, 5.17214, 5.67214; 4.17224, 5.17224, 6.17224)$$

- Expected span of busy period during the busy cycle

$$\mathcal{E}^{if}(T_1) = (2.42332, 2.92332, 3.42332; 1.92332, 2.92332, 3.92332)$$

- Expected span of repair period due to passive breakdown during the busy cycle $\mathcal{E}^{if}(T_2)$

$$\mathcal{E}^{if}(T_2) = (0.44824, 0.94824, 1.44824; -0.05176, 0.94824, 1.94824)$$

- Expected span of delayed period due to active breakdown during the busy cycle $\mathcal{E}(T_3)$

$$\mathcal{E}^{if}(T_3) = (3.52215, 4.02215, 4.52215; 3.02215, 4.02215, 5.02215)$$

- Expected span of delayed period during the busy cycle $\mathcal{E}(T_4)$

$$\mathcal{E}^{if}(T_4) = (1.02975, 1.52975, 2.02975; 0.52975. 1.52975, 2.52975)$$

❖ Breakdown probability of the server

$$\mathbb{P}^{if}{}_{R} = (-0.07298, 0.42702, 0.92702; -0.57298, 0.42702, 1.42702)$$

❖ Probability that the server is in working state

$$\mathbb{P}^{if}{}_{W} = (0.02846, 0.52846, 1.02846; -0.47154, 0.52846, 1.52846\,)$$

6. Results and Discussion

The following Table I provides a comparison of effectiveness measures under fuzzy and intuitionistic fuzzy environment:

Table I. Comparison of performance measures

Triangular Fuzzy (TFN)	← Environment → Measures ↓	Intuitionistic Triangular Fuzzy (ITFN)
(0.91571, 1.91571, 2.91571)	Busy period orbital mean queue length	(1.41571, 1.91571, 2.41571; 0.91571, 1.91571, 2.91571)
(2.11087, 3.11087, 4.11087)	Idle period orbital mean queue length	(2.61087, 3.11087, 3.61087; 2.11087, 3.11087, 4.11087)
(4.5488, 5.5488, 6.5488)	Active breakdown period orbital mean queue length	(5.0488, 5.5488, 6.0488; 4.5488, 5.5488, 6.5488)
(1.35298, 2.35298, 3.35298)	Passive breakdown period orbital mean queue length	(1.85298, 2.35298, 2.85298; 1.35298, 2.35298, 3.35298)
(1.91596, 2.91596, 3.91596)	Average orbital size during delayed time	(2.41596, 2.91596, 3.41596; 1.91596, 2.91596, 3.91596)
(14.84432, 15.84432, 16.84432)	Mean number of customers in the orbit	(15.34432, 15.84432, 16.34432; 14.84432, 15.84432, 16.84432)
(15.19691, 16.19691, 17.19691)	Mean number of system customers	(15.69691, 16.19691, 16.69691; 15.19691, 16.19691, 17.19691)
(13.30912, 14.30912, 15.30912)	Mean orbital waiting time of a tagged customer who finds the server unavailable	(13.80912, 14.30912, 14.80912; 13.30912, 14.30912, 15.30912)
(14.31896, 15.31896, 16.31896)	Average busy cycle length	(14.81896, 15.31896, 15.81896; 14.31896, 15.31896, 16.31896)
(4.17224, 5.17224, 6.17224)	Expected span of idle period during busy cycle	(4.67214, 5.17214, 5.67214; 4.17224, 5.17224, 6.17224)
(1.92332, 2.92332, 3.92332)	Expected span of busy period during busy cycle	(2.42332, 2.92332, 3.42332; 1.92332, 2.92332, 3.92332)
(-0.05176, 0.94824, 1.94824)	Expected span of repair period due to passive breakdown during busy cycle	(0.44824, 0.94824, 1.44824; -0.05176, 0.94824, 1.94824)
(3.02215, 4.02215, 5.02215)	Expected span of delayed period due to active	(3.52215, 4.02215, 4.52215; 3.02215, 4.02215, 5.02215)

	breakdown during busy cycle	
(0.52975, 1.52975, 2.52975)	Expected span of delayed period during busy cycle	(1.02975, 1.52975, 2.02975; 0.52975, 1.52975, 2.52975)
(-0.57298, 0.42702, 1.42702)	Breakdown probability of the server	(-0.07298, 0.42702, 0.92702; -0.57298, 0.42702, 1.42702)
(-0.47154, 0.52846, 1.52846)	Probability that the server is in working state	(0.02846, 0.52846, 1.02846; -0.47154, 0.52846, 1.52846)

From the above table we can say that mean number of customers in the orbit in fuzzy case lies in the range from 14.84432 to 16.84432 with most reliable size to be 15.84432. This measure, in intuitionistic case, has the range from 15.34432 to 16.34432 with most dependable size to be same as in the fuzzy case. In other words, left and right spread from most dependable value are respectively 14.84432 and 16.84432 in fuzzy case. In the intuitionistic scenario, the left and right spread of membership function are respectively 15.34432 and 16.34432 respectively and the same spreads for non-membership function are respectively 14.84432 and 16.84432. Other measures can be interpreted in same way.

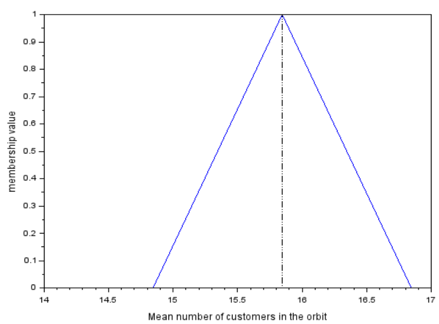

Figure 2: Mean number of customers in the orbit $\mathbb{N}_{ORB}^{f}$

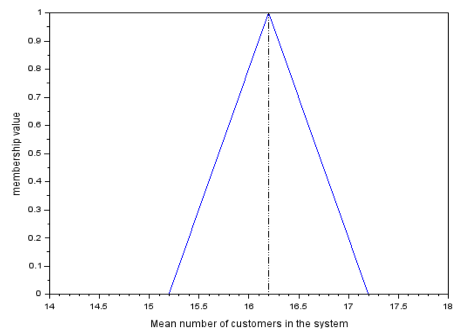

Figure 3: Mean of customers in the system $\mathbb{N}_{Sys}^{f}$

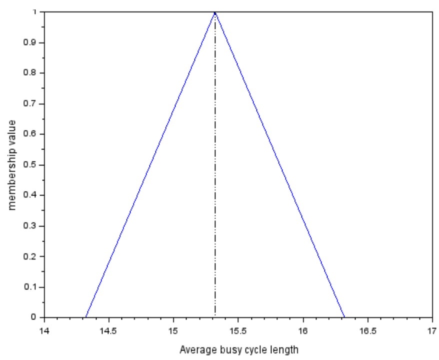

Figure 4: *Average busy cycle length* $\mathcal{E}^{f}(T)$

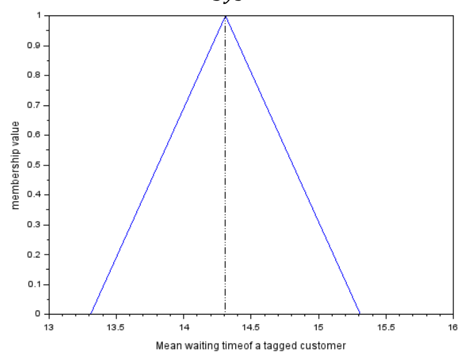

Figure 5: *Mean wait time of tagged customer* $₩_{orb}^{f}$

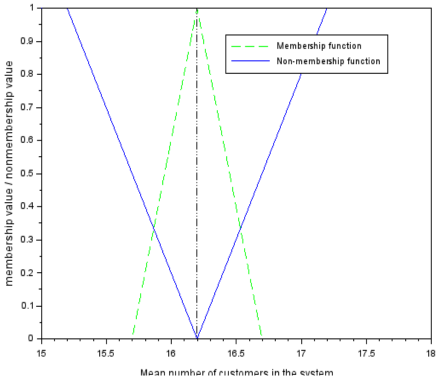

Figure 6: *Int. mean of customers in the system* $\mathbb{N}_{sys}^{if}$

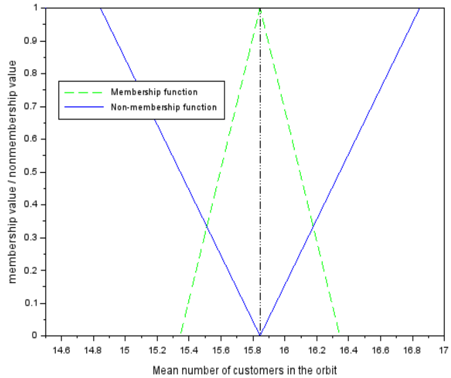

Figure 7: *Int. mean of customers in the orbit* $\mathbb{N}_{ORB}^{if}$

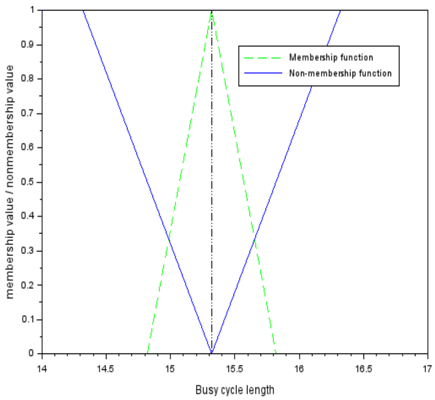

Figure 8: Int. busy cycle length $\mathcal{E}^{\text{if}}(T)$

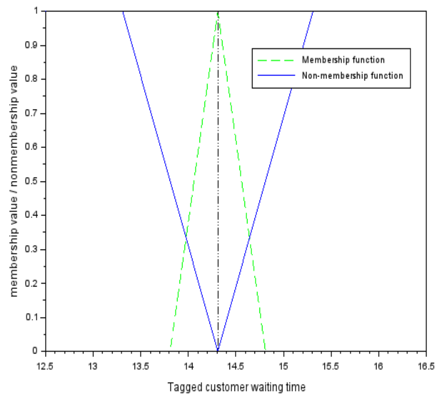

Figure 9: Int. tagged customer waiting time $₩_{orb}^{if}$

Graphical interpretations of above results are given in the above figures (only the graphs of some of the important measures are presented, because others can be drawn in the similar fashion and so unnecessary increase in the number of pages is checked):

7. Conclusion

We have fruitfully applied fuzzy and intuitionistic fuzzy mathematics to obtain necessary probabilities of the system as well effectiveness measures. We have verified stability inequality under both the environment. Also, we have endorsed our discussion with several graphs under both the scenario- fuzzy and intuitionistic fuzzy scenario. Future plan is to study the same system under various other fuzzy system, e.g., trapezoidal fuzzy/neutrosophic fuzzy/ intuitionistic fuzzy numbers etc.

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Appendix:

(see Aarthi[36] for details)

Fuzzy Arithmetic:

Let $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3)$ be two triangular fuzzy numbers with their parametric form, respectively, as $(\mathfrak{m}_f, \alpha_f, \beta_f)$ and $(\mathfrak{m}_g, \alpha_g, \beta_g)$, where $\mathfrak{m}_f = f_2, \alpha_f = f_2 - f_1, \beta_f = f_3 - f_2, \mathfrak{m}_g = g_2, \alpha_g = g_2 - g_1, \beta_g = g_3 - g_2$.

Then the four arithmetic operations are defined as follows:

$f \blacksquare g = \left(\mathfrak{m}_f \blacksquare \mathfrak{m}_g, \max(\alpha_f, \alpha_g), \min(\beta_f, \beta_g)\right)$, where $\blacksquare \in \{+, -, \times, \div\}$.

Rank of any triangular fuzzy number $f = (f_1, f_2, f_3)$ is given by $Rank(f) = \frac{f_1 + 4f_2 + f_3}{6}$.

Comparison between two triangular fuzzy numbers $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3)$ are given below:

$$f \succ g \Leftrightarrow Rank(f) > Rank(g);\ f \prec g \Leftrightarrow Rank(f) < Rank(g);$$
$$f = g \Leftrightarrow Rank(f) = Rank(g);$$

Intuitionistic Fuzzy Arithmetic:

Let $f = (f_1, f_2, f_3; f_1', f_2, f_3')$ and $g = (g_1, g_2, g_3; g_1', g_2, g_3')$ be two intuitionistic triangular fuzzy numbers with their parametric form, respectively, as $(\mathfrak{m}_f, \alpha_f, \beta_f; \mathfrak{m}_f, \alpha_f', \beta_f')$ and $(\mathfrak{m}_g, \alpha_g, \beta_g; \mathfrak{m}_g, \alpha_g', \beta_g')$, where $\mathfrak{m}_f = f_2, \alpha_f = f_2 - f_1, \beta_f = f_3 - f_2, \mathfrak{m}_g = g_2, \alpha_g = g_2 - g_1, \beta_g = g_3 - g_2, \alpha_f' = f_2 - f_1', \beta_f' = f_3' - f_2, \alpha_g' = g_2 - g_1', \beta_g' = g_3' - g_2$ etc.

Then the four arithmetic operations are defined as follows:

$f \blacksquare g =$
$\left(\mathfrak{m}_f \blacksquare \mathfrak{m}_g, \max(\alpha_f, \alpha_g), \min(\beta_f, \beta_g); \mathfrak{m}_f \blacksquare \mathfrak{m}_g, \max(\alpha_f', \alpha_g'), \min(\beta_f', \beta_g')\right)$,
where $\blacksquare \in \{+, -, \times, \div\}$.

Magnitude of any intuitionistic triangular fuzzy number

$f = (f_1, f_2, f_3; f_1', f_2, f_3')$ is given by $Mag(f) = \frac{f_1 + f_1' + 2f_2 + f_3 + f_3'}{6}$.

Comparison between two intuitionistic triangular fuzzy numbers f and g are given below:

$$f \succ g \Leftrightarrow Mag(f) > Mag(g);\ f \prec g \Leftrightarrow Mag(f) < Mag(g);$$
$$f = g \Leftrightarrow Mag(f) = Mag(g).$$