DOI: 1061294/jiaps2024.2816 ISSN 0974 - 9373

Vol. 28 No. 1 (2024) **J. Int. Acad. Phys. Sci.** pp. 63-70

On an Extension of the Generalized Hurwitz-Lerch Zeta Function using Generalized Wright Function

Archna Jaiswal and S.K. Raizada
Department of Mathematics and Statistics,
Dr. Rammanohar Lohia Avadh University Ayodhya, (U.P.), India
Email: archnacpa815@gmail.com ; skr.65@rediffmail.com

(Received January 28, 2024, Accepted March 05, 2024)

Abstract: In the present paper we have given an extension of the Generalized Hurwitz-Lerch Zeta Function $\phi_{\alpha,\gamma;\delta}(z, s, p)$. Using the Generalized Wright Function $W_{\alpha,\beta}^{\gamma,\delta}(z)$ we have obtained two integral representations, a summation formula and a differential formula for the newly introduced function. All results are given in the form of theorems. we have also discussed the corollaries of one of our main theorems. To strengthen our main results, we have also considered some important special cases.

Keywords: Generalized Hurwitz-Lerch Zeta Function, Binomial Series, Eulerian Integral, Wright Function.

1. Introduction

A class of Mathematical Functions that arise in the solution of various classical problems of mathematical physics are termed as Special Functions, for example, some Special Functions arise in solving the equation of heat flow or wave propagation in cylindrical co-ordinates, and in many other such physical problems.

Special functions have also applications in number theory, for example, the Hypergeometric functions are useful in constructing conformal mapping of polygonal regions whose sides are circular areas.

In the recent past, some applications have also seen in quantum mechanics and in the angular momentum theory for example Gegenbauer polynomials are used in the developments of four -dimensional spherical harmonics.

A number of researchers like H.M. Srivastava[1], Mridula Garg & Shyam kalla[2] have given generalizations and extensions of some well-known Special Functions and Polynomials.

In this sequence, Nadeem and Usman[3] have studied the analytical properties of the Hurwitz-Lerch Zeta Function in 2020 as:

$$(1.1) \qquad \phi_{\alpha,\gamma;\delta}\,(z, s, p; q, r) = \sum_{n=0}^{\infty} \frac{B_{q,r}\,(\gamma+m\,,\delta-\gamma)}{B\,(\gamma,\delta-\gamma)}\ \frac{z^n}{(n+p)^s}$$

$q \geq 0, r \geq 0; \alpha, \gamma \in C; s, z \in C; p, \delta \neq 0,-1,-2,...$ when $|z| < 1$; $R\,(s + \delta - \alpha\,) > 0$ when $|z|=1$.

Also, in the year 2020 Kalib and Moshen[4] have given the extended Gamma and Beta Functions in terms of Generalized Wright Function and have obtained various properties as:

$$(1.2) \qquad {}^{w}\Gamma_p^{(\alpha,\beta;\gamma,\delta)}(z) = \int_0^{\infty} t^{Z-1}\, w_{\alpha,\beta}^{\gamma,\delta}\left(-t - \frac{p}{t}\right) dt,$$

where $Re(z) > 0\,, \alpha, \beta, \gamma, \delta \in C$; $\alpha > -1, \delta \neq 0, -1, -2, \ \ . . . ,\ p \geq 0$.

$$(1.3) \qquad {}^{w}B_p^{(\alpha,\beta;\gamma,\delta)}(y, z) = \int_0^1 t^{y-1}(1-t)^{z-1}\, w_{\alpha,\beta}^{\gamma,\delta}\left(-\frac{p}{t(1-t)}\right) dt,$$

where $Re(y) > 0\,, Re(z) > 0;\ \alpha, \beta, \gamma, \delta \in \mathbb{C}$; $\alpha > -1, \delta \neq 0,-1,-2,...;\ p \geq 0$.

Motivated by the above recent works, in this present paper, we have defined an extension of the generalized Hurwitz-Lerch Zeta function in terms of the generalized Wright function.

Mridula Garg & Shyam kalla[2] in the year 2008, have introduced and defined the generalized Hurwitz – Lerch Zeta function as:

$$(1.4) \qquad \phi_{\alpha,\gamma;\delta}\,(z, s, p) = \sum_{n=0}^{\infty} \frac{(\alpha)_n\,(\gamma)_n}{(\delta)_n\ n!}\ \frac{z^n}{(n+p)^s}$$

$(\alpha, \gamma \in C; s, z, \in C); p, \delta \neq \{ 0,-1,-2,...\}$ when $|z| < 1$ and $R\,(s+\delta - \alpha\,) > 0$ when $|z| =1$.

This function given in equation (1.4) in the extension to the general Hurwitz-Lerch Zeta function defined by Srivastava & Choi[5]:

$$(1.5) \qquad \phi\,(z, s, p) = \sum_{n=0}^{\infty} \frac{z^n}{(n+p)^s}$$

$(p \in C \setminus Z_0^- ; s \in C); p \neq \{ 0,-1,-2,...\}$ when $|z| < 1$ and $R(s) > 1$ when $|z| = 1$. contains, as its special cases, not only the Riemann Zeta function, Hurwitz Zeta function and Lerch Zeta function:

$$(1.6) \qquad \zeta\,(s) = \sum_{n=1}^{\infty} \frac{1}{(n)^s}, \quad (R(s) > 1)$$

$$(1.7) \qquad \zeta\,(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \phi\,(z, s,1), (R(s) > 1; a \neq 0, -1, -2, \ . . .)$$

and

(1.8) $l_s(\xi) = \sum_{n=1}^{\infty} \frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi}\phi(e^{2\pi i\xi}, s, 1); \xi \in R; R(s) > 1; a \neq 0,-1,-2,..$

But, also other functions such as the polylogarithmic function $Li_s(z)$ defined as[6]:

(1.9) $Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{(n)^s}$ z ϕ (z, s,1), (s ∈ C when |z| < 1; R (s) > 1 when |z| =1)

And the generalized Zeta function[5]:

(1.10) $\phi(\xi, a, s) = \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \phi(e^{2\pi i\xi}, s, 1)$, (ξ ∈ R; R (s) > 1; $a \neq 0, -1, -2, \ldots$)

which was first studied by Lipschitz and Lerch in connection with Dirchlet's famous theorem on primes in arithmetic progressions.

In this paper, we have given the extension of the generalized Hurwitz-Lerch Zeta function defined in equation (1.4) above in terms of generalized Wright function[6] in the following manner:

(1.11) $$\phi_{\alpha,\gamma;\delta}(z, s, p) = W_{\alpha,\beta}^{\gamma,\delta}(z) \sum_{n=0}^{\infty} (\alpha)_n \frac{\Gamma(\alpha n+\beta)}{(n+p)^s},$$

(α, β, γ , δ ∈ C; s, z ∈ C; δ ≠ 0)(-1,-2,-3, . . . z ∈C) and p, δ ≠ { 0,-1,-2,. . .} when |z| < 1 , with α = -1; R (s+δ − α) > 0 when |z| =1.

We have investigated Integral Representations, Summation formula and Differential formula for our extended generalized Hurwitz-Lerch Zeta function.

2. Preliminaries

The Generalized Wright Function $W_{\alpha,\beta}^{\gamma,\delta}(z)$ defined by Moustafa El- Shahed and Ahemd Salem[6]:

(2.1) $$W_{\alpha,\beta}^{\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n \Gamma(\alpha n+\beta)} \frac{z^n}{n!}$$

(α, β, γ , δ ∈ C; s, z, ∈ C α > −1 , δ ≠ 0) (-1,-2,-3, . . . z ∈ C) and |z| < 1, with α = -1

The Eulerian Integral is given as[7]

(2.2) $$\frac{1}{(n+p)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(n+p)t} \, dt$$

(min R(s) , R(p) > 0, n ∈ N_0)

The Binomial Series:

(2.3) $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

where $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

The Integral representation of the Pochhammar Symbol $(\alpha)_n$ is defined as:

(2.4) $$(\alpha)_n = \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha+n-1}\, e^{-y}\, dy$$

where Pochhammer Symbol are defined for (n ∈ C):

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma a} = \begin{cases} 1 & (n = 0; a \neq 0) \\ a(a+1)(a+2) \quad - \quad - \quad - \quad \{a+(n-1)\} & (n \in N; a \in C) \end{cases}$$

$N = \{1,2,3 \ \ldots \}$ and $Z_0^- = \{0,-1,-2, \ \ldots \}$

and

$$(a)_{n+k} = (a)_n\, (a+n)_k$$

where C is the set of complex numbers, N is the set of natural numbers, R is the set of real numbers and Z is the set of integer numbers.

3. Main Results

Integral Representation

Theorem 3.1: *The following integral representation for* $\phi_{\alpha,\gamma;\delta}(z, s, p)$ *holds*:

(3.1) $$\phi_{\alpha,\gamma;\delta}(z, s, p) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, e^{-pt}\ W_{\alpha,\beta}^{\gamma,\delta}(z) \sum_{n=0}^\infty (\alpha)_n\ \Gamma(\alpha n+\beta)(e^{-t})^n\, dt$$

where $R(s) > 0$, $R(p) > 0$ *provided* $|z| \leq 1$; $R(s) > 1$, *provided* $z = 1$.

Proof: Using (2.2) on right-hand side of (1.4) we get:

$$\phi_{\alpha,\gamma;\delta}(z, s, p) = \sum_{n=0}^\infty \frac{(\alpha)_n(\gamma)_n}{(\delta)_n\, n!}\, z^n\, \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, e^{-(n+p)t}\, dt\,)$$

Interchanging the order of integration and summation on the right-hand of side in above eq^n we get:

$$\phi_{\alpha,\gamma;\delta}(z, s, p) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{n=0}^\infty \frac{(\alpha)_n(\gamma)_n}{(\delta)_n\, n!}\, z^n\, e^{-(n+p)t}\, dt$$

or

$$\phi_{\alpha,\gamma;\delta}(z, s, p) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, e^{-pt} \sum_{n=0}^\infty \frac{(\alpha)_n(\gamma)_n}{(\delta)_n\, n!}\, z^n\, e^{-nt}\, dt$$

or

$$\phi_{\alpha,\gamma;\delta}(z, s, p) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, e^{-pt} \sum_{n=0}^\infty \frac{(\alpha)_n(\gamma)_n}{(\delta)_n\, n!}\, (e^{-t})^n z^n\, dt$$

Multiplying & Dividing by $\Gamma(\alpha n+\beta)$ on right-hand side in above equation, we get:

$$\phi_{\alpha,\gamma;\delta}(z, s, p) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-pt} \sum_{n=0}^\infty \frac{(\alpha)_n (\gamma)_n \Gamma(\alpha n+\beta)}{(\delta)_n \, n! \, \Gamma(\alpha n+\beta)} (e^{-t})^n z^n \; dt$$

or

$$\phi_{\alpha,\gamma;\delta}(z, s, p) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-pt} \sum_{n=0}^\infty \frac{(\alpha)_n (\gamma)_n}{(\delta)_n \, \Gamma(\alpha n+\beta)} \frac{z^n}{n!} \Gamma(\alpha n + \beta)(e^{-t})^n \, dt$$

or

$$\phi_{\alpha,\gamma;\delta}(z, s, p) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-pt} \; W_{\alpha,\beta}^{\gamma,\delta}(z) \sum_{n=0}^\infty (\alpha)_n \; \Gamma(\alpha n + \beta)(e^{-t})^n \, dt,$$

which is the desired result, which we wanted to prove.

Corollary 1: When $\gamma = \delta$ in eq^n no. (3.1) we get:

$$\phi_{\alpha}(z, s, p) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-pt} \; W_{\alpha,\beta}(z) \sum_{n=0}^\infty (\alpha)_n \; \Gamma(\alpha n + \beta)(e^{-t})^n \; dt,$$

where R (s) > 0 , R (p) > 0 provided $|z| \leq 1$; R (s) > 1, provided z =1 and $W_{\alpha,\beta}(z)$ is given in the book[6].

Theorem 3.2: *The following integral representation for* $\phi_{\alpha,\gamma;\delta}(z, s, p)$ *holds true:*

(3.2) $$\phi_{\alpha,\gamma;\delta}(z, s, p) = \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha+n-1} e^{-y} \; W_{\alpha,\beta}^{\gamma,\delta}(z) \sum_{n=0}^\infty \frac{\Gamma(\alpha n+\beta)}{(n+p)^s} \, dy$$

R(s) > 0, R(p) > 0 *provided* $|z| \leq 1$; R (s) > 1, *provided* z =1.

Proof: Using (2.4) for $(\alpha)_n$ on right-hand side of (1.4) we get:

$$\phi_{\alpha,\gamma;\delta}(z, s, p) = \sum_{n=0}^\infty \frac{(\gamma)_n}{(\delta)_n} \frac{z^n}{(n+p)^s n!} \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha+n-1} e^{-y} \, dy$$

Multiplying & Dividing by $\Gamma(\alpha n + \beta)$ on right- hand side in above equation we get:

$$\phi_{\alpha,\gamma;\delta}(z, s, p) = \sum_{n=0}^\infty \frac{(\gamma)_n}{(\delta)_n} \frac{z^n}{(n+p)^s n!} \frac{\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta)} \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha+n-1} e^{-y} \, dy$$

Interchanging the order of integration and summation on the right-hand of side in above eq^n we get:

$$\phi_{\alpha,\gamma;\delta}(z, s, p) = \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha+n-1} e^{-y} \sum_{n=0}^\infty \frac{(\gamma)_n}{(\delta)_n} \frac{z^n}{(n+p)^s n!} \frac{\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta)} \, dy$$

or

$$\phi_{\alpha,\gamma;\delta}(z, s, p) = \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha+n-1} e^{-y} \sum_{n=0}^\infty \frac{(\gamma)_n}{(\delta)_n \, \Gamma(\alpha n+\beta)} \frac{z^n}{n!} \frac{\Gamma(\alpha n+\beta)}{(n+p)^s} \, dy$$

or

$$\phi_{\alpha,\gamma;\delta}(z, s, p) = \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha+n-1} e^{-y} \; W_{\alpha,\beta}^{\gamma,\delta}(z) \sum_{n=0}^\infty \frac{\Gamma(\alpha n+\beta)}{(n+p)^s} \, dy$$

Hence theorem 3.2 i.e, result (3.2) is established.

Corollary 2: When $\gamma = \delta$ in (3.2) we get:

$$\phi_\alpha(z, s, p) = \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha+n-1} e^{-y} \; W_{\alpha,\beta}(z) \sum_{n=0}^{\infty} \frac{\Gamma(\alpha n+\beta)}{(n+p)^s} \, dy$$

R (s) > 0, R(p) > 0 *provided* |z| ≤ 1 ; R (s) > 1, provided z =1 and $W_{\alpha,\beta}(z)$ is given in the book[6].

Summation Formula

Theorem 3.3: *The following summation formula for $\phi_{\alpha,\gamma;\delta}$ (z, s, p) holds:*

(3.3) $$\phi_{\alpha,\gamma;\delta}(z, s, p-t) = \sum_{r=0}^{\infty} \frac{(s)_r}{r!} \left\{ \phi_{\alpha,\gamma;\delta}(z, s+r, p) \right\} t^r$$

where R (s) > 0, R (p) > 0 *provided* |z| ≤ 1 R (s) > 1, *provided* z =1 *and* (|t| < |p| ; s ≠ 1)

Proof: Applying (1.11) on the left-hand side of (3.3), we get:

$$\phi_{\alpha,\gamma;\delta}(z, s, p-t) = W_{\alpha,\beta}^{\gamma,\delta}(z) \sum_{n=0}^{\infty} (\alpha)_n \frac{\Gamma(\alpha n+\beta)}{(n+p-t)^s}$$

or

$$\phi_{\alpha,\gamma;\delta}(z, s, p-t) = W_{\alpha,\beta}^{\gamma,\delta}(z) \sum_{n=0}^{\infty} (\alpha)_n \frac{\Gamma(\alpha n+\beta)}{(n+p)^s} \left(1 - \frac{t}{n+p}\right)^{-s}$$

Using (2.3) in the above equation, we get:

$$\phi_{\alpha,\gamma;\delta}(z, s, p-t) = \quad \sum_{r=0}^{\infty} \frac{(s)_r}{r!} \left\{ \sum_{n=0}^{\infty} (\alpha)_n \frac{(\gamma)_n}{(\delta)_n \, \Gamma(\alpha n+\beta)} \frac{z^n}{n!} \frac{\Gamma(\alpha n+\beta)}{(n+p)^{s+r}} \right\} t^r$$

$$\phi_{\alpha,\gamma;\delta}(z, s, p-t) = \sum_{r=0}^{\infty} \frac{(s)_r}{r!} \left\{ \sum_{n=0}^{\infty} (\alpha)_n \frac{(\gamma)_n}{(\delta)_n} \frac{z^n}{n! \; (n+p)^{s+r}} \right\} t^r$$

Using (1.4) on the right-hand side of the above equation, we get:

$$\phi_{\alpha,\gamma;\delta}(z, s, p-t) = \sum_{r=0}^{\infty} \frac{(s)_r}{r!} \left\{ \phi_{\alpha,\gamma;\delta}(z, s+r, p) \right\} t^r \; ,$$

which proves theorem 3.3 i.e result (3.3).

Differential Formula

Theorem 3.4: *The following differential formula for $\phi_{\alpha,\gamma;\delta}$* (z, s, p) *holds*:

(3.4) $$\frac{d}{dz}\{\phi_{\alpha,\gamma;\delta}(z, s, p)\} = \frac{\gamma}{\delta} W_{\alpha,\alpha+\beta}^{\gamma+1,\delta+1}(z) \sum_{n=0}^{\infty} (\alpha)_{n+1} \frac{\Gamma\{\alpha(n+1)+\beta\}}{(n+1+p)^s}$$

where R(s) > 0, R(p) > 0 *provided* |z| ≤ 1 R(s) > 1, *provided* z =1 *and* (|t| < |p| ; s ≠ 1).

Proof: Differentiation with respect to z on both sides of (1.4), we get:

$$\frac{d}{dz}\{\phi_{\alpha,\gamma;\delta}(z, s, p)\} = \sum_{n=0}^{\infty} \frac{n \; (\alpha)_n \, (\gamma)_n}{(\delta)_n \; n!} \frac{z^{n-1}}{(n+p)^s}$$

or

$$\frac{d}{dz}\{\phi_{\alpha,\gamma;\delta}(z, s, p)\} = \sum_{n=0}^{\infty} \frac{(\alpha)_n \, (\gamma)_n}{(\delta)_n \, (n-1)!} \frac{z^{n-1}}{(n+p)^s}$$

Multiplying & Dividing by $\Gamma(\alpha n+\beta)$ on right-hand side in above equation, we get:

$$\frac{d}{dz}\{\phi_{\alpha,\gamma;\delta}(z,s,p)\} = \sum_{n=0}^{\infty}\frac{(\alpha)_n\,(\gamma)_n}{(\delta)_n}\,\frac{z^{n-1}}{(n-1)!}\,\frac{\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta)}\,\frac{1}{(n+p)^s}$$

Replacing n by (n+1) on right – hand side in above eq^n we get:

$$\frac{d}{dz}\{\phi_{\alpha,\gamma;\delta}(z,s,p)\} = \sum_{n=0}^{\infty}\frac{(\alpha)_{n+1}\,(\gamma)_{n+1}}{(\delta)_{n+1}}\,\frac{z^{n}}{n!}\,\frac{\Gamma\{\alpha(n+1)+\beta\}}{\Gamma\{\alpha(n+1)+\beta\}}\,\frac{1}{(n+1+p)^s}$$

$\{\text{Using } (a)_{n+1} = a(a+1)_n\}$

or

$$\frac{d}{dz}\{\phi_{\alpha,\gamma;\delta}(z,s,p)\} = \frac{\gamma}{\delta}\sum_{n=0}^{\infty}\frac{(\gamma+1)_n}{(\delta+1)_n\,\Gamma\{\alpha n+\alpha+\beta\}}\,\frac{z^{n}}{n!}\,\frac{(\alpha)_{n+1}}{(n+1+p)^s}\,\Gamma\{\alpha(n+1)+\beta$$

or

$$\frac{d}{dz}\{\phi_{\alpha,\gamma;\delta}(z,s,p)\} = \frac{\gamma}{\delta}W^{\gamma+1,\delta+1}_{\alpha,\alpha+\beta}(z)\ \sum_{n=0}^{\infty}(\alpha)_{n+1}\frac{\Gamma\{\alpha(n+1)+\beta\}}{(n+1+p)^s}$$

{Using eq^n no. (2.1)}

Hence theorem 3.4 is proved.

Corollary 3: When $\gamma=\delta$ in (3.4), we get:

$$\frac{d}{dz}\{\phi_{\alpha,\delta;\delta}(z,s,p)\} = W_{\alpha,\alpha+\beta}(z)\sum_{n=0}^{\infty}(\alpha)_{n+1}\frac{\Gamma\{\alpha(n+1)+\beta\}}{(n+1+p)^s}$$

where $R(s)>0$, $R(p)>0$ provided $|z|\le 1$, $R(s)>1$, provided z =1 and ($|t|<|p|$; $s\neq 1$) and $W_{\alpha,\beta}(z)$ is given in the book[6].

4. Special Cases

Case 1: If we put $\gamma=\delta$ in (1.11), we obtain:

$$\phi_{\alpha,\delta;\delta}(z,s,p) = W^{\delta,\delta}_{\alpha,\beta}(z)\ \sum_{n=0}^{\infty}(\alpha)_n\frac{\Gamma(\alpha n+\beta)}{(n+p)^s}$$

($\alpha,\beta,\delta\in C$; $s,z,\in C$, $\delta\neq 0$) (-1,-2,-3, . . . $z\in C$) and $p,\delta\neq\{0,-1,-2,\ldots\}$ when $|z|<1$, with $\alpha=-1$; $R(s+\delta-\alpha)>0$ when $|z|=1$ and $W^{\delta,\delta}_{\alpha,\beta}(z)$ or $W_{\alpha,\beta}(z)$ is given in the book[6].

Case 2: If we put $\delta=\alpha$ in (1.11), we obtain:

$$\phi_{\alpha,\gamma;\alpha}(z,s,p) = W^{\gamma,\alpha}_{\alpha,\beta}(z)\ \sum_{n=0}^{\infty}(\alpha)_n\frac{\Gamma(\alpha n+\beta)}{(n+p)^s}$$

($\alpha,\beta,\gamma\in C$; $s,z,\in C$) (-1,-2,-3, ..., $z\in C$) and $p\neq\{0,-1,-2,\ldots\}$ when $|z|<1$, with $\alpha=-1$; $R(s-\alpha)>0$ when $|z|=1$.

Case 3: If we put $\gamma=\delta=1$ in (1.11), we obtain:

$$\phi_{\alpha,1;1}(z,s,p) = W_{\alpha,\beta}^{1,1}(z)\ \sum_{n=0}^{\infty}(\alpha)_n\ \frac{\Gamma(\alpha n+\beta)}{(n+p)^s}$$

($\alpha, \beta \in C$; $s, z, \in C$) (-1,-2,-3, ... $z \in C$) and $p \neq \{0,-1,-2,...\}$ when $|z| < 1$, with $\alpha = -1$; R $(s-\alpha) > 0$ when $|z| = 1$ and $W_{\alpha,\beta}(z)$ is given in the book[6] .

Case 4: If we put $\delta = \alpha = 1$ in (1.11), we obtain:

$$\phi_{1,\gamma;1}(z,s,p) = W_{1,\beta}^{\gamma,1}(z)\ \sum_{n=0}^{\infty}(1)_n\ \frac{\Gamma(n+\beta)}{(n+p)^s}$$

or

$$\phi_{1,\gamma;1}(z,s,p) = W_{1,\beta}^{\gamma,1}(z)\ \sum_{n=0}^{\infty}\ \frac{n!\,\Gamma(n+\beta)}{(n+p)^s}$$

($\beta, \gamma \in C$; $s, z, \in C$) (-1,-2,-3, ... $z \in C$) and $p \neq \{0,-1,-2,...\}$ when $|z| < 1$, with $\alpha = -1$; R $(s-\alpha) > 0$ when $|z| = 1$.

5. Conclusions

We have introduced the extension on the Generalized Hurwitz-Lerch Zeta Function using Generalized Wright Function and thereafter we have obtained t Integral Representation, for this function $\phi_{\alpha,\gamma;\delta}(z,s,p)$. and then some of the special cases of our main results are also considered which given rise to some other new interesting results.

References

1. H.M. Srivastava; A New Family of the λ – Generalized Hurwitz-Lerch Zeta Function with applications, 2014.
2. Mridula Garg and Shyam kalla; A Further Study of General Hurwitz-Lerch Zeta Function, 2008.
3. R. Nadeem, T. Usman; Analytical Properties of the Hurwitz-Lerch Zeta Function, 2020.
4. M.A.H. Kalib, F.B.F. Moshen; Extended Gamma and Beta Functions in Terms of Generalized Wright function, 2020.
5. H.M. Srivastava, J Choi; Series Associated with the Zeta and Related Functions, 2001.
6. H.M. Srivastava, J Choi; Zeta and q-Zeta Functions and Associated Series and Integrals, 2012.
7. Moustafa El- Shahed and Ahemd Salem; An Extension of Wright Function and Its Properties, 2015.
8. H.M. Srivastava, Tibor, K.Pogany; Integral Computational representations of the extended Hurwitz-Lerch Zeta Function.