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On ϑ-Curvature Tensor of a Generalized Randers Space

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Abstract: The purpose of the present paper is to find the relation between ϑ-curvatures with respect to Cartan connection $C\Gamma$ of a Finsler space $F^n=(M^n, L)$ and a Finsler space $F^{*n}=(M^n, L^*)$ whose metric L^* is derived from the metric L of F^n by $L^{*2}(x,y)=L^2(x,y)+(b_i y^i)^2$, where $b_i(x,y)$ is an $h-$vector. The Finsler space F^{*n} is called as generalized Randers space. The expression for ϑ-curvature of F^{*n} are obtained when F^n is S-3-like and S-4-like.

Keywords: Finsler metric, Randers space, $h-$vector, ϑ-curvature tensor, S-3-like Finsler space and S-4-like Finsler space.

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1. Introduction

Let $F^n=(M^n, L)$ be an n-dimensional Finsler space, where M^n is an n-dimensional differentiable manifold and $L(x,y)$ is the fundamental metric function. Matsumoto[1] introduced a Finsler space (M^n, L^*) whose fundamental metric function L^* is defined as:

$$L^{*2}(x,y)=L^2(x,y)+(b_i y^i)^2. \tag{1.1}$$

where the vector b_i is a function of positional coordinates x^i only. If $L(x,y)$ is a metric function of a Riemannian space then $L^*(x,y)$ reduces to the metric function of a Randers space[2-5]. Such a Finsler metric was first introduced by Randers[6] from the stand point of general theory of relativity and was applied to the theory of electron microscope by Ingarden[7]. The geometrical properties of such space have been used for various works. In all these works the vector b_i is assumed to be the function of positional coordinates only.

Izumi[8] while studying a conformal transformation[9] of a Finsler space, introduced the $h-$vector b_i which is ϑ covariantly constant with respect to Cartan connection $C\Gamma$ and $LC_{hij}\,b^h = \rho\, h_{ij}$. The $h-$vector b_i is not only a function of positional coordinates x^i but also a function of directional arguments y^i. In fact, $L\left(\partial b_i / \partial y^j\right) = \rho\, h_{ij}$. Here $b_i(x,y)$ is an $h-$vector in (M^n, L).

Let b_i be an $h-$vector in the Finsler space (M^n, L) and (M^n, L^*) be another Finsler space whose fundamental metric function $L^*(x,y)$ is defined by

(1.2) $$L^{*2}(x,y) = L^2(x,y) + \beta^2(x,y),$$

where $\beta(x,y) = b_i y^i$. Let us call the Finsler space $F^{n*} = (M^n, L^*)$ as generalized Randers space. To distinguish the geometrical objects of F^{n*} from those of the Finsler space F^n, we shall put a $*$ sign on the corresponding objects of F^n.

For an $h-$vector b_i we have the following lemmas[8]:

Lemma 1: *If* b_i *is an* $h-$*vector then the function* ρ *and* $l_i^* = b_i - \rho l_i$ *are independent of* y.

Lemma 2: *The magnitude of an* $h-$*vector* b_i *is independent of* y.

2. Preliminaries

Let b_i be a vector field in the Finsler space (M^n, L). If b_i satisfy the conditions[10],

(2.1) (i) $b_i\,|_j = 0$ (ii) $LC_{ij}^h b_h = \rho h_{ij}$,

then the vector field b_i is called an $h-$vector. Here $|_j$ denotes the covariant differentiation with respect to Cartan connection $C\Gamma$, C_{hij} is the Cartan C-tensor, h_{ij} is the angular metric tensor and ρ is a function given by

(2.2) $$\rho = (n-1)^{-1} LC^i b_i,$$

where $C^i = C_{jk}^i g^{jk}$ and $g^{jk} g_{ki} = \delta_i^j$.

From (2.1) we get

(2.3) $$\frac{\partial b_i}{\partial y^j} = L^{-1}\rho h_{ij}.$$

Differentiating $\beta = b_i y^i$ partially with respect to y^i and using (2.3), we have

$$\frac{\partial \beta}{\partial y^i} = b_i.$$

Differentiation of (1.2) with respect to y^i yields

(2.4) $$l^*_i = \left(\frac{L}{L^*}\right) l_i + \left(\frac{\beta}{L^*}\right) b_i.$$

After differentiating equation (2.4) with respect to y^j and using (2.3), we get

(2.5) $$l^*_i l^*_j + h^*_{ij} = \sigma h_{ij} + l_i l_j + b_i b_j,$$

where $\sigma = \left(1 + \frac{\beta\rho}{L}\right)$.

From (2.5), we may deduce

(2.6) $$h^*_{ij} = \sigma h_{ij} + A_0 \beta^2 l_i l_j + A_0 L^2 b_i b_j - A_0 \beta L \left(l_i b_j + l_j b_i\right),$$

where $A_0 = \left(\frac{1}{L^2 + \beta^2}\right)$.

Transvection of (2.6) with h^{*ip} yields the following

$$h^{*ip}\left[\sigma h_{ij} + A_0 \beta^2 l_i l_j + A_0 L^2 b_i b_j - A_0 \beta L \left(l_i b_j + l_j b_i\right)\right] = h^{*ip} h^*_{ij}$$

which implies

(2.7) $$h^{*ip}\left[\sigma h_{ij} + A_0 \beta^2 l_i l_j + A_0 L^2 b_i b_j - A_0 \beta L \left(l_i b_j + l_j b_i\right)\right] = \delta^p_j.$$

Again transvecting both sides of the above equation by $h^{jk} b_k$, we get

(2.8) $$h^{*ip} b_i b^m = \frac{b^p b^m}{\sigma + \frac{L^2 b^2 + \beta^2 - \beta L \mu}{A_0}},$$

where $\mu = h^{ij}\left(l_i b_j + l_j b_i\right)$.

Transvecting (2.7) by h^{jk} and applying (2.8), we get

$$h^{*mp} = \beta h^{mp} - \frac{L^2 \beta b^p b^m}{A_0 + L^2 \beta b^2}. \tag{2.9}$$

Thus, we have:

Theorem 2.1: *The covariant and contravariant components of the angular metric tensor of the generalized Randers space F^{*n} are given by (2.6) and (2.9) respectively.*

From equation (2.5) and $g_{ij} = h_{ij} + l_i l_j$ one gets

$$g^*_{ij} = \sigma g_{ij} + (1-\sigma) l_i l_j + b_i b_j. \tag{2.10}$$

From equation (2.10) we obtain the contravariant components of the fundamental metric tensor (g^*_{ij}) of F^{*n} in the following form:

$$g^{*ij} = \sigma^{-1} g^{ij} - A_1(l^i b^j + l^j b^i) + A_2 l^i l^j + A_3 b^i b^j, \tag{2.11}$$

where $A_1 = \frac{(1-\sigma)\beta}{L} A_3$,

$$A_2 = (1-\sigma)\left(b^2 + \sigma\right) A_3$$

and

$$A_3 = \frac{1}{\left\| \frac{\sigma\left\{(1-\sigma)\beta^2 - L^2\right\}}{L^2} - b^2 \right\|}.$$

Theorem 2.2: *The covariant components g^*_{ij} and the contravariant components g^{*ij}* of the *metric tensor of $F^{*n}(g^*_{ij})$ are given by (2.10) and (2.11) respectively.*

From equation (2.10) and $C_{ijk} = (½)\partial g_{ij}/\partial y^k$, we get

$$C^*_{ijk} = \sigma C_{ijk} + \frac{\rho}{2L}\left[h_{ij} m_k + h_{jk} m_i + h_{ki} m_j\right], \tag{2.12}$$

where $m_i = b_i - \frac{\beta}{L} l_i$.

Let us assume that

(2.13) $$C^*_{ijk} = \sigma C_{ijk}$$

holds. Then, from equations (2.12) and (2.13), we get

(2.14) $$\rho(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) = 0.$$

This implies at least one of the following conditions:

(2.15) (a) $\rho = 0$ (b) $h_{ij} m_k + h_{jk} m_i + h_{ki} m_j = 0$,

If (2.15a) is true, equation (2.3) implies $\frac{\partial b_i}{\partial y^j} = 0$, i.e. the vector b_i is independent of y^i, which contradicts our assumption that b_i are functions of (x^i, y^i). Hence (2.15 a) cannot be true. Therefore, we have (2.15(b)), which, in view of Walker's Lemma, implies $m_i = 0$ for $h_{ij} \neq 0$. Since $m_i = b_i - \frac{\beta}{L} l_i$, $m_i = 0$ implies $b_i = \frac{\beta}{L} l_i$. Using this result in (2.1(ii)), we get $\beta C^r_{ij} l_r = \rho h_{ij}$. But $C^r_{ij} l_r = C_{ijr} l^r = 0$. Hence $\rho h_{ij} = 0$, which implies $\rho = 0$ for $h_{ij} \neq 0$. Again, we get a contradiction. Thus, we have:

Theorem 2.3: *There exists no generalized Randers space F^{*n} whose torsion tensor is proportional to the torsion tensor of the Finsler space F^n.*

3. Connection Coefficients

Corresponding to γ^i_{jk} of F^n, we define connection coefficients γ^{*i}_{jk} of F^{*n} as follows:

(3.1) $$\gamma^{*i}_{jk} = \frac{1}{2} g^{*ih} \left(\frac{\partial g^*_{jh}}{\partial y^k} + \frac{\partial g^*_{kh}}{\partial y^j} - \frac{\partial g^*_{jk}}{\partial y^h} \right).$$

By virtue of (2.11), the connection coefficients γ^{*i}_{jk} in F^{*n} can be expressed as

(3.2) $$\gamma^{*i}_{jk} = \sigma g^{*ih}\gamma_{jhk} + \frac{\rho}{2L} g^{*ih}[Q_{jhk} + Q_{khj} - Q_{jkh}],$$

where $Q_{jhk} = h_{jh}m_k + h_{hk}m_j + h_{kj}m_h$ and $\gamma_{jhk} = C_{jhk} + C_{khj} - C_{jkh}$.

Theorem 3.1: *The connection coefficients* γ^{*i}_{jk} *of* F^{*n} *are given by (3.2).*

Multiplying (3.2) by $y^j y^k$ and using $\gamma^i_{jk} y^j y^k = 2G^i$, we get

(3.3) $$G^{*i} = \sigma g^{*ih} G_h + \frac{\rho}{2L} g^{*ih} y^j y^k [Q_{jhk} + Q_{khj} - Q_{jkh}],$$

where $\gamma_{jhk} y^j y^k = 2G_h$.

Differentiating (3.3) partially with respect to y^m and y^p, we get G^{*i}_{mp} (Berwald connection coefficients).

4. $\vartheta-$ Curvature Tensor

Definition 4.1: The $\vartheta-$ curvature tensor S_{hijk} of $F^n = (M^n, L)$ with respect to Cartan connection $C\Gamma$ is defined by[1]

(4.1) $$S_{hijk} = C_{hkm}C^m_{ij} - C_{hjm}C^m_{ik}.$$

From (2.1), (2.11), (2.12) and (2.14), we get

(4.2) $$\begin{aligned} C^{*h}_{ij} &= C^h_{ij} + C_0(h_{ij}m^h + h^h_j m_i + h^h_i m_j) - C_1[h_{ij} l^h C_2 + m_i m_j l^h] \\ &\quad + C_3[C_2 h_{ij} b^h + m_i m_j b^h], \end{aligned}$$

where

$$C_0 = \frac{\rho}{2L\sigma}, \quad C_1 = \frac{(1-\sigma)\beta\rho}{L^2} A_3, \quad C_2 = \frac{1}{2}\left(b^2 - \frac{\beta^2}{L^2}\right) + \sigma$$

and A_3 being defined in (2.11).

From equations (2.12) and (4.2), we have

(4.3) $$\begin{aligned} C^*_{hkm}C^{*m}_{ij} &= C_{hkm}C^m_{ij} + \mu_1 h_{ij}h_{hk} + R_1 h_{hk} m_i m_j + R_2 h_{ij} m_h m_k \\ &\quad + R_0 cycl(C_{ijk} m_h) + R_0^2 A_3 cycl(h_{jh} m_i m_k), \end{aligned}$$

where

$$\mu_1=\left(b^2-\frac{\beta^2}{L^2}\right)\left(\frac{\rho^2\sigma}{L^2}A_3+\frac{\rho^2}{4L^2\sigma}+\frac{\rho^2}{4L^2}A_3\right)+\frac{\rho^2\sigma^2}{L^2}A_3,\quad R_0=\frac{\rho}{2L},$$

$$R_1=\frac{\rho^2}{2L^2}A_3(\sigma+1)+\frac{\rho^2}{2L^2\sigma}\ \text{and}\ R_2=\frac{\rho^2}{2L^2\sigma}+\frac{\rho^2}{2L^2}A_3\left\{\sigma+\frac{1}{2}\left(b^2-\frac{\beta^2}{LH2}\right)\right\}.$$

From (4.3) and (4.1), we obtain:

Theorem 4.1: *The $\vartheta-$ curvature tensor (S^*_{hijk}) of the generalized Randers space F^{*n} has the form*

(4.4) $$S^*_{hijk}=\sigma S_{hijk}+h_{ij}e_{hk}+h_{hk}(e_{ij}+1)-h_{ik}e_{hj}-h_{hj}e_{ik},$$

where $e_{ij}=\frac{1}{2}\mu_1 h_{ij}+\mu_2 m_i m_j$ *and* $\mu_2=\frac{\rho^2}{L^2}A_3\left\{\sigma+\frac{1}{4}+\left(b^2-\frac{\beta^2}{L^2}\right)\right\}$.

Now we shall consider some special cases of $\vartheta-$ curvature tensor S_{hijk}:

Case 1: $S3$-like Finsler space

Definition 4.2: A Finsler space $F^n(n\geq 4)$ with fundamental function $L(x,y)$ is called $S3$-like[11-13] if the $\vartheta-$ curvature tensor S_{hijk} is of the form

(4.5) $$S_{hijk}=\frac{S}{(n-1)(n-2)}(h_{hj}h_{ik}-h_{hk}h_{ij}),$$

where the scalar S is a function of positional co-ordinates only.

From equation (2.6), (4.4) and (4.5) we have

Theorem 4.3: *The $\vartheta-$ curvature tensor* S^*_{hijk} of a $S3$-like Finsler space $F^{*n}(n\geq 4)$ has the form

(4.6) $$S^*_{hijk}=\frac{S}{(n-1)(n-2)}\left\{\sigma^2(h_{hj}h_{ik}-h_{hk}h_{ij})+P_{hijk}\right\},$$

where $$P_{hijk}=\sigma(h_{ik}-h_{ij})\left\{\beta^2A_0l_hl_k+ALb_hb_k-A\beta L(l_hb_k+l_kb_h)\right\}.$$

Theorem 4.4: *The generalized Randers space $F^{*n}(n\geq 4)$ of an $S3$-like Finsler space F_n is $S3$-like if* $P_{hijk}=0$.

Case 2: $S4$-like Finsler space

Definition 4.3: A Finsler space $F_n\,(n\geq 5)$ with fundamental function $L(x, y)$ is called $S4$-like[11, 12] if the $\vartheta-$ curvature tensor S_{hijk} is of the form

(4.7) $$L^2 S_{hijk} = h_{hj} M_{ik} + h_{ik} M_{hj} - h_{hk} M_{ij} - h_{ij} M_{hk},$$

where M_{ij} is a symmetric and indicatory tensor.

From equation (2.6), (4.4) and (4.7), we have

Theorem 4.5: *The $\vartheta-$ curvature tensor of a generalized Randers space of $F_n^*\,(n\geq 5)$ Can be expressed as* :

(4.8) $$L^{*2} S^*_{hijk} = [\sigma(h_{hj} M_{ik} + h_{ik} M_{hj} - h_{hk} M_{ij} - h_{ij} M_{hk}) + U_{hijk}],$$

where $$U_{hijk} = T_{(o)hijk} + T_{(1)hijk} + T_{(2)hijk} + T_{(3)hijk},$$

$$T_{(o)hijk} = A_o[\beta^2 l_h l_j + L^2 b_h b_j - \beta L(l_h b_j + l_j b_h)] M_{ik},$$

$$T_{(1)hijk} = A_o[\beta^2 l_i l_k + L^2 b_i b_{kj} - \beta L(l_i b_k + l_k b_i)] M_{jh},$$

$$T_{(2)hijk} = A_o[\beta^2 l_h l_k + L^2 b_h b_k - \beta L(l_h b_k + l_k b_h)] M_{ij},$$

$$T_{(3)hijk} = A_o[\beta^2 l_i l_j + L^2 b_i b_j - \beta L(l_i b_j + l_j b_i)] M_{hk}.$$

Theorem 4.6: *The generalized Randers space of $F_n^*\,(n\geq 5)$ of $S4$-like Finsler space is $S4$-like if $U_{hijk} = 0$.*

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