

Existence Theorems for Relational Theoretic (Ψ, Φ) -Interpolative Contractions

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Abstract: In this paper, we establish some fixed point results for conditional interpolative contractive mappings using $\mathcal{R}$ -admissible maps and other relational metrical concepts. Our findings enhance existing interpolative contractions found in the literature. We provide examples to illustrate our main results, especially in cases where previous research outcomes are not applicable.

Keywords: Interpolative contractions, Relational metric space, $\mathcal{R}$ -admissible.

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1. Introduction

The core of metric fixed-point theory is all about finding new ways to prove fixed-point theorems. These theorems ensure the existence of fixed points when certain conditions are met. The Banach Contraction Principle serves as the foundational concept in metric fixed-point theory, ensuring the existence of a unique fixed point for any contraction map defined on a complete metric space. It is noteworthy that mappings following the Banach Contraction Principle inherently exhibit continuity. Consequently, a natural question arises, can discontinuous maps in a complete metric space, satisfying analogous contractive conditions, still admit a fixed point?

Kannan¹ provided a resounding affirmative answer, pioneering a novel type of contraction. This marked a significant departure from the classical Banach contraction principle, opening doors to exploring the existence of fixed points for a wider class of maps, (see Ćirić² -Reich³ and Rus⁴, Hardy and Rogers⁵). Subsequently, Karapinar⁶ refined this concept by introducing the generalized Kannan-type contraction through an interpolative approach. A self-map $T : X \rightarrow X$ is said to be an interpolative Kannan-type contraction

for a metric space (X, d) , if there are constants $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$, such that

$$(1.1) \quad d(Tx, Ty) \leq \lambda[d(x, Tx)]^\alpha \cdot [d(y, Ty)]^{1-\alpha} \text{ for all } x, y \in X \setminus F(T),$$

where $F(T) = \{x \in X \text{ such that } Tx = x\}$, and established that such interpolative Kannan-type contractions possess a fixed point within a complete metric space.

In a recent development, Karapinar *et al.*^{7,8} introduced the notion of interpolative Reich–Rus–Ćirić and Hardy–Rogers type contractions, further enriching the theoretical landscape through the lens of interpolation. These advancements not only broadened the scope of applicable contraction principles but also unveiled new avenues for exploring the existence of fixed points in various metric spaces, such as b -metric spaces, rectangular b -metric spaces, orthogonal metric spaces, and many others.

In a recent significant advancement, Alam and Imdad⁹ have established a relational-theoretic version of the Banach contraction principle and have expanded its range of applicability. Our principal objective in this work is to illustrate certain relational theoretic interpolative fixed-point outcomes by introducing a novel generalized concept of $\mathcal{R}$ -admissibility. This is accomplished by leveraging the locally T -transitivity property of binary relations along with some other relational metrical concepts.

2. Preliminaries

Before presenting our main results, it is important to introduce formal notations that will be used throughout this paper. A relational set is represented by the pair $(X, \mathcal{R})$, where X is a non-empty set, and $\mathcal{R}$ is a binary relation on X . In the context of a metric space, denoted as (X, d) , we refer to $(X, d, \mathcal{R})$ as a relational metric space (RMS, for short). Under this framework, $F(T)$ denotes the set of fixed points of the self-mapping T , and $X(T, \mathcal{R})$ represents the subset of X consisting of elements x for which $(x, Tx) \in \mathcal{R}$, and we set $X_{\mathcal{R}} = \{(x, y) \in \mathcal{R} \text{ such that } x, y \notin F(T)\}$. These formalized notations ensure precision and consistency throughout our subsequent analyses and discussions.

Definition 2.1.⁹ In the context of a relational set $(X, \mathcal{R})$.

- (i) Any two elements $x, y \in X$ are considered $\mathcal{R}$ -comparative if $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. This relationship is symbolically represented as $[x, y] \in \mathcal{R}$.
- (ii) A sequence $\{x_k\} \subset X$ satisfies the condition $(x_k, x_{k+1}) \in \mathcal{R}$ for all $k \in \mathbb{N}_0$, is referred to as an $\mathcal{R}$ -preserving sequence.

- (iii) $\mathcal{R}$ is designated as T -closed when it satisfies the condition that if (x, y) belongs to $\mathcal{R}$, then (Tx, Ty) also belongs to $\mathcal{R}$, for any $x, y \in X$.
- (iv) $\mathcal{R}$ is referred to as d -self-closed under the condition that whenever there exists a $\mathcal{R}$ -preserving sequence $\{x_k\}$ such that $\{x_k\}$ converges to x w.r.t d , we can always find a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ such that $[x_{k_n}, x]$ belongs to $\mathcal{R}$ for all $n \in \mathbb{N}$.

Definition 2.2.¹⁰ $(X, d, \mathcal{R})$ is considered $\mathcal{R}$ -complete if every sequence in X , which is both $\mathcal{R}$ -preserving and Cauchy, converges.

Definition 2.3.¹⁰ A self-map T defined on X is termed $\mathcal{R}$ -continuous at $x \in X$, if any $\mathcal{R}$ -preserving sequence $\{x_k\}$ converges to x w.r.t. d , implies Tx_k converges to Tx w.r.t. d . Furthermore, if T exhibits this behavior at every point in X , it is simply categorized as $\mathcal{R}$ -continuous.

Definition 2.4¹¹ Consider a self-mapping T defined on X . If for every $\mathcal{R}$ -preserving sequence $\{x_n\} \subset T(X)$, with a range denoted as $E = \{x_n : n \in \mathbb{N}\}$, $\mathcal{R}|_E$ is transitive, then T is designated as locally T -transitive.

Definition 2.5.¹² Let T be a self-map on X , and $\alpha: X \times X \rightarrow R^+$ be a function, then T is said to be α -admissible if, $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$, for all $x, y \in X$.

Definition 2.6. Let T be a self-map defined on a non-empty set X , endowed with a binary relation $\mathcal{R}$. Then T is said to be $\mathcal{R}$ -admissible if there exists a function $\alpha: X \times X \rightarrow [0, \infty)$, such that

- (i) $\alpha(x, y) \geq 1$, for all $(x, y) \in \mathcal{R}$,
- (ii) $\mathcal{R}$ is T closed.

Remark 2.7. From the above two definitions, we can observe that if T is α -admissible, it also holds that T is $\mathcal{R}$ -admissible when considering $\mathcal{R} = \{(x, y) \in X^2 : \alpha(x, y) \geq 1\}$. However, it should be noted that the converse is not necessarily true, as illustrated in the following example.

Example 2.8. Let $X = \{0, 1, 2, 3\}$, $\alpha: X \times X \rightarrow R^+$ by

$$\alpha(x, y) = \begin{cases} 2, & (x, y) \in \{(0, 1), (1, 2), (2, 3)\} \\ 1, & (x, y) \in \{(0, 2), (1, 1), (2, 1), (2, 2)\} \\ \frac{2}{x+5}, & \text{otherwise.} \end{cases}$$

and $T: X \rightarrow X$ is defined by $T0 = 0, T1 = 2, T2 = 1$, and $T3 = 3$.

In this example, it is evident that $\alpha(2, 3) \geq 1$, but $\alpha(T2, T3) = \alpha(1, 3) \not\geq 1$, indicating that T is not α -admissible. Now, let's consider the binary relation $\mathcal{R}$ defined as

$$\mathcal{R} = \{(0, 1), (0, 2), (1, 2), (2, 1), (1, 1), (2, 2)\}.$$

It is straightforward to observe that $\mathcal{R}$ is T -closed, and for all $x, y \in X$ with $(x, y) \in \mathcal{R}$, $\alpha(x, y) \geq 1$. Therefore, T is $\mathcal{R}$ -admissible.

Concerning the functions ψ and $\phi: [0, \infty) \rightarrow [0, \infty)$, we lay out the following conditions:

- (a) ψ is continuous and $\psi(x) = 0$ if and only if $x = 0$,
- (b) ψ is non-decreasing,
- (c) ϕ is l.s.c. with $\phi(0) = 0$,
- (d) $\psi(x) > \phi(x)$, for all $x > 0$.

3. Main Results

This section initiates with the introduction of our novel concept of $\mathcal{R}(\psi, \phi)$ -interpolative Kannan type contraction, $\mathcal{R}(\psi, \phi)$ -interpolative Ciric-Reich-Rus type contraction, and $\mathcal{R}(\psi, \phi)$ -interpolative Hardy-Rogers type contraction, respectively.

Definition 3.1. Let $(X, d, \mathcal{R})$ be an RMS. A mapping $T : X \rightarrow X$ is characterized as an $\mathcal{R}(\psi, \phi)$ -interpolative Kannan type contraction if there exist functions $\psi, \phi: [0, \infty) \rightarrow [0, \infty)$, function $r: X \times X \rightarrow [0, +\infty)$ and a parameter $\alpha \in (0, 1)$, such that

$$(3.1) \quad \psi(r(x, y)d(Tx, Ty)) \leq \phi(d(x, Tx)^\alpha \cdot d(y, Ty)^{1-\alpha}),$$

for all $(x, y) \in X_{\mathcal{R}}$.

Definition 3.2. Let $(X, d, \mathcal{R})$ be an RMS. A mapping $T : X \rightarrow X$ is characterized as an $\mathcal{R}(\psi, \phi)$ -interpolative Ciric-Reich-Rus type contraction if there exist functions $\psi, \phi: [0, \infty) \rightarrow [0, \infty)$, function $r: X \times X \rightarrow [0, +\infty)$ and parameters $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$, such that

$$(3.2) \quad \psi(r(x, y)d(Tx, Ty)) \leq \phi(d(x, y)^\alpha d(x, Tx)^\beta \cdot d(y, Ty)^{1-\alpha-\beta}),$$

for all $(x, y) \in X_{\mathcal{R}}$.

Definition 3.3. Let $(X, d, \mathcal{R})$ be an RMS. A mapping $T : X \rightarrow X$ is characterized as an $\mathcal{R}(\psi, \phi)$ -interpolative Hardy-Rogers type contraction if there exist functions $\psi, \phi: [0, \infty) \rightarrow [0, \infty)$, function $r: X \times X \rightarrow [0, +\infty)$ and parameters $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$, such that

$$(3.3) \quad \psi(r(x, y)d(Tx, Ty)) \leq \phi\left(d(x, y)^\alpha d(x, Tx)^\beta \cdot d(y, Ty)^\gamma \cdot \left(\frac{d(x, Ty) + d(y, Tx)}{2}\right)^{1-\alpha-\beta-\gamma}\right),$$

for all $(x, y) \in X_{\mathcal{R}}$.

From the aforementioned definitions, it is evident that every interpolative Kannan contraction implies $\mathcal{R}(\psi, \phi)$ -interpolative Kannan type contraction, every Interpolative Ciric-Reich-Rus type contraction implies $\mathcal{R}(\psi, \phi)$ -interpolative Ciric-Reich-Rus type contraction and every interpolative Hardy-Rogers type contraction implies $\mathcal{R}(\psi, \phi)$ -interpolative Hardy-Rogers type contraction. This can be achieved by considering $\mathcal{R}$ as the universal relation and taking some specific values of the control functions ψ and ϕ . However, the converse is not necessarily true, as demonstrated in the following example.

Example 3.4. Let $X = [0, 1]$, and define the mapping $T: X \rightarrow X$ by

$$Tx = \begin{cases} \frac{t^2}{2}, & \text{if } t \leq \frac{4}{5} \\ \frac{1}{2}, & \text{if } t > \frac{4}{5}, \end{cases}$$

additionally, consider the function $\psi(t) = \begin{cases} \frac{t^2}{2}, & \text{if } t \leq \frac{4}{5} \\ \frac{1}{2}, & \text{if } t > \frac{4}{5} \end{cases}$ and $\phi(t) = \frac{2t^2}{5}$.

In this example, it is observed for the pair $x = \frac{9}{10}$ and $y = \frac{1}{10}$, there exists no pair of parameters $\alpha \in (0, 1)$ and $\lambda \in [0, 1]$ for which condition (2) holds, indicating that T is not an interpolative Kannan-type contraction.

Next, we verify that T satisfies condition (3). Consider the binary relation $\mathcal{R}$ defined as $\mathcal{R} = \{(x, y) \in X^2: \min\{x, y\} \geq \frac{1}{2}\}$.

It is evident from the definition of $\mathcal{R}$ that $(x, y) \in \mathcal{R}$ only if $x, y \in \left[\frac{1}{2}, 1\right]$.

We proceed to show that condition (1) holds for all $x, y \in \left[\frac{1}{2}, 1\right]$.

Let $\alpha = \frac{1}{2}$, then

Case I: Suppose $x, y \in \left[\frac{1}{2}, \frac{4}{5}\right]$, then

$$\begin{aligned} \psi(d(Tx, Ty)) &= \frac{|x^2 - y^2|}{20} \\ &\leq \frac{1}{20} (|2x - x^2| \cdot |2y - y^2|) \\ &\leq \frac{2}{5} \left(\left| \frac{2x - x^2}{2} \right|^{\frac{1}{2}} \cdot \left| \frac{2y - y^2}{2} \right|^{\frac{1}{2}} \right)^2 \\ &= \phi(d(x, Tx)^\alpha \cdot d(y, Ty)^{1-\alpha}). \end{aligned}$$

Case II: $x \in \left[\frac{1}{2}, \frac{4}{5}\right]$ and $y \in \left[\frac{4}{5}, 1\right]$,

$$\begin{aligned}\psi(d(Tx, Ty)) &= \frac{1}{20}(x^2 - 1) \\ &\leq \frac{2}{20}(|2x - x^2| \cdot |2y - 1|) \\ &\leq \frac{2}{5} \left(\left| \frac{2x - x^2}{2} \right|^{\frac{1}{2}} \cdot \left| \frac{2y - 1}{2} \right|^{\frac{1}{2}} \right)^2 \\ &= \phi(d(x, Tx)^\alpha \cdot d(y, Ty)^{1-\alpha}).\end{aligned}$$

The case when $x, y \in \left[\frac{4}{5}, 1\right]$ is trivial, while the case when $x \in \left[\frac{4}{5}, 1\right]$ and $y \in \left[\frac{1}{2}, \frac{4}{5}\right]$ is identical to Case II. Hence T satisfies (3.1) for all $x, y \in X_{\mathcal{R}}$.

We now proceed to introduce our results concerning the existence of fixed points for a self-mapping T satisfying conditions (3.1), (3.2), and (3.3).

Theorem 3.5. *Let T be a $\mathcal{R}(\psi, \phi)$ -interpolative Kannan type contraction defined on a RMS $(X, d, \mathcal{R})$ where $\mathcal{R}$ is locally T -transitive, and there exist functions ψ and ϕ satisfying conditions (a), (b), (c), (d) and further assume:*

- (c1) T is $\mathcal{R}$ admissible,
- (c2) (X, d) is $\mathcal{R}$ -complete,
- (c3) either T is $\mathcal{R}$ -continuous or $\mathcal{R}$ is d -self-closed,
- (c4) $X(T, \mathcal{R})$ is non-empty.

Then, there exists at least one fixed point of T in X .

Proof: Under the assumption (c4), suppose that x_0 be a chosen element of $X(T, \mathcal{R})$. Define the sequence x_n of Picard iterates with the initial point x_0 , i.e. $x_n = T^n x_0$ for all $n \in N_{\mathbb{0}}$. As $(x_0, Tx_0) \in \mathcal{R}$ and T is $\mathcal{R}$ -admissible, it follows that $(T^n x_0, T^{n+1} x_0) \in \mathcal{R}$. Consequently, $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in N_{\mathbb{0}}$, and this yields that the sequence $\{x_n\}$ is $\mathcal{R}$ -preserving and $(x_n, x_{n+1}) \geq 1$. On applying contractive condition (3.1), we arrive at

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\leq \psi(r(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n)) \\ &\leq \phi(d(x_{n-1}, x_n)^\alpha \cdot d(x_n, x_{n+1})^{1-\alpha}) \\ &< \psi(d(x_{n-1}, x_n)^\alpha \cdot d(x_n, x_{n+1})^{1-\alpha})\end{aligned}$$

which implies that $d(x_n, x_{n+1})^\alpha < d(x_{n-1}, x_n)^\alpha$.

This leads us to the conclusion that $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ and as a result $\{d(x_n, x_{n+1})\}$ forms a decreasing sequence. Therefore, there exists $z \geq 0$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$, thus

$$\begin{aligned}
\psi(z) &\leq \liminf \psi(d(x_n, x_{n+1})) \\
&\leq \limsup \psi(r(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n)) \\
&\leq \limsup \phi(d(x_{n-1}, x_n)^\alpha \cdot d(x_n, x_{n+1})^{1-\alpha}) \\
&\leq \phi(z)
\end{aligned}$$

using condition (b), we have $\psi(x) > \phi(x)$ for all $x > 0$, thus $z = 0$, and therefore $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$.

Now, to establish that $\{x_n\}$ is a Cauchy sequence, we make a counter assumption. Suppose it is not, then there exists a positive real number $\epsilon > 0$ along with sub-sequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, with $n_k > m_k \geq k$, such that

$$(3.4) \quad d(x_{m_k}, x_{n_k}) \geq \epsilon, \text{ for all } k \in N.$$

Selecting n_k as the smallest integer exceeding m_k such that (3.4) holds, we deduce that

$$(3.5) \quad d(x_{m_k}, x_{n_k-1}) < \epsilon.$$

Using triangular inequality and (3.4), (3.5) we obtain that

$$\begin{aligned}
\epsilon &\leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\
&< \epsilon + d(x_{n_k-1}, x_{n_k}).
\end{aligned}$$

On taking the limit $k \rightarrow +\infty$ and utilizing the fact that $\lim_{n \rightarrow +\infty} d(x_n, x_{n+1}) = 0$, we obtain

$$(3.6) \quad \lim_{k \rightarrow +\infty} d(x_{m_k}, x_{n_k}) = \epsilon.$$

By using triangular inequality, we obtain that

$$|d(x_{m_{k+1}}, x_{n_{k+1}}) - d(x_{m_k}, x_{n_k})| \leq d(x_{m_k}, x_{m_{k+1}}) + d(x_{n_k}, x_{n_{k+1}}),$$

letting limit $k \rightarrow +\infty$ in the above inequality and employing (3.6), we obtain the following

$$\lim_{k \rightarrow +\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \lim_{k \rightarrow +\infty} d(x_{m_k}, x_{n_k}) = \epsilon.$$

Since $\{x_n\}$ is $\mathcal{R}$ -preserving and $\{x_n\} \subset T(X)$, by the locally T -transitivity of $\mathcal{R}$, we have $(x_{m_k}, x_{n_k}) \in \mathcal{R}$. Thus, we can deduce

$$\begin{aligned}
\psi(d(x_{m_{k+1}}, x_{n_{k+1}})) &\leq \psi(r(x_{m_k}, x_{n_k})d(Tx_{m_k}, Tx_{n_k})) \\
&\leq \phi(d(x_{m_k}, x_{m_{k+1}})^\alpha \cdot d(x_{n_k}, x_{n_{k+1}})^{1-\alpha}),
\end{aligned}$$

which implies, $\psi(\epsilon) \leq \phi(0)$, leading to the conclusion that $\epsilon = 0$, a contradiction. Hence, $\{x_n\}$ is indeed a Cauchy sequence. Since (X, d) is $\mathcal{R}$ -complete metric space, there exists $z \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = z.$$

Now if T is $\mathcal{R}$ -continuous, we have $x_{n+1} = Tx_n \rightarrow Tz$. By the uniqueness of the limit, we conclude that $Tz = z$, i.e. z is a fixed point of T .

On the other hand, if $\mathcal{R}$ is d -self-closed. we again utilize the fact that $\{x_n\}$ is $\mathcal{R}$ -preserving and $\{x_n\} \rightarrow z$. This implies the existence of a sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, z] \in \mathcal{R}$, $\forall k \in \mathbb{N}_0$. If $(x_{n_k}, z) \in \mathcal{R}$, then on applying condition (3.1), we have

$$(3.7) \quad \begin{aligned} \psi(d(Tx_{n_k}, Tz)) &\leq \psi(r(x_{n_k}, z)d(Tx_{n_k}, Tz)) \\ &\leq \phi(d(x_{n_k}, x_{n_{k+1}}))^\alpha \cdot d(z, Tz)^{1-\alpha}, \end{aligned}$$

taking the limit $k \rightarrow \infty$, in (3.7), we obtain $\psi(d(z, Tz)) = 0$. This is only possible if $d(z, Tz) = 0$ which implies $z = Tz$. i.e. z is a fixed point of T . Similarly, if $(z, x_{n_k}) \in \mathcal{R}$, then by utilizing the symmetry of d , we once again obtain the same conclusion.

Theorem 3.6. *Let T be a $\mathcal{R}(\psi, \phi)$ -Interpolative Ciric-Reich-Rus type contraction defined on a RMS $(X, d, \mathcal{R})$ where $\mathcal{R}$ is locally T -transitive. Suppose that conditions (c1) - (c4) hold, and there exist functions ψ and ϕ satisfying conditions (a), (b), (c), and (d). Then there exists at least one fixed point of T in X .*

Proof: By following the steps outlined in Theorem 3.5, we can obtain an $\mathcal{R}$ - $\mathcal{R}$ -preserving sequence $\{x_n\}$ i.e. $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}_0$. Applying contractive condition (3.2), we have

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq \psi(r(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n)) \\ &\leq \phi(d(x_{n-1}, x_n)^\alpha d(x_{n-1}, x_n)^\beta \cdot d(x_n, x_{n+1})^{1-\alpha-\beta}) \\ &< \psi(d(x_{n-1}, x_n)^\alpha d(x_{n-1}, x_n)^\beta \cdot d(x_n, x_{n+1})^{1-\alpha-\beta}), \end{aligned}$$

which implies that,

$$d(x_n, x_{n+1})^{\alpha+\beta} < d(x_{n-1}, x_n)^{\alpha+\beta}$$

thus, we get $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ and consequently $\{x_n\}$ is a decreasing sequence, so there exists $z \geq 0$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$, thus

$$\begin{aligned} \psi(z) &\leq \liminf \psi(d(x_n, x_{n+1})) \\ &\leq \limsup \psi(r(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n)) \end{aligned}$$

$$\leq \limsup \phi \left(d(x_{n-1}, x_n)^\alpha \cdot d(x_{n-1}, x_n)^\beta \cdot d(x_n, x_{n+1})^{1-\alpha-\beta} \right) \\ \leq \phi(z).$$

Form condition (d) we have $\psi(x) > \phi(x)$, for all $x > 0$ thus $z = 0$, and therefore $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$.

Now to prove that $\{x_n\}$ is a Cauchy sequence. Suppose not then there exists $\epsilon > 0$ and sub-sequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $k \leq m_k < n_k$, such that we have

$$\lim_{n \rightarrow \infty} d(x_{m_k}, x_{n_k}) = \lim_{n \rightarrow \infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon.$$

As $\{x_n\}$ is $\mathcal{R}$ -preserving and $\{x_n\} \subset T(X)$, by locally T -transitivity of $\mathcal{R}$, we have $(x_{m_k}, x_{n_k}) \in \mathcal{R}$. Hence

$$\psi \left(d(x_{m_{k+1}}, x_{n_{k+1}}) \right) \leq \psi \left(r(x_{m_k}, x_{n_k}) d(Tx_{m_k}, Tx_{n_k}) \right) \\ \leq \phi \left(d(x_{m_k}, x_{n_k})^\alpha \cdot d(x_{m_k}, x_{m_{k+1}})^\beta \cdot d(x_{n_k}, x_{n_{k+1}})^{1-\alpha-\beta} \right)$$

which implies, $\psi(\epsilon) \leq (0)$, consequently we get $\epsilon = 0$, a contradiction. Which implies that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is $\mathcal{R}$ -complete metric space, there exists $z \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = z.$$

Now if T is $\mathcal{R}$ -continuous. Then we have $x_{n+1} = Tx_n \rightarrow Tz$. By uniqueness of the limit, we have $Tz = z$ i.e. z is a fixed point of T . On the other hand, if $\mathcal{R}$ is d -self closed, applying the same method as in Theorem 3.1 leads us to conclude that z is a fixed point of T .

Theorem 3.7. *Let T be a $\mathcal{R}(\psi, \phi)$ -Interpolative Hardy-Rogers type contraction defined on a RMS $(X, d, \mathcal{R})$ where $\mathcal{R}$ is locally T -transitive. Suppose that conditions (c1) - (c4) hold, and there exist functions ψ and ϕ satisfying conditions (a), (b), (c), and (d). Then there exists at least one fixed point of T in X .*

Proof: By following the steps outlined in Theorem 3.5, we can obtain an $\mathcal{R}$ - $\mathcal{R}$ -preserving sequence $\{x_n\}$ i.e. $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}_0$. Applying contractive condition (3.3), we have

$$\psi(d(x_n, x_{n+1})) \leq \psi(r(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n))$$

$$\begin{aligned}
&\leq \phi \left(d(x_{n-1}, x_n)^\alpha d(x_{n-1}, x_n)^\beta \cdot d(x_n, x_{n+1})^\gamma \cdot \right. \\
&\quad \left. \left(\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} \right)^{1-\alpha-\beta-\gamma} \right) \\
&< \psi(d(x_{n-1}, x_n)^\alpha d(x_{n-1}, x_n)^\beta \cdot d(x_n, x_{n+1})^{1-\alpha-\beta}).
\end{aligned}$$

Suppose if $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$ then we get,

$$d(x_n, x_{n+1})^{\alpha+\beta} < d(x_n, x_{n+1})^{\alpha+\beta},$$

a contradiction. Therefore $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$, and consequently $\{x_n\}$ is a decreasing sequence, so there exists $z \geq 0$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$, thus

$$\begin{aligned}
\psi(z) &\leq \liminf \psi(d(x_n, x_{n+1})) \\
&\leq \limsup \psi(r(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n)) \\
&\leq \limsup \phi \left(d(x_{n-1}, x_n)^\alpha \cdot d(x_{n-1}, x_n)^\beta \cdot d(x_n, x_{n+1})^\gamma \cdot \right. \\
&\quad \left. \left(\frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2} \right)^{1-\alpha-\beta-\gamma} \right) \\
&\leq \phi(z).
\end{aligned}$$

Using condition (d) we have, $\psi(x) > \phi(x)$ for all $x > 0$ thus $z = 0$, and therefore $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$.

Now to prove that $\{x_n\}$ is a Cauchy sequence. Suppose not then there exists $\epsilon > 0$ and sub-sequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $k \leq m_k < n_k$, such that we have

$$\lim_{n \rightarrow \infty} d(x_{m_k}, x_{n_k}) = \lim_{n \rightarrow \infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon.$$

As $\{x_n\}$ is $\mathcal{R}$ -preserving and $\{x_n\} \subset T(X)$, by locally T -transitivity of $\mathcal{R}$, we have $(x_{m_k}, x_{n_k}) \in \mathcal{R}$. Hence

$$\begin{aligned}
\psi(d(x_{m_{k+1}}, x_{n_{k+1}})) &\leq \psi(r(x_{m_k}, x_{n_k})d(Tx_{m_k}, Tx_{n_k})) \\
&\leq \phi \left(d(x_{m_k}, x_{n_k})^\alpha \cdot d(x_{m_k}, x_{m_{k+1}})^\beta \cdot d(x_{n_k}, x_{n_{k+1}})^\gamma \cdot \right. \\
&\quad \left. \left(\frac{d(x_{m_k}, x_{n_{k+1}}) + d(x_{n_k}, x_{m_{k+1}})}{2} \right)^{1-\alpha-\beta-\gamma} \right),
\end{aligned}$$

which implies, $\psi(\epsilon) \leq (0)$, consequently we get $\epsilon = 0$, a contradiction. Which implies that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is $\mathcal{R}$ -complete metric space, there exists $z \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = z.$$

Now if T is $\mathcal{R}$ -continuous. Then we have $x_{n+1} = Tx_n \rightarrow Tz$. By uniqueness of the limit, we have $Tz = z$ i.e. z is a fixed point of T . On the other hand, if

$\mathcal{R}$ is d -self closed, applying the same method as in Theorem 3.1 leads us to conclude that z is a fixed point of T .

4. Conclusion

In conclusion, this research establishes fixed point results for relational theoretic interpolative contractions. The introduced contraction condition allows more flexibility by focusing on compliance among related elements instead of the entire space. Our findings encompass various binary relations that have been traditionally utilized in well-established metrical fixed-point theorems, demonstrating their broad applicability. Overall, this work extends the existing literature, opening new possibilities for studying interpolative contractions and their applications in mathematics and related fields.

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