

Shifted Chebyshev Collocation Method for the Solution of Boundary Layer Flow and Heat Transfer over an Exponential Shrinking Surface

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Abstract: The present investigation is to analysis the solution of steady state boundary layer flow and heat transfer past an exponential shrinking surface. The governing equations describes physics of the problem and are transformed into system of nonlinear ordinary differential equations through suitable similarity transformations. The similarity solutions of these equations are obtained by using semi-numerical technique, viz. shifted Chebyshev collocation method. The occurrence of dual solution is observed for both velocity and temperature profiles. The analysis shows, for a particular range of mass suction parameter, boundary layer flow over an exponential shrinking sheet is exist and enhancement in mass suction parameter increases the skin friction coefficient for the upper branch solution and reverse behavior is observed for lower branch solution. Whereas, both the solutions of temperature profiles decrease with increasing values of the Prandtl number. The computed results are validated with the earlier finding and are comparable.

Keywords: Boundary layer flow, Heat transfer, Exponentially shrinking sheet, Dual solution, Spectral method.

1. Introduction

The study of boundary layer flow past a stretching/shrinking sheet receives more attention from last few decades, as they have numerous applications in several engineering and industrial manufacturing processes, some of these are metal spinning, hot rolling, metal and polymer extrusion, paper and glass-fiber production, wire drawing etc.^{1,2}. The analysis of boundary layer flow generated by movement of plate in a quiescent ambient fluid is first reported by Sakiadis^{3,4}. Crane⁵ investigated the steady two-dimensional boundary layer flow over a linearly stretching surface and

reported the solution in its closed form. Many researchers⁶⁻⁸ extended the Crane⁵ model by implementing different physical conditions on it. The behavior of boundary layer flows past a surface stretched nonlinearly was examined by Vajravelu⁹ and Cortell¹⁰.

While studying the nature of liquid film over an unsteady stretching sheet, Wang¹¹ found an unusual kind of flow due to shrinking sheet. Further, the condition for similarity solution exist and is unique was established by Miklavcic and Wang¹² and observed the dependency flow on mass suction that is imposed externally. Awati and Bujurke¹³ discussed the magnetohydrodynamic (MHD) viscous boundary layer flow over a porous nonlinear shrinking sheet. Analytic solution of MHD flow over shrinking sheet is reported by Hayat *et al.*¹⁴. The study of boundary layer flow due to linear and nonlinear stretching/shrinking sheet problems was carried out in the literature¹⁵⁻²⁰ by using different kinds of techniques like Dirichlet series solution, Mayksen method, Merkin method and method of stretching of variables techniques.

Form past few decades, the investigation of heat transfer process is an interesting field of research and receives the attention of many researchers, as the rate of cooling is an important factor in manufacture of quality products in many industrial and engineering processes. The first investigation of steady boundary layer flow over a continuous exponentially stretching surface is carried out by Magyari and Keller²¹, where wall temperature varies exponentially. The numerical demonstration of viscous boundary layer flow past an exponentially stretching sheet is reported by Elbashbeshy²². Further, Khan and Sanjayanand²³ extended the work of Elbashbeshy²² to viscoelastic fluid flow past an exponentially stretching surface along with heat transfer. Partha *et al.*²⁴ analyzed the influence of viscous dissipation on mixed convection flow over an exponentially stretching sheet. Sajid and Hayat²⁵ extended the model²⁴ to examine the effect of thermal radiation on viscous flow past an exponentially stretching surface and obtained analytical solution through homotopy analysis method and the numerical solution of this model²⁵ is reported by Bidin and Nazar²⁶. Recently, Bhattacharyya²⁷ analyzed viscous boundary layer flow past an exponentially shrinking surface and obtained the dual solution.

The spectral methods are efficient and elegant semi-numerical techniques, important feature of these techniques is to obtain the numerical solution of linear/ nonlinear differential and integral equations. These methods are specialization of class of techniques known as weighted residuals and are categories as global methods. These methods mainly divided into three kinds of schemes based on the choice of test functions as Galerkin, Collocation and

Tau methods. In the present investigation the collocation method is adopted using shifted Chebyshev polynomials and choice of these polynomials is highly encouraged, as they satisfy minimax property, Runge phenomenon and shows the dominance over the Chebyshev polynomials. Clenshaw²⁸ investigated the Chebyshev expansion series solution method for the solution of linear ordinary differential equations (LODE). The extension of this work for obtaining the solution of LODE with variable coefficients was demonstrated by Sezer and Kaynak²⁹ and called the technique as Chebyshev matrix method. The important applications and analysis of Chebyshev polynomials were reported by authors³⁰⁻³¹ in detail. Akyuz and Sezer³² demonstrated the Chebyshev collocation method (CCM) to achieve the desirable solution of LODE. The advancement of Akyuz and Sezer³² work for higher-order nonlinear ordinary differential equations was carried out by Dascioglu and Yaslan³³. Recently, Awati *et al.*³⁴ discussed the CCM technique and made the analysis of dual solutions obtained for nonlinear system of equations.

In present paper, the two-dimensional viscous boundary layer flow and heat transfer over an exponentially shrinking sheet is discussed. The governing equations are converted into self-similar nonlinear ordinary differential equations through relevant similarity variables of an exponential form and solution of these equations are obtained by adopting semi-numerical method viz. shifted Chebyshev collocation method. Also, the influence of mass suction parameter and Prandtl number on velocity and temperature profiles are presented through graphs.

2. Mathematical Formulation

Consider the steady incompressible two-dimensional boundary layer flow of a Newtonian fluid past an exponentially shrinking sheet. From boundary layer approximations theory, the governing equations of continuity, momentum and energy equations are expressed as²⁷:

$$(2.1) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$(2.2) \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}.$$

and

$$(2.3) \quad u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial y^2}.$$

where u and v denotes the x and y components of the velocity field respectively and ν, T, ρ, k and C_p denote the kinematic viscosity, temperature,

fluid density, thermal conductivity and specific heat of the fluid respectively.

The relevant boundary conditions are given by

$$(2.4) \quad u = U_w(x) = -c \exp\left(\frac{x}{L}\right), \quad v = v_w(x) = v_0 \exp\left(\frac{x}{2L}\right), \quad T = T_w = T_\infty + T_0 \exp\left(\frac{x}{2L}\right)$$

at $y = 0$, and

$$(2.5) \quad u \rightarrow 0, \quad T \rightarrow T_\infty \quad \text{as } y \rightarrow \infty,$$

where U_w represents shrinking velocity with $c > 0$ (shrinking constant), v_w is the mass transfer with v_0 being constant ($v_0 > 0$ and $v_0 < 0$ for mass injection and suction respectively), T_w and T_∞ be the temperature at the surface and free stream temperature respectively, T_0 is the rate of temperature gained along the surface and is constant.

Now, introduce the stream function $\psi(x, y)$ as

$$(2.6) \quad u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}.$$

The continuity equation (2.1) satisfied trivially from equation (2.6) and then the equations (2.2) and (2.3) take the form

$$(2.7) \quad \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3},$$

$$(2.8) \quad \frac{\partial \psi}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial y} = \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial y^2}.$$

The corresponding boundary conditions (2.4) becomes

$$(2.9) \quad \frac{\partial \psi}{\partial y} = U_w(x), \quad \frac{\partial \psi}{\partial x} = -v_w(x) \quad \text{at } y = 0, \quad \frac{\partial \psi}{\partial y} \rightarrow 0 \quad \text{as } y \rightarrow \infty.$$

Now, define the similarity transformations as²⁷

$$(2.10) \quad \eta = y \sqrt{\frac{c}{2\nu L}} \exp\left(\frac{x}{2L}\right), \quad \psi = \sqrt{2\nu L c} f(\eta) \exp\left(\frac{x}{2L}\right) \quad \text{and} \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}.$$

Utilizing equation (2.10), the equations (2.7)-(2.9) can be expressed as

$$(2.11) \quad f''' + f f'' - 2f'^2 = 0,$$

$$(2.12) \quad \theta'' + \text{Pr}(f\theta' - f'\theta) = 0.$$

The resultant boundary conditions are

$$(2.13) \quad f(\eta) = S, \quad f'(\eta) = -1, \quad \theta(\eta) = 1 \quad \text{at } \eta = 0,$$

and

$$(2.14) \quad f'(\eta) \rightarrow 0, \quad \theta(\eta) \rightarrow 0 \quad \text{as } \eta \rightarrow \infty,$$

where $\text{Pr} = \mu C_p / k$ and $S = -v_0 / \sqrt{vc/2L}$ are the Prandtl number and mass transfer parameter respectively, $S > 0$ and $S < 0$ corresponds to mass suction and mass injection respectively.

3. Method of Solution

The equations (2.11)-(2.14) are transformed from infinite domain into shifted Chebyshev polynomials domain *i.e.* $[0,1]$ for the implementation of SCCM utilizing following transformations

$$(3.1) \quad \xi = \frac{\eta}{\eta_{\infty}},$$

$$(3.2) \quad f(\eta) = \eta_{\infty} f(\xi), \quad \theta(\eta) = \eta_{\infty} \theta(\xi),$$

where η_{∞} denotes the boundary layer thickness, using equations (3.1)-(3.2), the equations (2.11)-(2.12) take the form

$$(3.3) \quad f''' + \eta_{\infty}^2 f f'' - 2\eta_{\infty}^2 f'^2 = 0,$$

and

$$(3.4) \quad \theta'' + \eta_{\infty}^2 \text{Pr} f \theta' - \eta_{\infty}^2 \text{Pr} f' \theta = 0.$$

The corresponding boundary conditions become

$$(3.5) \quad \left\{ \begin{array}{l} f(\xi) = \frac{S}{\eta_{\infty}}, \quad f'(\xi) = -1, \quad \theta(\bar{\eta}) = \frac{1}{\eta_{\infty}} \quad \text{at } \xi = 0 \\ f'(\xi) = 0, \quad \theta(\xi) = 0 \quad \text{as } \xi = 1 \end{array} \right\},$$

To obtain the solution of equations (3.3)-(3.4), we seek a solution in the form of truncated shifted Chebyshev series expansion (TSCSE) as expressed below

$$(3.6) \quad f(\xi) = \sum_{r=0}^N a_r T_r^*(\xi), \quad \theta(\xi) = \sum_{r=0}^N b_r T_r^*(\xi),$$

$$(3.7) \quad f^{(s)}(\xi) = \sum_{r=0}^N a_r^{(s)} T_r^*(\xi), \quad \theta^{(s)}(\xi) = \sum_{r=0}^N b_r^{(s)} T_r^*(\xi),$$

where, $T_r^*(\xi)$ is shifted Chebyshev polynomial of degree r , $a_r, a_r^{(s)}, b_r$ and $b_r^{(s)}$ are shifted Chebyshev coefficients need to be determine and N be any positive integer. Consider the Chebyshev collocation points in their natural order, defined as

$$(3.8) \quad \xi_p = \frac{1}{2} \left[1 - \cos \left(\frac{p\pi}{N} \right) \right], \quad p = 0, 1, 2, \dots, N.$$

Evaluating equations (3.4) and (3.5) at the collocation points, the matrix form of resultant system is expressed as:

$$(3.9) \quad f(\xi) = T^*(\xi)X, \quad f^{(s)}(\xi) = T^*(\xi)X^{(s)},$$

where $f(\xi) = [f(\xi_i)]_{0 \leq i \leq N}^T, f^{(s)}(\xi) = [f^{(s)}(\xi_i)]_{0 \leq i \leq N}^T, T^*(\xi) = [T^*_i(\xi_j)]_{0 \leq i, j \leq N}^T,$

$$X = [a_i]_{0 \leq i \leq N}^T \text{ and } X^{(s)} = [a_i^{(s)}]_{0 \leq i \leq N}^T.$$

Using the relation between unknown Chebyshev coefficients²⁹, the final version of equations (3.3)-(3.4) in compact matrix form is expressed as:

$$(3.10) \quad 4T^*M^3X + \eta_\infty^2(\hat{T}\hat{X}T^*M^2X - 2\hat{T}\hat{M}\hat{X}T^*MX) = 0,$$

$$(3.11) \quad 4T^*M^2Y + \eta_\infty^2\text{Pr}(\hat{T}\hat{X}T^*MY - \hat{T}\hat{M}\hat{X}T^*Y) = 0,$$

where

$$M = \begin{pmatrix} 0 & 1/2 & 0 & 3/2 & 0 & 5/2 & \cdots & m_1 \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & m_2 \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & m_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(N+1) \times (N+1)}.$$

For even N : $m_1 = m_3 = 0, m_2 = N$ and for odd N : $m_1 = \frac{N}{2}, m_2 = 0, m_3 = 2m_1$. $T^* = T^*(\xi)$, $\hat{T} = \text{diag}[T^*(\xi_i)]_{0 \leq i \leq N}$, $\hat{M} = \text{diag}(M)$, $\hat{X} = \text{diag}(X)$ and $Y = [b_i]_{0 \leq i \leq N}^T$.

The equations (3.10) and (3.11) can be expressed as

$$(3.12) \quad AU = B,$$

where A is matrix of order $(2N+2) \times (2N+2)$ and U being the matrix of unknowns of corresponding order. Similarly matrix form of (3.5) is given by

$$(3.13) \quad \hat{A}U = \hat{B}.$$

Using the equation (3.13) in equation (3.12) and solving the obtained nonhomogeneous, nonlinear system of algebraic equations with help of suitable technique gives the unknown Chebyshev coefficients. Now, the solution on the required domain can be obtained with help of equations (3.2) and (3.6), satisfies boundary conditions approximately.

4. Result and discussion

The numerical investigation of steady boundary layer flow over an exponentially shrinking sheet has been analysed. As compared to linear case, in exponential case shrinking rate is faster. Hence in an exponentially shrinking case the amount of vorticity generated is greater than linear shrinking case. For Newtonian fluids authors^{12,35} investigation shows that, when mass suction parameter $S \geq 2$, two-dimensional flow with wall mass transfer past a shrinking sheet is possible. The analysis of Bhattacharyya²⁷ found that in an exponentially shrinking sheet, flow is possible only when $S \geq 2.66684$ and if $S < 2.66684$ similarity solution does not exist. Figures 1 and 2 represents the values of skin friction coefficients $f''(0)$ and surface

temperature gradient $-\theta'(0)$ for corresponding values of S . The present results are in good agreement with earlier findings²⁷ with $S \geq 2.26676$ for possible fluid flow to exist. From both the figures it is observed that, for first solution values of $f''(0)$ and $-\theta'(0)$ increases with increasing values of S , but for second solution reverse behaviour is observed with little variations in the values of $-\theta'(0)$ as S increases.

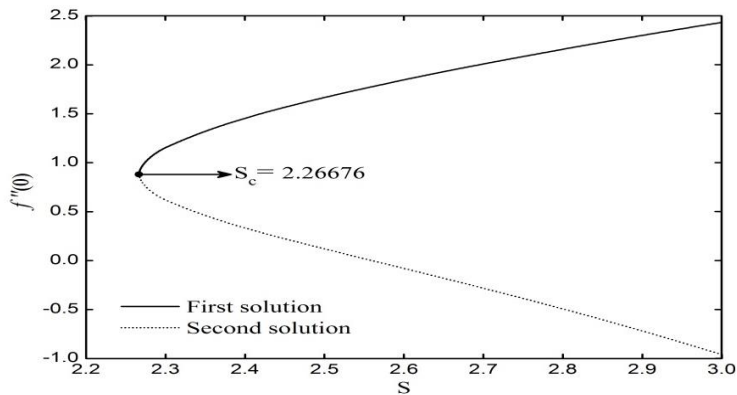


Figure 1. Skin friction coefficient for different values of S

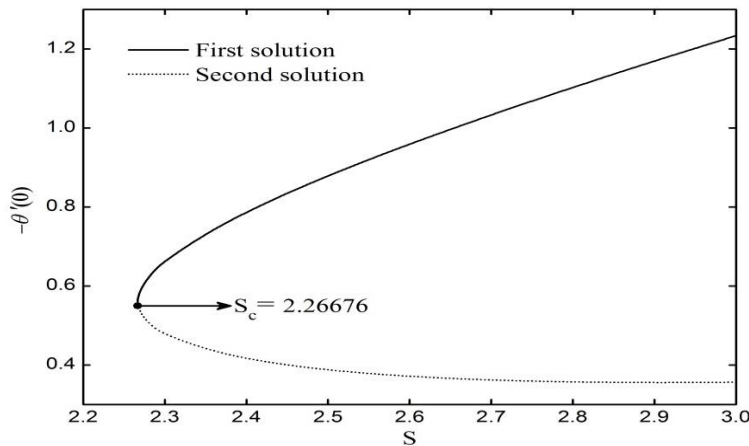


Figure 2. Temperature gradient at the sheet for different values of S

Figures 3 and 4 depict the influence of S on nondimensional velocity profiles $f'(\eta)$ and shear stress profiles $f''(\eta)$ respectively. In Figure 3 with increasing values of S , enhancement in $f'(\eta)$ for first solution and reduction in $f'(\eta)$ for second solution is noticed. This results in thinner the momentum boundary layer for first solution as compared to second solution. Whereas, shear stress profiles of first solution increases initially, get reverted in mid-region and merge in far-field region with enhancement in S and opposite nature of these profiles is seen for second solution.

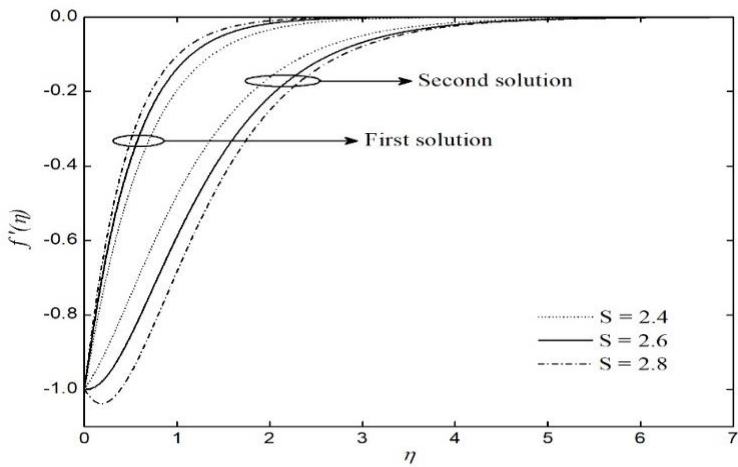


Figure 3. Velocity profiles for different values of S

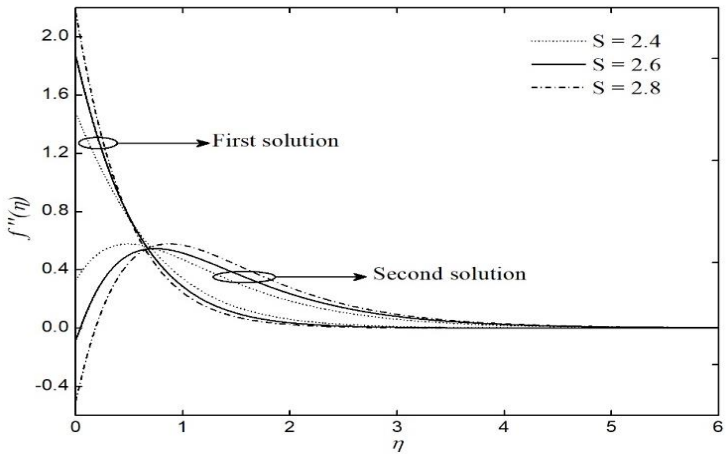


Figure 4. Shear stress profiles for different values of S

Figure 5 represents the impact of S on temperature profiles $\theta(\eta)$. For first solution, the values of $\theta(\eta)$ decreases with increasing values of S and the value of $\theta(\eta)$ increases with S for second solution, results into thinner thermal boundary layer of first solution in comparison with second solution. The influence of Prandlt number Pr on $\theta(\eta)$ is depicted in Figure 6, both the solutions show the similar nature i.e. temperature profiles of both solutions decrease with increasing values of Pr . The thickness of thermal boundary layer is thicker for second solution as compared to first solution for all values of Pr . The inverse relation between Pr and thermal conductivity shows that, fluids having larger Pr have the slower heat diffusion, whereas faster heat diffusion is observed in fluids having lower Pr .

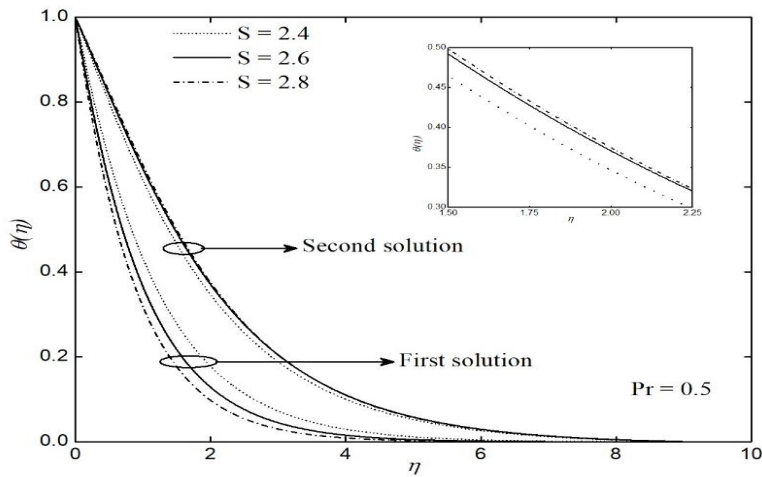


Figure 5. Temperature profiles for different values of S .

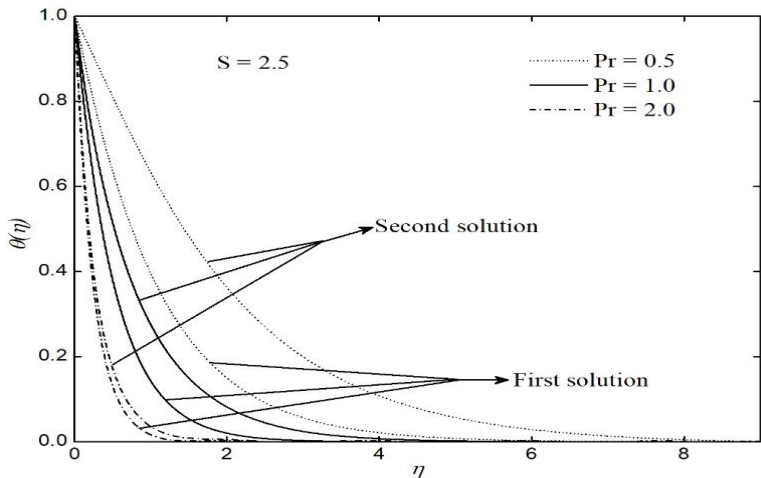


Figure 6. Temperature profiles for different values of Pr .

5. Conclusion

In present investigation, SCCM technique is used to achieve the desirable solution of similarity equations for the analysis of heat transfer and boundary layer flow over a sheet shrunk exponentially and these studies reveals that the possible steady flow over exponentially shrinking surface takes place under the condition that mass suction parameter $S \geq 2.26676$. The obseravation shows, the occurance of dual solution for both velocity and temperature profiles, for increasing values of S velocity increases for the first solution and decreases for the second solution. In case of temperature distributions reverse behavior is noticed for increasing values of S . The effect

of Prandtl number on temperature profiles shows that thermal boundary layer thickness becomes thinner with increasing values of Pr .

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