

A Study of Certain New Connections in an Almost Complex Space-I

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Abstract: In this paper, while studying some transformations, we have defined several new connection parameters and studied their properties in an almost complex space as well as Kahler space. We have further defined and studied some curvature properties based on several curvature tensors and identities satisfied by them.

Keywords: Symmetric and skew-symmetric Tensor, Complex Space, Hermitian Metric Tensor, Ricci Tensor.

1. Introduction

Let (M^n, g) be an even dimensional space of class C^{r+1} , with coordinate neighbourhoods (U, Σ^h) , $(h, i, j, \dots = 1, 2, \dots, n)$ and let g_{ji} and F_i^h be respectively the components of the Hermitian metric tensor and an almost complex structure tensor of M^n , such that

$$(1.1) \quad F_j^i F_i^h = -\delta_j^h$$

and

$$(1.2) \quad F_j^t F_i^s g_{ts} = g_{ji}$$

If $F_{ji} = F_j^t g_{ti}$, we can observe that $F_{ji} = -F_{ij}$ and $F_j^t F_{ht} = g_{jh}$. Let K_{kjih} , K_{ji} and K be respectively the curvature tensor, Ricci tensor and curvature scalar of M^n , we have¹:

$$(1.3) \quad H_{kj} = (1/2)K_{kjih}F^{ih},$$

$$(1.4) \quad H_{hj} = K_{hs}F_j^s, \quad K_{kj} = -H_{ks}F_j^s$$

and

$$(1.5) \quad K = -H_{kj}F^{kj}$$

Let ∇ be a linear connection in a Riemannian manifold M^n of dimension $n (\geq 2)$, then the quarter-symmetric connection is defined and expressed locally by Golab² as follows:

$$(1.6) \quad \Gamma_{ji}^h = \{h_{ji}\} + T_j^h p_i - T_{ji} p^h$$

where $\{h_{ji}\}$ are Christoffel symbols, T_{ji} is a symmetric tensor and p_i is a 1-form such that $p^h = g^{hj} p_j$.

Various properties of quarter-symmetric connections have been studied by several authors including Rastogi^{3,4}, Mishra and Pandey⁵ and Yano and Imai⁶. Yano and Imai gave the definition of most general form of quarter-symmetric connection as follows:

$$(1.7) \quad \Gamma_{ji}^h = \{h_{ji}\} + U_j^h p_i - p_j V_i^h - U_{ji} p^h,$$

where U_{ji} and V_{ji} are respectively symmetric and skew-symmetric parts of an arbitrary tensor T_{ji} , $U_j^h = U_{jt} g^{th}$, $V_i^h = V_{jt} g^{th}$.

The connection Γ_{ji}^h given by equation (1.7) is such that the torsion tensor is given by $T_j^h p_i - T_i^h p_j$ and curvature tensor R_{kji}^h is expressible as

$$(1.8) \quad R_{kji}^h = K_{kji}^h - \zeta_{(j,k)} [U_k^h p_{ji} + p_k^h U_{ji} - p_i \nabla_k U_j^h + p^h \nabla_k U_{ji} + V_i^h \nabla_k p_j \\ + p_j \nabla_k V_i^h - V_t^h p_j p_i U_k^t + U_{jt} V_i^t p_k p^h + U_k^h V_i^t p_j p_t - p_k p^t V_t^h U_{ji}],$$

where $p_{ji} = \nabla_j p_i - U_{jt} p_t p_i + (1/2) U_{ji} p_t p^t$, $p_i^h = p_{it} g^{th}$ and $\zeta_{(j,k)}$ means interchange of j and k and subtraction.

2. Connection Parameter ${}^1\Gamma_{ji}^h$.

Definition 2.1. A transformation which transforms Christoffel symbols to a connection ${}^1\Gamma_{ji}^h$ given by equation (2.1)

$$(2.1) \quad {}^1\Gamma_{ji}^h = \{h_{ji}\} - p_j F_i^h$$

shall be called F -transformation.

If ${}^1\nabla_m$ represents covariant derivative based on connection ${}^1\Gamma_{ji}^h$, we can easily establish for a tensor T_j^h

$$(2.2) \quad {}^1\nabla_m T_j^h = \nabla_m T_j^h - p_m (T_j^r F_r^h - T_r^h F_j^r)$$

which leads to

$$(2.3) \quad {}^1\nabla_m g_{ji} = 0, \quad {}^1\nabla_m F_i^h = \nabla_m F_i^h$$

From equation (2.3), we can observe that

Theorem 2.1. *The covariant derivative ${}^1\nabla_m$ based on coefficients of connection ${}^1\Gamma_{ji}^m$ is a metric connection as well as it satisfies ${}^1\nabla_m F_i^h = \nabla_m F_i^h$.*

Let ${}^1R_{kji}^h$ be the curvature tensor based on ${}^1\Gamma_{ji}^h$, then we can easily obtain from equation (1.8), for $U_{ji} = 0$ and $V_i^j = F_i^j$,

$$(2.4) \quad {}^1R_{kji}^h = K_{kji}^h - \zeta_{(j,k)}[F_i^h \nabla_k p_j + p_j \nabla_k F_i^h],$$

which leads to

$$(2.5) \quad {}^1R_{ji} = K_{ji} - F_i^h (\nabla_h p_j - \nabla_j p_h) - p_j \nabla_h F_i^h + p_h \nabla_j F_i^h$$

and

$$(2.6) \quad {}^1R = K - 2F^{hj} \nabla_h p_j - 2p_j \nabla_h F^{hj}$$

Hence:

Theorem 2.2. *If ${}^1R_{kji}^h$, ${}^1R_{ji}$ and ${}^1R$ are respectively the curvature tensor, Ricci tensor and curvature scalar based on connection ${}^1\Gamma_{ji}^h$ defined by equation (2.1), they are expressed by equations (2.4), (2.5) and (2.6).*

From equation (2.4), we can establish following curvature identity

$$(2.7a) \quad [{}^1R_{kji}^h + p_i (\nabla_j F_k^h - \nabla_k F_j^h) + F_i^h (\nabla_k p_j - \nabla_j p_k)] + \text{cycl.}(i, j, k) = 0$$

while from equation (2.5) we can obtain

$$(2.7b) \quad \zeta_{(i,j)} [{}^1R_{ji} + F_i^h (\nabla_h p_j - \nabla_j p_h) + p_j \nabla_h F_i^h - p_h \nabla_j F_i^h] = 0.$$

Multiplying equation (2.5) by F_k^i and using equation (1.4), we can obtain on simplification

$$(2.8a) \quad {}^1H_{jk} = H_{jk} - (\nabla_k p_j - \nabla_j p_k) - F_k^i (p_j \nabla_h F_i^h - p_h \nabla_j F_i^h),$$

and

$$(2.8b) \quad \zeta_{(j,k)} [{}^1H_{jk} - H_{jk} + 2\nabla_k p_j + F_k^i (p_j \nabla_h F_i^h - p_h \nabla_j F_i^h)] = 0$$

where we have used ${}^1R_{ji} F_k^i = {}^1H_{jk}$. Hence:

Theorem 2.3. *If the tensor ${}^1H_{jk} = {}^1R_{ji} F_k^i$ and $H_{jk} = K_{ji} F_k^i$, they are related by equation (2.8a) and satisfy equation (2.8b).*

Now we shall consider some special cases

Case I. If p_i is a gradient vector, equations (2.4), (2.5) and (2.6) respectively give

$$(2.9a) \quad {}^1R_{kji}^h = K_{kji}^h - \zeta_{(j,k)}[p_j \nabla_k F_i^h],$$

$$(2.9b) \quad {}^1R_{ji} = K_{ji} - p_j \nabla_h F_i^h + p_h \nabla_j F_i^h$$

$$(2.9c) \quad {}^1R = K - 2p_j \nabla_h F^{hj}.$$

Hence

Corollary 1. *If p_i is a gradient vector, equations (2.4), (2.5) and (2.6) respectively reduce to equations (2.9a), (2.9b) and (2.9c).*

From equation (2.9) a, we can obtain the following curvature identity

$$(2.10a) \quad [{}^1R_{kji}^h - p_i(\nabla_k F_j^h - \nabla_j F_k^h)] + \text{cycl.}(k, j, i) = 0.$$

From equation (2.9) b, we can obtain

$$(2.10b) \quad {}^1R_{ji} - {}^1R_{ji} = p_i \nabla_h F_j^h - p_j \nabla_h F_i^h + p_h(\nabla_j F_i^h - \nabla_i F_j^h)$$

which leads to

$$(2.10c) \quad \zeta_{(j,k)}[{}^1H_{jk} - H_{jk} - (\nabla_h F_i^h)(F_j^i p_k) - (\nabla_j F_i^h)(F_k^i p_h)] = 0$$

From equation (2.10) a and (2.10) b and (2.10) c, we can establish

Theorem 2.4. *If p_i is a gradient vector the curvature tensor ${}^1R_{kji}^h$, Ricci tensor ${}^1R_{ji}$ and tensor ${}^1H_{jk}$ satisfy equations (2.10a), (2.10b) and (2.10c).*

Further from these equations, we can establish

Theorem 2.5. *If p_i is a gradient vector*

i) *the tensor ${}^1R_{ji}$ will be symmetric in j and i if and only if*

$$p_i \nabla_h F_j^h + p_h \nabla_j F_i^h = p_j \nabla_h F_i^h + p_h \nabla_i F_j^h,$$

ii) *the tensor ${}^1H_{jk}$ will be symmetric in j and k if and only if*

$$\zeta_{(j,k)}[H_{jk} + (\nabla_h F_i^h)(F_j^i p_k) + (\nabla_j F_i^h)(F_k^i p_h)] = 0.$$

Case II. If the given space is a Kahler space, tensor F_i^h satisfies $\nabla_k F_i^h = 0$, therefore equations (2.4), (2.5) and (2.6) will satisfy

$$(2.11a) \quad {}^1R_{kji}^h = K_{kji}^h - \zeta_{(j,k)}[F_i^h \nabla_k p_j],$$

$$(2.11b) \quad {}^1R_{ji} = K_{ji} - F_i^h(\nabla_h p_j - \nabla_j p_h)$$

$$(2.11c) \quad {}^1R = K - 2F^{hj}(\nabla_h p_j)$$

From equations (2.11a,b), we can obtain

$$(2.12a) \quad [{}^1R_{kji}^h + F_i^h(\nabla_k p_j - \nabla_j p_k)] + \text{cycl.}(i, j, k) = 0$$

$$(2.12b) \quad {}^1R_{ji} - {}^1R_{ji} = [F_j^h(\nabla_h p_i - \nabla_i p_h) - F_i^h(\nabla_h p_j - \nabla_j p_h)]$$

and

$$(2.12c) \quad \zeta_{(j,k)}[{}^1H_{jk} - H_{jk} - (\nabla_k p_j - \nabla_j p_k)] = 0.$$

Hence:

Corollary 2. *In case of a Kahler space Yano⁶ and Rastogi⁵, the curvature tensor and Ricci tensor based on equation (2.1) satisfy equations (2.12a), (2.12b) and (2.12c) respectively.*

From equations (2.12) b and (2.12) c, we can establish

Theorem 2.6. *In case of a Kahler space i) the tensor ${}^1R_{ji}$ shall be symmetric in i and j if and only if $\zeta_{(i,j)}\{F_j^h(\nabla_h p_i - \nabla_i p_h)\} = 0$, ii) the tensor ${}^1H_{jk}$ will be symmetric in i and j if and only if we have*

$$\zeta_{(j,k)}\{2\nabla_k p_j + H_{jk}\} = 0.$$

Case III. If p_i is a gradient vector as well as given space is a Kahler space, equations (2.3), (2.4) and (2.5) respectively reduce to

$$(2.13) \quad {}^1R_{kji}^h = K_{kji}^h, {}^1R_{ji} = K_{ji} \text{ and } {}^1R = K.$$

Hence:

Corollary 3. *If p_i is a gradient vector as well as the given space is a Kahler space, equations (2.3), (2.4) and (2.5) give equation (2.13).*

3. Curvature Tensors $*R_{kji}^h$

In case of a Kahler space from equations (2.11a) and (2.11b) by eliminating $\nabla_k p_j - \nabla_j p_k$, we can obtain and define curvature tensor $*R_{kji}^h$ as follows:

$$(3.1) \quad *R_{kji}^h = {}^1R_{kji}^h + n^{-1}F_i^h F_t^r {}^1R_{kjr}^t = K_{kji}^h + n^{-1}F_i^h F_t^r K_{kjr}^t,$$

which leads to

$$(3.2) \quad *R_{ji} = {}^1R_{ji} + n^{-1}F_i^h F_t^r {}^1R_{hjr}^t = K_{ji} + n^{-1}F_i^h F_t^r K_{hjr}^t$$

and

$$(3.3) \quad *R = {}^1R + n^{-1}F^{hj} F_t^r {}^1R_{hjr}^t = K + n^{-1}F^{hj} F_t^r K_{hjr}^t$$

Hence:

Theorem 3.1. *In a Kahler space, the F-transformation given by equation (2.1) is such that, the curvature tensor, Ricci tensor and curvature scalar $*R_{kji}^h$, $*R_{ji}$ and $*R$ defined by equations (3.1), (3.2) and (3.3) respectively are F- invariant entities.*

From equations (3.2) and (3.3), we can obtain on simplification by virtue of equations (1.3), (1.4) and (1.5) and $*K = -*H_{kj}F^{kj}$

$$(3.4) \quad *H_{jm} = (n+2).n^{-1}H_{jm}, \quad *K = (n+2).n^{-1}K$$

Hence:

Theorem 3.2. *In a Kahler space, The F-transformation given by equation (2.1) is such that the Ricci tensor and curvature scalar satisfying equations (3.2) and (3.3), also satisfy equation (3.4).*

4. Curvature Tensor $*^1R_{kji}^h$

Multiplying equation (2.1) by F_t^i , we get on simplification

$$(4.1) \quad p_j = n^{-1}[\{h_{jt}\} - {}^1\Gamma_{jt}^h]F_h^t$$

which leads to

$$(4.2) \quad *^1\Gamma_{ji}^h = {}^1\Gamma_{ji}^h - n^{-1} {}^1\Gamma_{jt}^h F_r^t F_i^h = \{h_{ji}\} - n^{-1}\{r_{jt}\}F_r^t F_i^h$$

Hence:

Theorem 4.1. *If F-transformation of a Christoffel symbol is given by equation (2.1), the connection $*^1\Gamma_{ji}^h$ defined by equation (4.2) is an invariant connection.*

If $*^1R_{kji}^h$ be the curvature tensor based on coefficient of connection $*^1\Gamma_{ji}^h$ defined by equation (4.2), on simplification we can obtain

$$\begin{aligned} (4.3) \quad *^1R_{kji}^h &= K_{kji}^h - \zeta_{(j,k)}n^{-1}[F_i^h F_r^t (\partial_k \{r_{jt}\}) + F_i^h \{r_{jt}\}(\partial_k F_r^t) \\ &\quad + F_r^t \{r_{jt}\}(\partial_k F_i^h) + F_r^t F_i^a \{h_{ka}\}\{r_{jt}\} \\ &\quad + F_r^t F_a^h \{a_{ji}\}\{r_{kt}\} - n^{-1}F_r^t F_s^u \delta_i^h \{r_{jt}\}\{s_{ju}\}], \\ &= {}^1R_{kji}^h - \zeta_{(j,k)}n^{-1}[F_i^h F_r^t (\partial_k {}^1\Gamma_{jt}^r) + F_i^h {}^1\Gamma_{jt}^r (\partial_k F_r^t) \\ &\quad + F_r^t {}^1\Gamma_{jt}^r (\partial_k F_i^h) + F_r^t F_i^a {}^1\Gamma_{ka}^h {}^1\Gamma_{jt}^r \\ &\quad + F_r^t F_a^h {}^1\Gamma_{ji}^a {}^1\Gamma_{kt}^r - n^{-1}F_r^t F_s^u \delta_i^h {}^1\Gamma_{kt}^r {}^1\Gamma_{ju}^s]. \end{aligned}$$

Hence:

Theorem 4.2. *The curvature tensor $*^1R_{kji}^h$ based on coefficient of connection $*^1\Gamma_{ji}^h$ and given by equation (4.2), is invariant under the F-transformation.*

From equation (4.3) using $*^1R_{kji}^h = {}^1R_{kji}$, we get on simplification

$$\begin{aligned}
 (4.4) \quad {}^1R_{ji} = & K_{ji} - n^{-1}[F_i^h F_r^t(\partial_h\{r_{jt}\} - \partial_j\{r_{ht}\}) + F_i^h(\{r_{jt}\}(\partial_h F_r^t) \\
 & - \{r_{ht}\}(\partial_j F_r^t)) + F_r^t(\{r_{jt}\}(\partial_h F_i^h) - \{r_{ht}\}(\partial_j F_i^h)) \\
 & + F_r^t F_i^a(\{h_{ha}\}\{r_{jt}\} - \{h_{ja}\}\{r_{ht}\}) + F_r^t F_a^h(\{a_{ji}\}\{r_{ht}\} \\
 & - \{a_{hi}\}\{r_{jt}\} - n^{-1}F_r^t F_s^u(\{r_{jt}\}\{s_{ju}\} - \{r_{jt}\}\{s_{iu}\})]
 \end{aligned}$$

which when multiplied by g^{ji} leads by virtue of ${}^1R_{ji} g^{ji} = {}^1R$ to

$$\begin{aligned}
 (4.5) \quad {}^1R = & K - n^{-1}[F^{hj}(F_r^t(\partial_h\{r_{jt}\} - \partial_j\{r_{ht}\}) + \{r_{jt}\}(\partial_h F_r^t) \\
 & - \{r_{ht}\}(\partial_j F_r^t)) + g^{ji}(F_r^t(\{r_{jt}\}(\partial_h F_i^h) - \{r_{ht}\}(\partial_j F_i^h)) \\
 & + F_r^t F_i^a(\{h_{ha}\}\{r_{jt}\} - \{h_{ja}\}\{r_{ht}\}) + F_r^t F_a^h(\{a_{ji}\}\{r_{ht}\} \\
 & - \{a_{hi}\}\{r_{jt}\}) - n^{-1}F_r^t F_s^u(\{r_{jt}\}\{s_{ju}\} - \{r_{jt}\}\{s_{iu}\})]
 \end{aligned}$$

Hence:

Theorem 4.3. *The Ricci tensor ${}^1R_{ji}$ and curvature scalar ${}^1R$ represented by equations (4.4) and (4.5) respectively are invariant under F -transformation.*

In case of Kahler manifolds, equations (4.3), (4.4) and (4.5) can be expressed as

$$\begin{aligned}
 (4.6) \quad {}^1R_{kji}^h = & K_{kji}^h - \zeta_{(j,k)} n^{-1}[F_i^h F_r^t(\partial_k\{r_{jt}\}) + F_r^t F_i^a\{h_{ka}\}\{r_{jt}\} \\
 & + F_r^t F_a^h\{a_{ji}\}\{r_{kt}\} - n^{-1}F_r^t F_s^u \delta_i^h\{r_{kt}\}\{s_{ju}\}], \\
 = & {}^1R_{kji}^h - \zeta_{(j,k)} n^{-1}[F_i^h F_r^t(\partial_k {}^1\Gamma_{jt}^r) + F_r^t F_i^a {}^1\Gamma_{ka}^h {}^1\Gamma_{jt}^r \\
 & + F_r^t F_a^h {}^1\Gamma_{ji}^a {}^1\Gamma_{kt}^r - n^{-1}F_r^t F_s^u \delta_i^h {}^1\Gamma_{kt}^r {}^1\Gamma_{ju}^s],
 \end{aligned}$$

$$\begin{aligned}
 (4.7) \quad {}^1R_{ji} = & K_{ji} - n^{-1}[F_i^h F_r^t(\partial_h\{r_{jt}\} - \partial_j\{r_{ht}\}) + F_r^t F_i^a(\{h_{ha}\}\{r_{jt}\} \\
 & - \{h_{ja}\}\{r_{ht}\}) + F_r^t F_a^h(\{a_{ji}\}\{r_{ht}\} - \{a_{hi}\}\{r_{jt}\}) \\
 & - n^{-1}F_r^t F_s^u(\{r_{jt}\}\{s_{ju}\} - \{r_{jt}\}\{s_{iu}\})]
 \end{aligned}$$

and

$$\begin{aligned}
 (4.8) \quad {}^1R = & K - n^{-1}[F^{hj}(F_r^t(\partial_h\{r_{jt}\} - \partial_j\{r_{ht}\}) + F_r^t F_i^a(\{h_{ha}\}\{r_{jt}\} \\
 & - \{h_{ja}\}\{r_{ht}\}) + F_r^t F_a^h(\{a_{ji}\}\{r_{ht}\} - \{a_{hi}\}\{r_{jt}\}) \\
 & - n^{-1}F_r^t F_s^u(\{r_{jt}\}\{s_{ju}\} - \{r_{jt}\}\{s_{iu}\})]
 \end{aligned}$$

From equation (4.7), we get by virtue of equations (1.3) and (1.4)

$$(4.9) \quad {}^1H_{jm} = H_{jm} + n^{-1}[F_r^t(\partial_m\{r_{jt}\} - \partial_j\{r_{mt}\} + \{h_{hm}\}\{r_{jt}\} \\ - \{h_{jm}\}\{r_{ht}\}) + F_r^t F_a^h F_m^i[\{a_{ji}\}\{r_{ht}\} \\ - n^{-1}(\{a_{hi}\}\{r_{jt}\} - \{a_{hi}\}\{r_{jt}\})]$$

Hence:

Theorem 4.4. *The tensors ${}^1H_{jm}$ and H_{jm} in a Kahler manifold are related by equation (4.9).*

5. Connection Parameter ${}^2\Gamma_{ji}^h$

Definition 5.1. A transformation which transforms Christoffel symbols to a connection ${}^2\Gamma_{ji}^h$ given by equation (5.1)

$$(5.1) \quad {}^2\Gamma_{ji}^h = \{h_{ji}\} + F_j^h q_i - F_{ji} q_h,$$

shall be called $*F$ -transformation.

If ${}^2\nabla_m$ represents covariant derivative based on connection ${}^2\Gamma_{ji}^h$, we can easily establish for a tensor T_j^h

$$(5.2) \quad {}^2\nabla_m T_{bc}^a = \nabla_m T_{bc}^a + T_{bc}^r(F_m^a q_r - F_{mr} q^a) - T_{rc}^a(F_m^r q_b \\ - F_{mb} q^r) - T_{br}^a(F_m^r q_c - F_{mc} q^r)$$

which leads to

$$(5.3a) \quad {}^2\nabla_m g_{ji} = 0,$$

and

$$(5.3b) \quad {}^2\nabla_m F_i^h = \nabla_m F_i^h + F_i^r(F_m^h q_r - F_{mr} q^h) - F_r^h(F_m^r q_i - F_{mi} q^r)$$

Hence:

Theorem 5.1. *From equations (5.3a), we can observe that ${}^2\nabla_m$ based on the coefficient of connection ${}^2\Gamma_{ji}^m$ is a metric connection as well as it satisfies equation (5.3b).*

Let ${}^2R_{kji}^h$ be the curvature tensor based on ${}^2\Gamma_{ji}^h$, then we can obtain

$$(5.4) \quad {}^2R_{kji}^h = K_{kji}^h + \zeta_{(j,k)}[F_j^h(\nabla_k q_i + F_{ki} q_a q^a) + q_i(\nabla_k F_j^h - F_k^a F_j^h q_a) \\ - q^h(\nabla_k F_{ji} + F_{ka} F_j^a q_i - F_{ka} F_{ji} q^a) - F_{ji} \nabla_k q^h],$$

which leads to

$$(5.5) \quad {}^2R_{ji} = K_{ji} + [\nabla_h(F_j^h q_i) + g_{ji} q^a q_a + q_i q_j - \nabla_h(q^h F_{ji}) \\ + \nabla_j(q^h F_{hi}) + 2q^h q_i F_{hj} + q^h q^a (F_{ha} F_{ji} - F_{ja} F_{hi})]$$

and

$$(5.6) \quad {}^2R = K + 2\nabla_h(q^j F_j^h) + (n + 2)q^2$$

Hence:

Theorem 5.2. *The curvature tensor ${}^2R_{kji}^h$, Ricci tensor ${}^2R_{ji}$ and curvature scalar ${}^2R$ based on coefficient of connection ${}^2\Gamma_{ji}^h$ are respectively given by equations (5.4), (5.5) and (5.6).*

From equation (5.4) for vector q_i to be a gradient vector, we can obtain

$$(5.7) \quad {}^2R_{kji}^h = K_{kji}^h + \zeta_{(j,k)}[F_j^h F_{ki} q_a q^a + q_i(\nabla_k F_j^h - F_k^a F_j^h q_a) \\ - q^h(\nabla_k F_{ji} + F_{ka} F_j^a q_i - F_{ka} F_{ji} q^a)],$$

which gives

$$(5.8) \quad {}^2R_{ji} = K_{ji} + q_a q^a g_{ji} + q_i q_j + q_i \nabla_h F_j^h + q^h (\nabla_j F_{hi} - \nabla_h F_{ji}) \\ - q^a q^h F_{ja} F_{hi})$$

and

$$(5.9) \quad {}^2R = K + 2q^j (\nabla_h F_j^h) + n q^2$$

Hence:

Theorem 5.3. *If p_i is a gradient vector, curvature tensor ${}^2R_{kji}^h$, Ricci tensor ${}^2R_{ji}$ and curvature scalar ${}^2R$ are expressed by equations (5.7), (5.8) and (5.9).*

From equation (5.8) we can easily obtain for ${}^2R_{ji} = {}^2R_{ji}$

$$(5.10) \quad q_i \nabla_h F_j^h - q^j \nabla_h F_j^h + q^h (\nabla_j F_{hi} - \nabla_i F_{hj} - 2\nabla_h F_{ji}) = 0$$

Hence:

Theorem 5.4. *If q_i is a gradient vector, Ricci tensor ${}^2R_{ji}$, will be symmetric in i and j if and only if equation (5.10) is satisfied.*

In case of a Kahler space equations (5.4), (5.5) and (5.6) can be expressed as

$$(5.11a) \quad {}^2R_{kji}^h = K_{kji}^h + \zeta_{(j,k)}[\nabla_k q_i + F_{ki} q_a q^a] + q_i q_a F_j^a F_k^h \\ - q^h F_{ka} (F_j^a q_i - F_{ji} q^a) - F_{ji} \nabla_k q^h,$$

which leads to

$$(5.11b) \quad {}^2R_{ji} = K_{ji} + F_j^h (\nabla_h q_i) + F_{hi} (\nabla_j q^h) - F_{ji} (\nabla_h q^h) + q_i q_j \\ + q^2 g_{ji} - F_{ja} F_{hi} q^a q^h$$

and

$$(5.11c) \quad {}^2R = K + (n+1)q^2 + F_h^j (\nabla_j q^h - F_{ja} q^a q^h) + F^{ha} (\nabla_h q_a)$$

Hence:

Theorem 5.5.: *In case of a Kahler space, ${}^2R_{kji}^h$, ${}^2R_{ji}$ and ${}^2R$ are expressed by the equations (5.11a), (5.11b) and (5.11c).*

Multiplying equation (5.1) by F_t^j and contracting the resulting equation for h and t , we get on simplification

$$(5.12) \quad q_i = (n-1)^{-1} F_h^j (\{h_{ji}\} - {}^2\Gamma_{ji}^h).$$

Substituting value of q_i in equation (5.1), we get on simplification a connection, which we shall denote by ${}^{*2}\Gamma_{ji}^h$ and which is defined by equation (5.13) as follows:

$$(5.13) \quad {}^{*2}\Gamma_{ji}^h = {}^2\Gamma_{ji}^h + (n-1)^{-1} F_m^t {}^2\Gamma_{tr}^m [\delta_i^r F_j^h - F_{ji} g^{rh}] \\ = \{h_{ji}\} + (n-1)^{-1} F_m^t \{m_{tr}\} [\delta_i^r F_j^h - F_{ji} g^{rh}].$$

Hence:

Theorem 5.6.: *If $*F$ -transformation of a Christoffel symbol is given by equation (5.1), the connection ${}^{*2}\Gamma_{ji}^h$ defined by equation (5.13) is an invariant connection.*

Let ${}^{*2}R_{kji}^h$, be the curvature tensor based on connection ${}^{*2}\Gamma_{ji}^h$, then we can obtain

$$(5.14) \quad {}^{*2}R_{kji}^h = K_{kji}^h + (n-1)^{-1} \zeta_{(j,k)} [(\partial_k F_m^t) F_j^h \{m_{ti}\} + (\partial_k \{m_{ti}\}) F_m^t F_j^h \\ + (\partial_k F_j^h) F_m^t \{m_{ti}\} - (\partial_k \{m_{tr}\}) F_{ji} g^{rh} F_m^t \\ - (\partial_k F_{ji}) \{m_{tr}\} g^{rh} F_m^t - (\partial_k g^{rh}) \{m_{tr}\} F_{ji} F_m^t \\ - (\partial_k F_m^t) \{m_{tr}\} g^{rh} F_{ji} + \{a_{ji}\} \{m_{tr}\} F_m^t (\delta_a^r F_k^h - F_{ka} g^{rh}) \\ + \{h_{ka}\} \{n_{us}\} F_n^u (\delta_i^s F_j^a - F_{ji} g^{sa}) \\ + (n-1)^{-1} [F_m^t F_n^u \{m_{tr}\} \{n_{us}\} (\delta_a^r F_k^h - F_{ka} g^{rh}) \\ \cdot (\delta_i^s F_j^a - F_{ji} g^{sa})]]$$

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