

CW-Translations of Homogeneous Finsler Spaces with (α, β) -metrics

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Abstract: In the present paper, we consider homogeneous Finsler spaces with (α, β) -metrics. Taking a vector field X on these spaces, we find the necessary condition for X to be the Killing vector field and extend the concept to Riemannian metric α . Next, we find the sufficient conditions for X to be a Killing vector field of constant length on the Finsler spaces with square metric and Randers change of square metrics. Further by dropping the restriction to the Killing vector fields, we study CW-translations of left-invariant Finsler metrics on compact Lie groups.

Keywords: Homogeneous Finsler space; square metric; Randers change of square metric; Killing vector field; CW-translation.

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1. Introduction

A Clifford-Wolf translation of a locally compact connected metric space (X, d) is an isometry ρ of (X, d) onto itself such that the function $d(x, \rho(x))$ is constant on X . Geometrically, a Clifford-Wolf translation is a metric space isometry such that every point in X moves by the same amount. Finding every Clifford-Wolf translation of an explicit Riemannian manifold is a significant issue. It was established by J. A. Wolf that the Riemannian isometry is a Clifford-Wolf translation if and only if it is bounded¹. Also, any Clifford-Wolf translation of a Riemannian manifold (M, g) must be trivial if the curvature is strictly negative. R. Azencott and E. Wilson², E. Heintze³ and many others have worked on homogeneous Riemannian manifold with non-positive curvature. S. Deng generalized J. A. Wolf's results to general Finsler spaces⁴.

The purpose of this article is to study the Clifford-Wolf translations of Finsler spaces with (α, β) -metrics. Let (M, F) be a Finsler space, where F is

positively homogeneous of degree one. CW-translations and constant length Killing vector fields are closely related. For a compact manifold, a Killing vector field of constant length generates CW-translation. In the homogeneous Finsler spaces with square metric and Randers change of square metric, we give some explicit sufficient and necessary conditions for an isometry to be a Clifford-Wolf translation.

The arrangement of the paper is as the following: In Section 2, we discuss some basic definitions and results on Finsler space, Clifford-Wolf translations, homogeneous Finsler space to be used in subsequent sections. In Section 3, we study the Killing vector fields on the Finsler spaces with square metric and Randers change of square metric and the isometries of flow generated by these vector fields on (M, F) . Further, in Section 4, by taking a Killing vector field X of constant length on Finsler spaces with square metric and Randers change of square metric and obtain a characterization of Clifford-Wolf. In Section 4, we apply the results to homogeneous Randers spaces and obtain some explicit description of Clifford-Wolf translations. Then we deal with left invariant square metric and Randers change of square metric on a compact simple Lie group.

2. Preliminaries

In order to investigate the aforementioned spaces, we must first address certain fundamental concepts and findings. We cite⁵⁻⁸ for notations and other information.

Definition 2.1⁸. *An integral curve is a differentiable curve $\gamma: I \rightarrow M$ on smooth manifold M , where I be any open interval containing 0, is such that for a vector field X on M the curve γ satisfies*

$$(2.1) \quad \gamma'(t) = X_{\gamma(t)} \quad \forall t \in I,$$

In other words, the tangent vector to at each point is equal to the value of X at that point. If $0 \in I$, the point $p = \gamma(0)$ is called the starting point of γ .

Assume that X is a vector field on the smooth manifold M , for each $p \in M$, X has a unique integral curve $\varphi^{(p)}: R \rightarrow M$ starting at p such that $(\varphi^{(p)})'(t) = X(\varphi^{(p)}(t)) \quad \forall t \in R$, and $\varphi^{(p)}(0) = p$. For each $t \in R$, define $\varphi_t: M \rightarrow M$ as $\varphi_t(p) = \varphi^{(p)}(t)$, i.e., $t \rightarrow \varphi_t(p)$ is an integral curve starting at p . The continuous R -action for the additive group R is $\varphi: R \times M \rightarrow M$ on M satisfying

$$\varphi(t, \varphi(s, p)) = \varphi(t + s, p), \quad \varphi(0, p) = p$$

is the global flow of X , $\varphi(t, p) = \varphi_t(p)$.

Definition 2.2. If g is invariant under the flow of a vector field X for a Riemannian space (M, g) , then the vector field X is said to be Killing vector field.

In other words, X is considered to be Killing if and only if $L_X g = 0$, L_X is the Lie derivative with respect to X .

For a Riemannian connection ∇ on M , the condition $(L_X g)(Y, Z) = 0$ is equivalent to $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for $Y, Z \in \chi(M)$.

Definition 2.3. A Killing vector field X on a Riemannian space (M, g) has constant length if every integral curve of X is a geodesic on (M, g) .

Definition 2.4. The injectivity radius of M at x is defined as $i_x = \inf_y i_y$, $y \in S_x M$ (indicatrix at x), and the injectivity radius i_M of M is defined by $i_M = \min\{i_x : x \in M\}$.

Definition 2.5. A Finsler metric on a manifold M is a function $F: TM \rightarrow R^+$, which is smooth on the slit tangent bundle $TM \setminus \{0\}$. In any local coordinates (x^i, y^j) for TM , F satisfies the following conditions:

- (1) $F(x, y) > 0$ for any $y \neq 0$.
- (2) $F(x, \lambda y) = \lambda F(x, y)$ for any $y \in T_x M$ and $\lambda > 0$.
- (3) The Hessian matrix defined by $g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}$ is positive definite.

The idea of (α, β) -metrics was first developed by M. Matsumoto⁹ in 1972. An (α, β) -metric on a connected smooth manifold M is a Finsler metric F constructed from a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and a one-form $\beta = b_i(x)y^i$ on M .

In fact, (α, β) -metrics are the generalizations of Randers metric. A simple non-Riemannian metric is the Randers metric given by $F = \alpha + \beta$ with $\|\beta\|_\alpha < 1$. Aside from Randers metrics, other interesting non-Riemannian metrics include Matsumoto, infinite series, square metrics, etc. Matsumoto metric was introduced by M. Matsumoto¹⁰.

Shen's lemma⁶ provides necessary and sufficient conditions for a function of α and β to be a Finsler metric.

Lemma 2.1. Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where ϕ is a smooth function on an open interval $(-b_0, b_0)$, α is a Riemannian metric and β is a 1-form with $\|\beta\|_\alpha < b_0$. Then F is a Finsler metric if and only if the following conditions are satisfied:

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad \forall |s| \leq b < b_0.$$

Definition 2.5. An isometry φ of the Finsler space (M, F) is a diffeomorphism on M satisfying:

$$(2.2) \quad F(x, y) = F(\varphi(x), d\varphi_x(y)) \quad \forall x \in M \text{ and } y \in T_x M.$$

Definition 2.6. Let (M, F) be a Finsler space and d the distance function on M which is not symmetric in general as F is not absolute homogeneous in general. An isometry ρ on M such that $d(x, \rho(x))$ is constant for all $x \in M$ is called Clifford-Wolf translation on M .

Definition 2.7. If there exists a CW-translation such that $\rho(a) = b$ for any $a, b \in M$ in a Finsler space (M, F) , then the space is said to be Clifford-Wolf homogenous.

Berestovskii and Nikonorov investigated the relationship between CW-translations and Killing vector fields with constant lengths in the Riemannian case. The crucial finding is that Killing vector fields of constant length have flow curves that are geodesics, allowing them to produce local one-parameter groups of CW translations when the parameter variable is close to 0. This observation holds true in the Finslerian case as well, however, Killing vector fields with constant lengths can only produce CW translations with positive small parameter variables when F is not reversible. In fact, we have the following interrelation theorems¹¹.

Theorem 2.1. Let (M, F) be a complete Finsler manifold with a positive injective radius. If X is a Killing vector field of constant length, then for all sufficiently small $t > 0$, the flow φ_t generated by X is a CW-translation.

Theorem 2.2. For a compact Finsler manifold (M, F) , there exists a $\delta > 0$, such that any CW-translation ρ with $d(x, \rho(x)) < \delta$ is generated by a Killing vector field of constant length.

Definition 2.8¹². A Finsler manifold (M, F) is called CW-homogeneous if, for any two points $x_1, x_2 \in M$, there is a CW-translation ρ such that $\rho(x_1) = x_2$. It is called restrictively CW-homogeneous if, for any point $x \in M$, there is a neighborhood V of x , such that, for any two points $x_1, x_2 \in V$, there is a CW-translation ρ mapping x_1 to x_2 .

Definition 2.9. Let G be a smooth manifold with an abstract group structure. G is said to be a Lie group, if the maps $i: G \rightarrow G$ and $\mu: G \times G \rightarrow G$ defined as $i(g) = g^{-1}$, and $\mu(g, h) = gh$ respectively, are smooth.

Let M be a smooth manifold and G , a Lie group. Then a smooth map $f: G \times M \rightarrow M$ such that $f(g_2, f(g_1, x)) = f(g_2 g_1, x)$, $\forall g_1, g_2 \in G$, and $x \in M$ is called a smooth action of G on M .

Definition 2.10. Let (M, F) be a connected Finsler space and $G = I(M, F)$ the group of isometries of (M, F) . If G acts transitively on M , then (M, F) is said to be a homogeneous Finsler space.

Theorem 2.3⁷. The group of isometries $G = I(M, F)$ of a Finsler space (M, F) is a Lie transformation group of M . For $a \in M$ and $I_a(M, F)$ be the isotropy subgroup of $I(M, F)$ at a , then $I_a(M, F)$ is compact.

Let (M, F) be a homogeneous Finsler space with group of isometries $G = I(M, F)$. For $a \in M$, let $H = I_a(M, F)$ be a closed isotropy subgroup of G which is compact and also H is a Lie group itself. M can be written as the quotient space G/H .

Definition 2.11¹⁸. Let G and H be the Lie groups, $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively. Then the direct sum decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, such that $\text{Ad}(h)(\mathfrak{m}) \subset \mathfrak{m} \forall h \in H$, where $\mathfrak{m}$ is a subspace of $\mathfrak{g}$, is called a reductive decomposition of $\mathfrak{g}$, and if such decomposition exists, then $(G/H, F)$ is called reductive homogeneous space.

As a result, each homogeneous Finsler space may be expressed as a connected Lie group's coset space with an invariant Finsler metric. Here, the Finsler metric F is a G invariant Finsler metric on M .

We may identify the tangent space $T_H(G/H)$ of the reductive homogeneous manifold G/H at the origin $eH = H$ with $\mathfrak{m}$ through the map

$$Y \rightarrow \frac{d}{dt} \exp(tY) H|_{t=0}, \quad Y \in \mathfrak{m},$$

since M is identified with G/H and Lie algebra of any Lie group G is viewed as $T_e G$.

3. Killing vector fields of constant length

It is proved that¹³ the group of isometries of a Finsler space (M, F) is a Lie transformation group on M . Its Lie algebra is the space of Killing vector fields. Equivalently, a smooth vector field on (M, F) is a Killing vector field if and only if the flow φ_t generated by X are isometries of (M, F) .

Theorem 3.1 A vector field X on a Finsler space (M, F) with the square metric $F = \frac{(\alpha + \beta)^2}{\alpha}$ is a Killing vector field for F , then $L_X \beta = 0$. Further, X is a Killing vector field for α if and only if it is a Killing vector field for the special metric $\bar{F} = \frac{\beta^2}{\alpha}$.

Proof. Let the vector field X be a Killing vector field for (M, F) , then its flow φ_t fixes F , i.e.

$$(3.1) \quad \phi_t^* F = F$$

Using Eqn. (3.1) for $F = \frac{(\alpha+\beta)^2}{\alpha} = \alpha + 2\beta + \frac{\beta^2}{\alpha}$,

$$(3.2) \quad \phi_t^* \alpha + 2\phi_t^* \beta + \phi_t^* \frac{\beta^2}{\alpha} = \alpha + 2\beta + \frac{\beta^2}{\alpha}, \quad \forall t$$

Replacing y by $-y$ in Eqn. (3.2),

$$(3.3) \quad \phi_t^* \alpha - 2\phi_t^* \beta + \phi_t^* \frac{\beta^2}{\alpha} = \alpha - 2\beta + \frac{\beta^2}{\alpha}, \quad \forall t$$

Subtracting Eqn. (3.3) from (3.2), we have

$$(3.4) \quad \phi_t^* \beta = \beta$$

implies

$$(3.5) \quad L_X \beta = 0.$$

To prove the further case, let us suppose X be a Killing vector field for α , i.e., $\phi_t^* \alpha = \alpha$, then from Eqn. (3.1) we have $\phi_t^* \frac{\beta^2}{\alpha} = \frac{\beta^2}{\alpha}$. Thus X is a Killing vector field for the special metric $\bar{F} = \frac{\beta^2}{\alpha}$.

For the converse part, assume that X is a Killing vector field of $\bar{F} = \frac{\beta^2}{\alpha}$. Then from the Eqns. (3.2) and (3.4), we have $\phi_t^* \alpha = \alpha$. Hence the proof. ■

Now, for the Randers change of square metric we have the following theorem:

Theorem 3.2 *A vector field X on a Finsler space (M, F) with Randers change of square metric $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$ be a Killing vector field for F , then $L_X \beta = 0$. Further, X is a Killing vector field for α if and only if it is a Killing vector field for the special metric $\bar{F} = \frac{\beta^2}{\alpha}$.*

Proof. The metric F can be written as $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta = \alpha + 3\beta + \frac{\beta^2}{\alpha}$. Following the same steps of theorem 3.1, we can prove the theorem.

4. Clifford-Wolf translations on Homogeneous Finsler spaces

Assume that (M, F) is a homogeneous Finsler space and its full isometry group $G = I(M, F)$ acts transitively on M . $\mathfrak{g}$ denotes the Lie algebra of G . The homogeneous space M can be identified with G/H , where H is a compact isotropy subgroup of G at a point x of M . The Finsler metric F is determined by the data at the tangent space $T_x \cong \mathfrak{g}/\mathfrak{h} = \mathfrak{m}$, i.e., α is determined by an

inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$, and the dual V of β is a vector of $\mathfrak{m}$, both invariant under the Ad -action of H .

If M is compact, then G must also be compact. With respect to the Killing form of $\mathfrak{g}$, the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ has an orthogonal decomposition. For any $X \in \mathfrak{g}$, its decomposition will be denoted $X = X_{\mathfrak{h}} + X_{\mathfrak{m}}$.

From Section 3, it is clear that, the problem of finding Clifford-Wolf translations can be reduced to the problem of finding Killing fields of constant length. When considered as an element of $\mathfrak{g}$, the differential of the length function $F(X)$ of a Killing vector field X can be calculated. The vector of X at each point can be pulled back to the chosen point, by some $g \in G$, and can be identified with $(\text{Ad}_g X)_{\mathfrak{m}}$ in $\mathfrak{m}$. The following theorems deduce the condition for a vector field to be of constant length:

Theorem 4.1 *Let X be a Killing vector field on a connected homogeneous Finsler space $(G/H, F)$, $F = \frac{(\alpha + \beta)^2}{\alpha}$ with $G = I(M, F)$. Then X is of constant length if it satisfies*

$$(4.1) \quad \frac{\langle [Y, \text{Ad}_g X]_{\mathfrak{m}}, (\text{Ad}_g X)_{\mathfrak{m}} \rangle}{\langle (\text{Ad}_g X)_{\mathfrak{m}}, (\text{Ad}_g X)_{\mathfrak{m}} \rangle^{\frac{1}{2}}} + 2 \langle [Y, \text{Ad}_g X]_{\mathfrak{m}}, V \rangle \left(1 + \frac{\langle (\text{Ad}_g X)_{\mathfrak{m}}, V \rangle}{\langle (\text{Ad}_g X)_{\mathfrak{m}}, (\text{Ad}_g X)_{\mathfrak{m}} \rangle^{\frac{1}{2}}} \right) - \frac{\langle [Y, \text{Ad}_g X]_{\mathfrak{m}}, (\text{Ad}_g X)_{\mathfrak{m}} \rangle \langle (\text{Ad}_g X)_{\mathfrak{m}}, V \rangle^2}{\langle (\text{Ad}_g X)_{\mathfrak{m}}, (\text{Ad}_g X)_{\mathfrak{m}} \rangle^{\frac{3}{2}}} = 0, \quad \forall Y \in \mathfrak{g}, \text{ and } g \in G.$$

Proof. The square metric F can be written as, $F = \frac{(\alpha + \beta)^2}{\alpha} = \alpha + 2\beta + \frac{\beta^2}{\alpha}$

$$(4.2) \quad F = \langle (\text{Ad}_g X)_{\mathfrak{m}}, (\text{Ad}_g X)_{\mathfrak{m}} \rangle^{\frac{1}{2}} + 2 \langle (\text{Ad}_g X)_{\mathfrak{m}}, V \rangle + \frac{\langle (\text{Ad}_g X)_{\mathfrak{m}}, V \rangle^2}{\langle (\text{Ad}_g X)_{\mathfrak{m}}, (\text{Ad}_g X)_{\mathfrak{m}} \rangle^{\frac{1}{2}}}$$

for a family $g_t = \exp(tY) \cdot g$, with $Y \in \mathfrak{g}$, the derivative of Eqn. (4.2) for t at 0 is

$$(4.3) \quad \frac{d}{dt} F|_{t=0} = \frac{\langle [Y, \text{Ad}_g X]_{\mathfrak{m}}, (\text{Ad}_g X)_{\mathfrak{m}} \rangle}{\langle (\text{Ad}_g X)_{\mathfrak{m}}, (\text{Ad}_g X)_{\mathfrak{m}} \rangle^{\frac{1}{2}}} + 2 \langle [Y, \text{Ad}_g X]_{\mathfrak{m}}, V \rangle \left(1 + \frac{\langle (\text{Ad}_g X)_{\mathfrak{m}}, V \rangle}{\langle (\text{Ad}_g X)_{\mathfrak{m}}, (\text{Ad}_g X)_{\mathfrak{m}} \rangle^{\frac{1}{2}}} \right) - \frac{\langle [Y, \text{Ad}_g X]_{\mathfrak{m}}, (\text{Ad}_g X)_{\mathfrak{m}} \rangle \langle (\text{Ad}_g X)_{\mathfrak{m}}, V \rangle^2}{\langle (\text{Ad}_g X)_{\mathfrak{m}}, (\text{Ad}_g X)_{\mathfrak{m}} \rangle^{\frac{3}{2}}}$$

Thus X is of constant length if it satisfies Eqn. (4.1). ■

For a family $g_t = g \cdot \exp(tY)$, with $Y \in \mathfrak{g}$, the condition (4.3) becomes

$$(4.4) \quad \frac{\langle Ad_g[Y, X]_m, (Ad_g X)_m \rangle}{\langle (Ad_g X)_m, (Ad_g X)_m \rangle^{\frac{1}{2}}} + 2\langle Ad_g[Y, X]_m, V \rangle \left(1 + \frac{\langle (Ad_g X)_m, V \rangle}{\langle (Ad_g X)_m, (Ad_g X)_m \rangle^{\frac{1}{2}}} \right) - \frac{\langle Ad_g[Y, X]_m, (Ad_g X)_m \rangle \langle (Ad_g X)_m, V \rangle^2}{\langle (Ad_g X)_m, (Ad_g X)_m \rangle^{\frac{3}{2}}} = 0,$$

Corollary 4.1 Let X be a Killing vector field of constant length on a connected homogeneous Finsler space $(G/H, F)$, $F = \frac{(\alpha+\beta)^2}{\alpha}$ with $G = I(M, F)$. If X has constant length with respect to α , then

$$(4.5) \quad \langle [Y, Ad_g X]_m, V \rangle = 0$$

Proof. If X has constant length with respect to α , i.e., $\langle (Ad_g X)_m, (Ad_g X)_m \rangle^{\frac{1}{2}} = \text{const}$, then from Eqn. (4.1), we have

$$\langle [Y, Ad_g X]_m, V \rangle \left(\langle (Ad_g X)_m, V \rangle + \langle (Ad_g X)_m, (Ad_g X)_m \rangle^{\frac{1}{2}} \right) = 0$$

Thus $\langle [Y, Ad_g X]_m, V \rangle = 0$. ■

Similarly, for the Randers change of square metric:

Theorem 4.2 Let X be a Killing vector field on a connected homogeneous Finsler space $(G/H, F)$, $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$ with $G = I(M, F)$. Then X is of constant length if it satisfies

$$(4.6) \quad 3\langle [Y, Ad_g X]_m, V \rangle + \frac{\langle [Y, Ad_g X]_m, (Ad_g X)_m \rangle + 2\langle [Y, Ad_g X]_m, V \rangle \langle (Ad_g X)_m, V \rangle}{\langle (Ad_g X)_m, (Ad_g X)_m \rangle^{\frac{1}{2}}} - \frac{\langle [Y, Ad_g X]_m, (Ad_g X)_m \rangle \langle (Ad_g X)_m, V \rangle^2}{\langle (Ad_g X)_m, (Ad_g X)_m \rangle^{\frac{3}{2}}} = 0, \quad \forall Y \in \mathfrak{g}, \text{ and } g \in G.$$

Corollary 4.2 Let X be a Killing vector field of constant length on a connected homogeneous Finsler space $(G/H, F)$, $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$ with $G = I(M, F)$. If X has constant length with respect to α , then

$$(4.7) \quad \langle [Y, Ad_g X]_m, V \rangle = 0$$

For a connected, compact Lie group G with Lie algebra $\mathfrak{g}$, a Finsler metric F on G is called left invariant if the left translation group $L(G)$ is contained

in the isometry group of F . The Lie algebra $\mathfrak{g}$ is naturally contained in the Lie algebra of the isometry group $I(G, F)$, i.e., the space of all Killing vector fields for (G, F) . A left invariant Finsler metric is completely determined by its restriction to $T_e G \cong \mathfrak{g}$ with α determined by an inner product on $\mathfrak{g}$ and β by an element of $\mathfrak{g}^*$. Let $\langle \cdot, \cdot \rangle$ be the inner product induced by α and, $\langle \cdot, \cdot \rangle_{bi}$ be a fixed $Ad(G)$ -invariant inner product on $\mathfrak{g}$ induced by a bi-invariant metric on G ; then the restriction of F to $T_e G$ can be written as $F(y) = \frac{(\sqrt{\langle y, y \rangle} + \langle y, V \rangle_{bi})^2}{\sqrt{\langle y, y \rangle}}$ for some fixed $V \in \mathfrak{g}$.

The main focus of our considerations is the study of Killing vector fields of constant length of a left invariant square metric. In general, it would be very difficult to classify all possible $I_0(G, F)$ s or equivalently their Lie algebras if the restriction $X \in \mathfrak{g}$ is not considered. The following theorem, however, shows that when G is simple and α is left invariant, $L(G)R(G)$ contains the identity component $I_0(G, F)$ of the entire group $I(G, F)$ of isometries for (G, F) .

Theorem 4.3 *For a compact, connected, simple Lie group G with F , a left invariant square metric, the 1-form β is also left invariant. Further if α is left invariant, then the identity component $I_0(G, F) \subset L(G)R(G)$.*

Proof. Let $F = \frac{(\alpha + \beta)^2}{\alpha} = \alpha + 2\beta + \frac{\beta^2}{\alpha}$ be a left invariant square metric. For an isometry ϕ of (G, F) , then, we have

$$(4.8) \quad F(d\phi(y)) = F(y), \quad \forall y \in G \text{ and } \forall y \in T_g(G),$$

$$\alpha(d\phi(y)) + 2\beta(d\phi(y)) + \frac{\beta^2(d\phi(y))}{\alpha(d\phi(y))} = \alpha(y) + 2\beta(y) + \frac{\beta^2(y)}{\alpha(y)}$$

Replacing y by $-y$ in Eqn. (4.8),

$$(4.9) \quad \alpha(d\phi(y)) - 2\beta(d\phi(y)) + \frac{\beta^2(d\phi(y))}{\alpha(d\phi(y))} = \alpha(y) - 2\beta(y) + \frac{\beta^2(y)}{\alpha(y)}$$

From Eqns. (4.8) and (4.9), we have $\beta(d\phi(y)) = \beta(y)$.

Further, if $\alpha(d\phi(y)) = \alpha(y)$, then we have $L(G) \subset I(G, F) \subset I(G, \alpha)$. Thus from¹³, as G is compact connected and simple, $I_0(G, \alpha) \subset L(G)R(G)$. Since $I_0(G, F) \subset I_0(G, \alpha)$, we have $I_0(G, F) \subset L(G)R(G)$. ■

Theorem 4.4 *For a compact, connected, simple Lie group G with F , a left invariant Randers change of square metric, the 1-form β is also left invariant. Further if α is left invariant, then the identity component $I_0(G, F) \subset L(G)R(G)$.*

Proof. Let $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta = \alpha + 3\beta + \frac{\beta^2}{\alpha}$ be a left invariant square metric.

For an isometry ϕ of (G, F) , then, we have

$$(4.10) \quad F(d\phi(y)) = F(y), \quad \forall g \in G \text{ and } \forall y \in T_g(G),$$

$$\alpha(d\phi(y)) + 3\beta(d\phi(y)) + \frac{\beta^2(d\phi(y))}{\alpha(d\phi(y))} = \alpha(y) + 3\beta(y) + \frac{\beta^2(y)}{\alpha(y)}$$

Replacing y by $-y$ in Eqn. (4.10),

$$(4.11) \quad \alpha(d\phi(y)) - 3\beta(d\phi(y)) + \frac{\beta^2(d\phi(y))}{\alpha(d\phi(y))} = \alpha(y) - 3\beta(y) + \frac{\beta^2(y)}{\alpha(y)}$$

From Eqns. (4.10) and (4.11), we have $\beta(d\phi(y)) = \beta(y)$.

Further, if $\alpha(d\phi(y)) = \alpha(y)$, then we have $L(G) \subset I(G, F) \subset I(G, \alpha)$.

Thus from¹³, as G is compact connected and simple, $I_0(G, \alpha) \subset L(G)R(G)$.

Since $I_0(G, F) \subset I_0(G, \alpha)$, we have $I_0(G, F) \subset L(G)R(G)$.

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