

Ricci Curvature Formula for a Homogeneous Finsler Space with (α, β) -metric

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Abstract: Curvature properties of homogeneous Finsler spaces with (α, β) -metrics are among the most significant topics in Finsler geometry. In this article, first we will discuss Ricci curvatures in Finsler geometry. We have obtained the formula for the Ricci curvature of homogeneous Finsler space with special metric, $F = \alpha + \sqrt{\alpha^2 + \beta^2}$. We have discussed the conditions to have vanishing S -curvature for the space $(G / H, F)$.

Keywords: Finsler space, Ricci curvature, special (α, β) -metric, vanishing S -curvature, Homogeneous Finsler space.

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1. Introduction

M. Matsumoto¹ introduced the concept of (α, β) -metric in Finsler geometry in 1972. This is the generalization of Randers metric, introduced by G. Randers². So many authors have worked on this concept³⁻⁵. In physics and biology, (α, β) -metrics have several applications. For (α, β) -metrics Riemannian and Ricci curvatures are given by Zhou⁶. A Finsler space (M, F) whose Ricci curvature is written as $Ric(x, y) = \lambda(x)F^2(x, y)$, where λ -smooth function on M , is called an Einstein metric⁷. In the paper, Cheng *et al.*⁸ have shown that the formulae given in⁶ are incorrect. Later, they have also given the corrected formulae for Ricci curvature and Riemannian curvature for (α, β) -metrics.

Aim of this article is to compute the explicit and applicable formula for the Ricci curvature of homogeneous Finsler space with of special (α, β) -metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$. This paper is described as in the following manner: In first part, we recall some basic definitions related to the Finsler space and homogeneous Finsler space. along with this we have discussed the condition

for positive definiteness of Finsler metric F . In part 2 and part 3, we talk over the definition of Ricci curvature, theorems and calculated some quantities related to the Ricci curvature formulae. At last, we established the explicit formula for Ricci curvature of homogeneous Finsler space with special metric, $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ as well as we have shown that $(G/H, F)$ having vanishing S -curvature.

Definition 1.1. Let M be a connected smooth manifold. If $\exists$ a continuous function $F: TM \rightarrow [0, \infty)$ such that F is smooth on tangent bundle $TM \setminus \{0\}$ and is restricted to the tangent space is a Minkowski norm. Then (M, F) is called a Finsler space, here in this case, F is called Finsler metric.

Definition 1.2. A connected Finsler space (M, F) is called homogeneous Finsler space if $I(M, F)$, the group of isometries of (M, F) , acts transitively on M .

Here is the Shen's lemma that gives necessary and sufficient conditions for an α and β function to be a Finsler metric:

Lemma 1.1.⁹ Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ is a Minkowski norm for any Riemannian metric α and 1-form β is bounded above with respect to α according to its length with $\|\beta\|_\alpha < b_0$ if and only if $\phi = \phi(s)$ satisfies the following conditions:

$$(1.1) \quad \phi(s) > 0, \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0,$$

Theorem 1.1.¹⁰ Let (M, F) be a Finsler space. Then $G = I(M, F)$, the group of isometries of M is a Lie transformation group of M . Let $a \in M$ and $I_a(M, F)$ be the isotropy subgroup of $I(M, F)$ at a . Then $I_a(M, F)$ is compact.

Let (M, F) be a homogeneous Finsler space, i.e., $G = I(M, F)$ acts transitively on M . For $a \in M$, let $H = I_a(M, F)$ be a closed isotropy subgroup of G which is compact. Then H is a Lie group itself being a closed subgroup of G . Here M can be written as the quotient space G/H .

Definition 1.3.¹¹ Let G be a Lie group and H , a Lie subgroup of G . Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively, $\mathfrak{h}$ is subalgebra of $\mathfrak{g}$. The direct sum decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{m}$ is a subspace of $\mathfrak{g}$ such that $Ad(h)(\mathfrak{m}) \subset \mathfrak{m}, \forall h \in H$, is called a reductive decomposition of $\mathfrak{g}$, and if such a decomposition exists, then $(G/H, F)$ is called reductive homogeneous space.

Definition 1.4.¹² The Ricci curvature of a Finsler metric F on a manifold is a scalar function $Ric: TM \rightarrow R$ with the homogeneity $Ric(\lambda y) = \lambda^2 Ric(y)$. A Finsler metric F on an n -dimensional manifold M is called an Einstein metric if there is a scalar function $\mu = \mu(x)$ on M such that $Ric = (n - 1)\mu F^2$.

2. Ricci curvature of homogeneous Finsler spaces

The Ricci curvature is an important geometrical entity. The notion of Riemannian curvature can be extended to Finsler metrics for Riemannian spaces. For a non-zero vector $z \in T_x M$, a linear map $R_z: T_x M \rightarrow T_x M$ is the Riemannian curvature⁸, which can be defined as follows:

$$R_z(u) = R_k^i(z)u^k \frac{\partial}{\partial x^i}, \quad u = u^i \frac{\partial}{\partial x^i},$$

where
$$R_k^i(z) = 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial z^k} z^j + 2G^j \frac{\partial^2 G^i}{\partial z^j \partial z^k} - \frac{\partial G^i}{\partial z^j} \frac{\partial G^j}{\partial z^k},$$

and G^i are geodesic coefficients given by

$$G^i = \frac{1}{4} g^{im} \{ (F^2)_{x^k z^m z^k} - (F^2)_{x^m} \}, \quad i = 1, 2, \dots, n.$$

Definition 2.5.⁸ Let (M, F) be a Finsler space. A scalar function $Ric: TM \rightarrow R$ such that $Ric(z) = tr(R_z)$ is called Ricci curvature of Finsler space (M, F) and $tr(R_z)$ is trace of its Riemannian curvature, where $z \in TM$.

Here are some notations that will be used in the next theorem:

$$\begin{aligned} r_{ij} &= \frac{b_{i;j} + b_{j;i}}{2}, & s_{ij} &= \frac{b_{i;j} - b_{j;i}}{2}, & r_j^i &= a^{iq} r_{qj}, \\ r_i &= b^q r_{qi} = b_j r_i^j, & s_i &= b^q s_{qi} = b_j s_i^j, & r &= r_{ij} b^i b^j = b^i r_i, \\ r_{00} &= r_{ij} z^i z^j, & r_{i0} &= r_{ij} z^j, & s_{i0} &= s_{ij} z^j, \\ r_0 &= r_i z^i, & s_0 &= s_i y^i, & s_j^i &= a^{iq} s_{qj}, \end{aligned}$$

where the contravariant derivative denotes as $'^j$, with respect to Levi-Civita connection on Riemannian metric α and $a^{ij} = (\alpha_{ij})^{-1}$, $b^i = a^{ij} b_j$.

Theorem 2.2.⁸ On Finsler space M , let F be an (α, β) -metric. Then

$$(2.1) \quad Ric(Z) = Ric^\alpha(Z) + RT_q^q$$

is its given Ricci curvature and Ric^α be Ricci curvature of α , where

$$\begin{aligned} RT_q^q &= \frac{1}{\alpha^2} \{ (n-1)K_1 + K_2 \} r_{00}^2 \\ &+ \frac{1}{\alpha^2} \left[((n-1)K_3 + K_4) r_{00} s_0 + ((n-1)K_5 + K_6) r_{00} r_0 \right. \\ &\quad \left. + ((n-1)K_7 + K_8) r_{00;0} \right] \\ &+ ((n-1)K_9 + K_{10}) s_0^2 + (r r_{00} - r_0^2) K_{11} + ((n-1)K_{12} + K_{13}) r_0 s_0 \\ &+ (r_{00} r_q^q - r_{0q} r_0^q b^q - r_{0q;0} b^q) K_{14} + ((n-1)K_{15} + K_{16}) r_{0q} s_0^q \\ &+ ((n-1)K_{17} + K_{18}) s_{0;0} + s_{0q} s_0^q K_{19} \end{aligned}$$

$$\begin{aligned}
& +\alpha[rs_0K_{20} + ((n-1)K_{21} + K_{22})s_qs_0^q] \\
& +\alpha\left((3s_qr_0^q - 2s_0r_q^q + 2r_qs_0^q - 2s_{0;q}b^q + s_{q;0}b^q)K_{23} + s_{0;q}^qK_{24}\right) \\
& +\alpha^2(s_qs^qK_{25} + s_q^is_i^qK_{26}),
\end{aligned}$$

here

$$\begin{aligned}
K_1 &= 2\Psi\Theta_s(B-s^2) - 2s\Psi\Theta + \Theta^2 - \Theta_s, \\
K_2 &= 2\Psi\Psi_{ss}(B-s^2)^2 - (6s\Psi\Psi_s + \Psi_{ss})(B-s^2) + 2s\Psi_s, \\
K_3 &= -4(2Q\Theta_s + Q_s\Theta)\Psi(B-s^2) + 4Q\Theta_s + 2Q_s\Theta + 4Q\Theta(s\Psi - \Theta) - 2\Theta_B, \\
K_4 &= -4\Psi(2Q\Psi_{ss} + Q_s\Psi_s + Q_{ss}\Psi_s^2)(B-s^2)^2 + (-4\Psi^2(Q-sQ_s) + 4Q_{ss}\Psi \\
& + 2Q_s\Psi_s + 4Q\Psi_{ss} - 2\Psi_{sB} + 20sQ\Psi\Psi_s)(B-s^2) + 2\Psi(Q-sQ_s) \\
& - 4\Psi_s - Q_{ss} - 10sQ\Psi_s \\
K_5 &= 4\Psi\Theta - 2\Theta_B, \quad K_6 = 2(2\Psi\Psi_s - \Psi_{sB})(B-s^2) - 2\Psi_s, \\
K_7 &= -\Theta, \quad K_8 = -\Psi_s(B-s^2), \\
K_9 &= 8Q\Psi(Q\Theta_s + Q_s\Theta)(B-s^2) + 4Q^2(\Theta^2 - \Theta_s) + 4Q(\Theta_B - Q_s), \\
K_{10} &= (4\Psi^2(2QQ_{ss} - Q_s^2) + 8Q\Psi(Q\Psi_{ss} + Q_s\Psi_s) - 4Q^2\Psi_s^2)(B-s^2)^2 \\
& + (-16sQ\Psi(Q\Psi_s + Q_s\Psi) - 4\Psi(2QQ_{ss} - Q_s^2) - 4Q(Q\Psi_{ss} + Q_s\Psi_s) \\
& + 4Q\Psi_{sB} + 4Q_s\Psi_B)(B-s^2) - 4s^2Q^2\Psi^2 + 4(2+3sQ)(Q\Psi_s + Q_s\Psi) \\
& - 8Q^2\Psi + 2QQ_{ss} - Q_s^2 + 4sQ\Psi_B \\
K_{14} &= 2\Psi, \quad K_{15} = 4Q\Theta, \quad K_{16} = 4(Q\Psi_s - Q_s\Psi)(B-s^2) + 2Q_s - 2(1+2sQ)\Psi, \\
K_{17} &= 2Q\Theta, \quad K_{18} = 2(Q\Psi_s + Q_s\Psi)(B-s^2) + 2sQ\Psi - Q_s, \\
K_{19} &= 2(1+sQ)Q_s - 2Q^2, \quad K_{20} = -8(\Psi^2 + \Psi_B)Q, \quad K_{21} = -4Q^2\Theta, \\
K_{22} &= 2Q\Psi - 4Q^2\Psi_s(B-s^2), \quad K_{23} = 2Q\Psi, \quad K_{24} = 2Q, \quad K_{25} = -4Q^2\Psi, \\
K_{26} &= -Q^2, \quad Q = \frac{\phi'}{(\phi-s\phi')}, \quad \Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi(\phi-s\phi' + (B-s^2)\phi'')}, \quad \Psi = \frac{\phi''}{2(\phi-s\phi' + (B-s^2)\phi'')}, \\
B &= b^2.
\end{aligned}$$

3. Ricci curvature of a homogeneous Finsler space with special (α, β) - metric

Now we calculate the Ricci curvature of a homogeneous Finsler space. Let us consider y be a G -invariant vector field in m corresponding to 1-form β of

length c and with respect to the restriction $\langle, \rangle$ of Riemannian metric α of an orthonormal basis $\{y_1, y_2, \dots, y_n = \frac{y}{c}\}$. Let Γ_{ij}^l be the Christoffel symbols given by $\nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^j} \right) = \Gamma_{ij}^l \left(\frac{\partial}{\partial x^l} \right)$. The structure constants of $\mathfrak{m}$ is $C_{ij}^l = \langle [y_i, y_j]_{\mathfrak{m}}, y_l \rangle$, $(1 \leq i, j, l \leq n)$, where $[y_i, y_j]_{\mathfrak{m}}$ denotes the projection of $[y_i, y_j]$ to $\mathfrak{m}$, and $f(l, i)$ can be defined by

$$f(l, i) := \begin{cases} 1, & l < i; \\ 0, & l \geq i. \end{cases}$$

As a prerequisite to the rest of the calculation, we need the following lemma:

Lemma 3.2. *At the origin $o = eH$, we have the following values:*

$$b_i = c\delta_{ni}, \quad s_{ij} = \frac{c}{2}C_{ij}^n, \quad s_j = \frac{c^2}{2}C_{nj}^n, \quad r_{ij} = -\frac{c}{2}(C_{ni}^j + C_{nj}^i),$$

$$s_{i;j} = c \left(s_{ni;j} + \frac{c}{2}C_{qi}^n \Gamma_{nj}^q \right), \quad r_{ij;l} = s_{ij;l} + b_{j;i;l},$$

$$s_{ij;l} = \frac{c}{2} \left(\frac{1}{2}C_{qj}^n (C_{li}^q + C_{iq}^l + C_{lq}^i) + \frac{1}{2}C_{iq}^n (C_{lj}^q + C_{jq}^l + C_{lq}^j) + C_{ji}^q C_{lq}^n \right),$$

$$\frac{1}{c}b_{i;j;l} = -\Gamma_{nj}^p \langle \nabla_{\widehat{y}_l} \widehat{y}_i, \widehat{y}_p \rangle - \Gamma_{np}^i \langle \nabla_{\widehat{y}_l} \widehat{y}_j, \widehat{y}_p \rangle + C_{ln}^p \langle \nabla_{\widehat{y}_p} \widehat{y}_j, \widehat{y}_l \rangle + \widehat{y}_l \langle \nabla_{\widehat{y}_n} \widehat{y}_j, \widehat{y}_l \rangle,$$

$$\Gamma_{ij}^l = f(i, j)C_{ij}^l + \langle \nabla_{\widehat{y}_i} \widehat{y}_j, \widehat{y}_l \rangle, \quad \langle \nabla_{\widehat{y}_i} \widehat{y}_j, \widehat{y}_l \rangle|_0 = -\frac{1}{2}(C_{jl}^i + C_{il}^j + C_{ij}^l),$$

$$\langle \nabla_{\widehat{y}_i} \widehat{y}_j, \widehat{y}_l \rangle \widehat{y}_k|_0 = \frac{1}{2} (C_{ka}^l C_{ij}^a + C_{ka}^j C_{il}^a + C_{ka}^i C_{jl}^a + C_{ij}^p C_{kl}^p + C_{il}^p C_{kj}^p + C_{jl}^p C_{ki}^p),$$

$$\frac{\partial \Gamma_{ij}^l}{\partial x^k}|_0 = \langle \nabla_{\widehat{y}_i} \widehat{y}_j, \widehat{y}_l \rangle \widehat{y}_k + f(k, j)C_{kj}^a \langle \nabla_{\widehat{y}_i} \widehat{y}_a, \widehat{y}_l \rangle + f(k, i)C_{ki}^p \langle \nabla_{\widehat{y}_p} \widehat{y}_j, \widehat{y}_l \rangle$$

$$-(\Gamma_{kp}^l + \langle \nabla_{\widehat{y}_k} \widehat{y}_l, \widehat{y}_p \rangle) \Gamma_{ij}^p, \quad i \geq j.$$

As a result of lemma (3.2), we calculate the quantities involved in the statement of theorem (2.2) at the origin of homogeneous Finsler spaces.

$$r_{00} = cC_{0n}^0, \quad s_0 = \frac{c^2}{2}C_{n0}^n, \quad r_0 = -\frac{c^2}{2}C_{n0}^n, \quad r_{00;0} = -cC_{0p}^0 (C_{np}^0 + C_{np}^p),$$

$$s_{0;0} = \frac{c^2}{2}C_{np}^n C_{0p}^0, \quad r = 0, r_q^q = -cC_{nq}^q, \quad r_{0q}r_0^q = \frac{c^2}{4}(C_{n0}^q + C_{nq}^0)^2,$$

$$r_{00;q}b^q = -\frac{c^2}{2}(C_{n0}^p + C_{np}^0)(-C_{n0}^p + C_{np}^0 + C_{0p}^n),$$

$$r_{0q;0}b^q = \frac{c^2}{2} \left[C_{np}^n C_{p0}^0 + \frac{1}{2}(C_{n0}^p + C_{np}^0)(C_{0n}^p + C_{p0}^n + C_{pn}^0) \right],$$

$$\begin{aligned}
r_{0q}s_0^q &= -\frac{c^2}{4}C_{q0}^n(C_{n0}^q + C_{nq}^0), \quad s_{0q}s_0 = -\frac{c^2}{4}(C_{q0}^n)^2, \quad s_qs_0^q = \frac{c^3}{4}C_{nq}^nC_{q0}^n, \\
s_qr_0^q &= -\frac{c^3}{4}C_{nq}^n(C_{nq}^0 + C_{n0}^q), \quad r_q s_0^q = -\frac{c^3}{4}C_{nq}^nC_{q0}^n, \quad s_qs^q = \frac{c^4}{4}(C_{nq}^n)^2, \\
s_{0;q}b^q &= \frac{c^3}{4}C_{np}^n(C_{0n}^p + C_{0p}^n + C_{np}^0), \quad s_{q;0}b^q = \frac{c^3}{4}C_{np}^n(C_{n0}^p + C_{0p}^n + C_{np}^0), \\
s_{0;q}^q &= \frac{c}{2}C_{p0}^nC_{qp}^q + \frac{c}{4}C_{qp}^n(C_{qp}^0 + C_{0p}^q + C_{0q}^p), \quad s_q^i s_i = -\frac{c^2}{4}(C_{qi}^n)^2.
\end{aligned}$$

For special metric, K_1 to K_{26} can be calculated as follows:

$$\begin{aligned}
\phi &= 1 + \sqrt{s^2 + 1}, \quad \phi' = \frac{s}{\sqrt{s^2 + 1}}, \quad \phi'' = \frac{1}{(s^2 + 1)^{\frac{3}{2}}}, \\
K_1 &= -\frac{3s^2 \left(((B+s)s^4 + (6B+14)s^2 + 4B+8)\sqrt{s^2+1} + (-s^4 + 2s^2 + 4B+8)(s^2+1)^2 \right)}{(4\phi^2\sqrt{s^2+1}(s^2\sqrt{s^2+1}+B+\phi)^3)}, \\
K_2 &= \frac{\left((3\sqrt{s^2+1}(s^6 - (7B+6)s^4 + (2B^2 - 3B - 6)s^2 + B^2 + 2B)\sqrt{s^2+1} + 2s^8) \right. \\
&\quad \left. + s^6(-4B+1) - (12B+9)s^4 + (3B^2 - 2B - 6)s^2 + B^2 + 2B \right)}{2(s^2\sqrt{s^2+1}+B+\phi)^4}, \\
K_3 &= \frac{\left(((3B+19)s^4 + (20B+52)s^2 + 16B+32)\sqrt{s^2+1} + (s^2+1)(3s^2+4)(-s^4 + 3s^2 + 4B+8) \right)s^3}{\phi^3\sqrt{s^2+1}(s^2\sqrt{s^2+1}+B+\phi)^3}, \\
K_4 &= -\frac{1}{(s^2+1)^{\frac{3}{2}}\phi^2(s^2\sqrt{s^2+1}+B+\phi)^5} \left(((3s^{14} - (101 + 72B)s^{12} - (683 + 12B^2 + 458B)s^{10} \right. \\
&\quad - 2(64B^2 + 676B + 939)s^8 + 2(15B^3 - 127B^2 - 1032B - 1351)s^6 + (-2133 - 171B^2 + 74B^3 - 1664B)s^4 + \\
&\quad (-672B - 876 - 29B^2 + 54B^3 - B^4)s^2 + 2(-72 + 15B^3 + 12B^2 + B^5 + \frac{9B^4}{2} - 48B))\sqrt{s^2+1} + 6s^{16} - (7 + 24B)s^{14} - 2(151 + 116B)s^{12} \\
&\quad - (1378 + 52B^2 + 989B)s^{10} - \frac{1}{2}(4478B - 24B^3 + 418B^2 + 6005)s^8 - \frac{1}{2}(7327 + 5678B - 90B^3 + 620B^2)s^6 + \\
&\quad \left. \frac{1}{2}(-5106 - 359B^2 + 12B^4 + 164B^3 - 3994B)s^4 + \frac{1}{2}(-1440B - 1896 - 25B^2 + 138B^3 + 20B^4)s^2 - (144 + 30B^3 + 24B^2 + B^5 + 9B^4 - 96B)s \right), \\
K_5 &= \frac{2s^3}{\phi(s^2\sqrt{s^2+1}+B+\phi)^2}, \quad K_6 = \frac{\phi(s^2+1)+2(B-s^2)}{\phi(s^2\sqrt{s^2+1}+B+\phi)^3}, \\
K_7 &= -\frac{s^3}{2\phi(s^2\sqrt{s^2+1}+B+\phi)}, \quad K_8 = \frac{3s\sqrt{(s^2+1)(B-s^2)}}{2\phi(s^2\sqrt{s^2+1}+B+\phi)^2},
\end{aligned}$$

$$\begin{aligned}
K_9 &= -\frac{1}{\phi^4\sqrt{s^2+1}(s^2\sqrt{s^2+1}+B+\phi)^3} (s((4s^{10} - 2s^9 + 24s^8 + (3B + 17)s^7 + \\
&\quad 4(6B + 20)s^6 + (18B + 52)s^5 + 4(3B^2 + 24B + 37)s^4 + 36(B + 2)^2s^2)\sqrt{s^2+1} - 3s^{11} + 8s^{10} + 2s^9 + 4(3B + 13)s^8 + (39 + 8B)s^7 + \\
&\quad 4(35 + 15B)s^6 + 4(17 + 6B)s^5 + 4(6B^2 + 39B + 53)s^4 - 2(B + 2)(B - 8)\phi s^3 + 4(B + 11)(B + 2)^2s^2 + 8(B + 2)^3\phi)), \\
K_{10} &= \frac{1}{\phi^2\sqrt{s^2+1}(s^2\sqrt{s^2+1}+B+\phi)^4} \left(\left((17s^{10} + (142 + 56B)s^8 + (144B + 299 - \right. \right. \\
&\quad 3B^2)s^6 + (215 + 88B - 17B^2)s^4 + (24 - 12B - 9B^2 + B^3)s^2) + \\
&\quad 2(s^{10} + 12(B + 3)s^8 + (40B + 104)s^6 + (2B^2 + 41B + 97)s^4 + \\
&\quad \left. \left. (-B^2 + 2B + 18)s^2)\sqrt{s^2+1} - (B + 3)(B + 2)^2(s^2 + \phi) \right) \right), \\
K_{11} &= -\frac{1}{(s^2\sqrt{s^2+1}+B+\phi)^2}, \quad K_{12} = -\frac{4s^4}{\phi^2(s^2\sqrt{s^2+1}+B+\phi)^2}, \\
K_{13} &= \frac{2s^2(-3s^4-14s^2+6B-1)+2(B+2)\phi}{\phi(s^2\sqrt{s^2+1}+B+\phi)^3}, \quad K_{14} = \frac{1}{s^2\sqrt{s^2+1}+B+\phi}, \\
K_{15} &= \frac{2s^4}{\phi^2(s^2\sqrt{s^2+1}+B+\phi)}, \\
K_{16} &= \frac{s^2\sqrt{s^2+1}(s^2+2)+6s^6+(-6B+7)s^4+(-5B+3)s^2+(B+2)\phi}{\phi\sqrt{s^2+1}(s^2\sqrt{s^2+1}+B+\phi)^2}, \\
K_{17} &= \frac{s^4}{\phi^2(s^2\sqrt{s^2+1}+B+\phi)}, \\
K_{18} &= \frac{-(s^4+2s^2)\sqrt{s^2+1}-(B+2)\phi+3s^6+(-3B+2)s^4-(4B+3)s^2}{\phi\sqrt{s^2+1}(s^2\sqrt{s^2+1}+B+\phi)^2}, \\
K_{19} &= \frac{-2(s^2\sqrt{s^2+1}-s^2-\phi)}{\phi^2\sqrt{s^2+1}}, \quad K_{20} = \frac{2s}{\phi(s^2\sqrt{s^2+1}+B+\phi)^2}, \\
K_{21} &= -\frac{2s^5}{\phi^3(s^2\sqrt{s^2+1}+B+\phi)}, \\
K_{22} &= \frac{(s(s-6s^4\sqrt{s^2+1}+6Bs^2\sqrt{s^2+1}+s^4+s^2\sqrt{s^2+1}+B\sqrt{s^2+1}+2s^2+B+2\sqrt{s^2+1}+2))}{\phi(\sqrt{s^2+1}(s^2+1)+B+1)^2}, \\
K_{23} &= \frac{s}{\phi(\sqrt{s^2+1}(s^2+1)+B+1)}, \quad K_{24} = \frac{2s}{\phi}, \quad K_{25} = -\frac{2s^2}{\phi^2(\sqrt{s^2+1}(s^2+1)+B+1)}, \\
K_{26} &= \frac{s^2}{\phi^2}, \quad Q = \frac{s}{\phi}, \quad Q_s = \frac{1}{\phi\sqrt{s^2+1}}, \quad Q_{ss} = -\frac{(\sqrt{s^2+1}+\phi)s}{(s^2+1)^2\phi^2},
\end{aligned}$$

$$\begin{aligned}
\Psi &= \frac{1}{2(s^2\sqrt{s^2+1}+B+\phi)}, \quad \Psi_s = -\frac{3s\sqrt{s^2+1}}{2(s^2\sqrt{s^2+1}+B+\phi)^2}, \\
\Psi_{ss} &= \frac{3((4s^4+3s^2-1)\sqrt{s^2+1}-3(B+1))}{2\sqrt{s^2+1}(s^2\sqrt{s^2+1}+B+\phi)^3}, \quad \Psi_B = -\frac{1}{2(s^2\sqrt{s^2+1}+B+\phi)^2}, \\
\Psi_{sB} &= \frac{3s\sqrt{s^2+1}}{(s^2\sqrt{s^2+1}+B+\phi)^3}, \quad \Theta = \frac{s^3}{2\phi\sqrt{s^2+1}(s^2\sqrt{s^2+1}+B+\phi)}, \\
\Theta_s &= \frac{-2(s^2\sqrt{s^2+1}-s^2-\phi)}{\phi^2(\sqrt{s^2+1}(s^2+1)+B+1)^2}, \quad \Theta_B = -\frac{s^3}{2\phi(s^2\sqrt{s^2+1}+B+\phi)^2}, \\
\Theta_{sB} &= -\frac{\left((-s^2(4s^2+1)+3(B+2))\sqrt{s^2+1}\right)}{2\phi^2\sqrt{s^2+1}((s^2\sqrt{s^2+1}+B+\phi)^3)}.
\end{aligned}$$

We obtain the following result from the above computation:

Theorem 3.3. *A compact homogeneous Finsler space G/H with G -invariant special metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$. Then the Ricci curvature is given by*

$$\begin{aligned}
Ric(Z) &= Ric^\alpha(Z) + \frac{c^2(C_{0n}^0)^2}{\alpha^2(Z)}\{(n-1)K_1 + K_2\} \\
&\quad + \frac{c^3C_{0n}^0C_{n0}^n}{2\alpha(Z)}\{(n-1)(K_3 - K_5) + K_4 - K_6\} \\
&\quad - \frac{cC_{0q}^0(C_{nq}^0 + C_{n0}^q)}{\alpha(Z)}\{(n-1)K_7 + K_8\} \frac{c^4(C_{n0}^n)^2}{4}\{(n-1)(K_9 - K_{12}) + K_{10} \\
&\quad \quad - K_{11} - K_{13}\} \\
&\quad - \frac{c^2}{4}\{4C_{0n}^0C_{nq}^q + (C_{n0}^q + C_{nq}^0)(2C_{nq}^0 + C_{0q}^n + 2C_{0n}^q) + 2C_{nq}^nC_{q0}^0\}K_{14} \\
&\quad + \frac{c^2}{4}C_{0q}^n(C_{n0}^q + C_{nq}^0)\{(n-1)K_{15} + K_{16}\} \\
&\quad + \frac{c^2}{2}C_{nq}^nC_{0q}^0\{(n-1)K_{17} + K_{18}\} \\
&\quad - \frac{c^2}{4}(C_{q0}^n)^2K_{19} + \frac{c^3}{4}\alpha(Z)C_{q0}^nC_{nq}^n\{(n-1)K_{21} + K_{22}\} + \frac{c^3}{4}\alpha(Z) \\
&\quad \times \{4C_{n0}^nC_{nq}^q - C_{nq}^n(4C_{nq}^0 - C_{0q}^n)\}K_{23} \\
&\quad \quad + \frac{c}{4}\alpha(Z)\{2C_{m0}^nC_{qm}^q + C_{qm}^nC_{qm}^0\}K_{24} \\
&\quad + \frac{c^2}{4}\alpha^2(Z)\{c^2(C_{nq}^n)^2K_{25} - (C_{qi}^n)^2K_{26}\}
\end{aligned}$$

where a non-zero vector Z in $\mathfrak{m}$.

4. Ricci curvature of Homogeneous Finsler spaces with (α, β) -metric has vanishing S -curvature

In this section we discuss the conditions to have vanishing S -curvature for the space $(G/H, F)$. For the sake of further study, we use the formula of S -curvature for homogeneous Finsler space with special metric, $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ given in the paper¹³.

We use the following theorem:

Theorem 4.4. *Let $(M = G/H, F)$ be a compact connected homogeneous Finsler space with G -invariant metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ on G/H . Then (M, F) has vanishing S -curvature if and only if $\langle [y, Y]_{\mathfrak{m}}, Y \rangle = 0, \forall Y \in \mathfrak{m}$ and $y \in \mathfrak{m}$ corresponds to the β , 1-form.*

Theorem 4.5. *Let $(M = G/H, F)$ be a compact connected homogeneous Finsler space with G -invariant special metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ on G/H . Suppose that (M, F) has vanishing S -curvature. Then Ricci curvature is given by*

$$Ric(Z) = Ric^\alpha(Z) - \frac{c^2}{4} (C_{q0}^n)^2 K_{19} + \frac{c}{4} \alpha(Z) (2C_{m0}^n C_{qm}^q + C_{qm}^n C_{qm}^0) K_{24} - \frac{c^2}{4} \alpha^2(Z) (C_{il}^n)^2 K_{26}$$

where $Z (\neq 0) \in \mathfrak{m}$ and K_{19}, K_{24}, K_{26} are already defined.

Proof. Assume that $(M = G/H, F)$ has vanishing S -curvature, from theorem 4.4,

$$\langle [y, Z]_{\mathfrak{m}}, Z \rangle = 0, \forall Z \in \mathfrak{m}.$$

$$\text{Therefore, } C_{0n}^0 = \langle Z, [y_n, Z]_{\mathfrak{m}} \rangle = \left\langle Z, \left[\frac{y}{c}, Z \right]_{\mathfrak{m}} \right\rangle = 0,$$

$$\begin{aligned} \text{and } C_{nj}^i + C_{ni}^j &= \left\langle y_i, [y_n, y_j]_{\mathfrak{m}} \right\rangle + \left\langle y_j, [y_n, y_i]_{\mathfrak{m}} \right\rangle \\ &= \left\langle y_i, \left[\frac{y}{c}, y_j \right]_{\mathfrak{m}} \right\rangle + \left\langle y_j, \left[\frac{y}{c}, y_i \right]_{\mathfrak{m}} \right\rangle \\ &= \frac{1}{c} \left\{ \left\langle y_i, [y, y_j]_{\mathfrak{m}} \right\rangle + \left\langle y_j, [y, y_j]_{\mathfrak{m}} \right\rangle + \left\langle y_j, [y, y_i]_{\mathfrak{m}} \right\rangle + \left\langle y_i, [y, y_i]_{\mathfrak{m}} \right\rangle \right\} \\ &= \frac{1}{c} \left\langle y_i + y_j, [y, y_j]_{\mathfrak{m}} \right\rangle + \left\langle y_i + y_j, [y, y_i]_{\mathfrak{m}} \right\rangle \\ &= \frac{1}{c} \left\langle y_i + y_j, [y, y_i + y_j]_{\mathfrak{m}} \right\rangle = 0, \quad \forall 1 \leq i, j \leq n. \end{aligned}$$

$$\text{Also } C_{ni}^n = \langle [y_n, y_i]_{\mathfrak{m}}, y_n \rangle = \left\langle \left[\frac{y}{c}, y_i \right]_{\mathfrak{m}}, \frac{y}{c} \right\rangle$$

$$\begin{aligned}
&= \frac{1}{c^2} \{ \langle [y, y_i]_m, y \rangle + \langle [y, y_i]_m, y_i \rangle \} \\
&= \frac{1}{c^2} \langle [y, y]_m + [y, y_i]_m, y + y_i \rangle \\
&= \frac{1}{c^2} \langle [y, y + y_i]_m, [y + y_i] \rangle = 0.
\end{aligned}$$

By theorem 3.3, The required result is obtained.

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