

On Symmetric, Semi-Symmetric and Quarter-Symmetric Riemannian Manifolds

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Abstract: In this paper, we have studied on Riemannian symmetric manifolds, Semi-symmetric manifolds and further defined and studied quarter-symmetric manifold of first and second kind, also discussed their properties.

Keywords: Riemannian manifolds, symmetric manifolds, Semi-symmetric manifolds, quarter-symmetric manifolds.

1. Introduction

Let (M^n, g) be an n -dimensional Riemannian manifold with coordinates $x^i (i = 1, \dots, n)$, positive definite Riemannian metric g , metric tensor $g_{ji}(x)$, Christoffel symbols $\{h_{ji}\}$, curvature tensor $K_{kji h}$, Ricci tensor $K_{ji} = K_{kji h} g^{kh}$ and scalar curvature $K = K_{ji} g^{ji}$, Yano¹. Let ∇_m denotes covariant derivative with respect to Christoffel symbols, then there exist following important Riemannian manifolds.

A Riemannian manifold M^n is called *symmetric Riemannian manifold*²⁻³, if the non-zero curvature tensor satisfies

$$(1.1) \quad \nabla_m K_{kjih} = 0.$$

A Riemannian manifold M^n is called *bi-symmetric manifold*, if it is not a symmetric manifold and its curvature tensor satisfies

$$(1.2) \quad \nabla_m \nabla_p K_{kjih} = 0.$$

A Riemannian manifold M^n is called *tri-symmetric manifold*, if its curvature tensor is neither zero nor it satisfies equation (1.2), but it satisfies

$$(1.3) \quad \nabla_m \nabla_p \nabla_q K_{kjih} = 0.$$

A Riemannian manifold M^n is called *semi-symmetric manifold*⁴⁻⁷, if the Riemannian curvature tensor satisfies $K(X, Y)K = 0$, for all tangent vectors X and Y , where the endomorphism $K(X, Y)$ operates on K as a derivation of the tensor algebra at each point of (M^n, g) . Locally this condition is expressed as⁴:

$$(1.4) \quad K_{ajih} K_{mpk}^a + K_{kaih} K_{mpj}^a + K_{kjah} K_{mpi}^a + K_{kjia} K_{mph}^a = 0$$

or equivalently as

$$(1.5) \quad \nabla_m \nabla_p K_{kjih} - \nabla_p \nabla_m K_{kjih} = 0.$$

The projective¹, conformal¹, Con-harmonic⁸ and con-circular¹ curvature tensors are respectively given as

$$(1.6) \quad P_{kjih} = K_{kjih} - (n-1)^{-1} \zeta_{(j,k)}(g_{kh} K_{ji}),$$

$$(1.7) \quad C_{kjih} = K_{kjih} - \zeta_{(j,k)}(g_{hj} L_{ki} + g_{ki} L_{hj}),$$

$$(1.8) \quad H_{kjih} = K_{kjih} - (n-2)^{-1} \zeta_{(j,k)}(g_{hk} K_{ji} + g_{ji} K_{hk})$$

and

$$(1.9) \quad Z_{kjih} = K_{kjih} - K\{n(n-1)\}^{-1} \zeta_{(j,k)}(g_{hk} g_{ji})$$

where $\zeta_{(j,k)}$ means interchange of indices j and k and subtraction and also

$$(1.10) \quad L_{ji} = (n-2)^{-1} \{(1/2)K(n-1)^{-1} g_{ji} - K_{ji}\}.$$

Yano, Sawaki⁹ and the Rastogi¹⁰ introduced following tensors:

$$(1.11) \quad W_{kjih} = aK_{jih} + \zeta_{(j,k)}(g_{kh} T_{ji} + g_{ji} T_{kh})$$

and

$$(1.12) \quad S_{kjih} = K_{kjih} + \zeta_{(j,k)}(\alpha g_{kh} K_{ji} + \beta g_{ji} K_{kh} + \gamma K g_{ji} g_{kh})$$

where $a, b, \alpha, \beta, \gamma$ are some constants and

$$(1.13) \quad T_{ji} = (1/2)(n-1)^{-1}\{a - (n-3)b\} K g_{ji} - b(n-2)L_{ji}.$$

The purpose of the present paper is to study symmetric Riemannian manifold, semi-symmetric Riemannian manifold and further define the quarter Riemannian manifold (M^n, g) of first and second kind and to discuss some of their properties.

2. Symmetric Riemannian (SR)-Manifold

Let us assume that a Riemannian manifold (M^n, g) satisfies condition of generalised recurrent manifold studied by De and Biswas¹¹ then we have:

$$(2.1) \quad \nabla_m K_{kjih} = a_m K_{kjih} + b_m \zeta_{(j,k)} \{g_{kh} g_{ji}\}$$

where a_m and b_m are non-zero vectors. Multiplying equation (2.1) by g^{kh} , we obtain

$$(2.2) \quad \nabla_m K_{ji} = a_m K_{ji} + b_m (n-1) g_{ji},$$

which leads to

$$(2.3) \quad \nabla_m K = a_m K + b_m n(n-1)$$

If we assume that $b_m = -Kn^{-1}(n-1)^{-1}a_m$, equation (2.3) gives that K is a constant scalar, while equation (2.2) and (2.1) respectively give

$$(2.4) \quad \nabla_m K_{ji} = a_m (K_{ji} - Kn^{-1} g_{ji})$$

and

$$(2.5) \quad \nabla_m K_{kjih} = a_m [K_{kjih} - Kn^{-1}(n-1)^{-1} \zeta_{(j,k)} \{g_{kh} g_{ji}\}]$$

Comparing equations (1.9) and (2.5), we get $\nabla_m K_{kjih} = a_m Z_{kjih}$, which leads to

Theorem 2.1: *A given Riemannian manifold (M^n, g) , which is generalised recurrent and satisfies $b_m = -Kn^{-1}(n-1)^{-1}a_m$, with vanishing con-circular curvature tensor is a symmetric Riemannian manifold.*

Let us assume that a Riemannian manifold (M^n, g) is a UV-recurrent manifold Rastogi⁷ :

$$(2.6) \quad \nabla_m K_{kjih} = a_m K_{kjih} + b_m \zeta_{(j,k)} \{U_{kh} V_{ji} + U_{ji} V_{kh}\}$$

then for $U_{ji} = g_{ji}$ and $V_{ji} = K_{ji}$ it satisfies

$$(2.7) \quad \nabla_m K_{kjih} = a_m K_{kjih} + b_m \zeta_{(j,k)} \{g_{kh} K_{ji}\}$$

which gives

$$(2.8) \quad \nabla_m K_{ji} = \{a_m + (n-1)b_m\}K_{ji}, \nabla_m K = \{a_m + (n-1)b_m\}K$$

Assuming $b_m = -(n-1)^{-1}a_m$, we can obtain from equation (2.7)

$$(2.9) \quad \nabla_m K_{kjih} = a_m[K_{kjih} - (n-1)^{-1}\zeta_{(j,k)}\{g_{kh}K_{ji}\}]$$

Comparing equations (1.6) and (2.9), we can obtain $\nabla_m K_{kjih} = a_m P_{kjih}$, which leads to

Theorem 2.2: *A given Riemannian manifold (M^n, g) , which is UV-recurrent such that $U_{ji} = g_{ji}$ and $V_{ji} = K_{ji}$ and satisfies $b_m = -(n-1)^{-1}a_m$ with vanishing projective curvature tensor is a symmetric Riemannian manifold.*

Let us assume that a Riemannian manifold (M^n, g) is such that it satisfies

$$(2.10) \quad \nabla_m K_{kjih} = a_m K_{kjih} + b_m \zeta_{(j,k)}\{g_{hj}L_{ki} + g_{ki}L_{hj}\}$$

then we can obtain

$$\begin{aligned} \nabla_m K_{ji} &= a_m K_{ji} - b_m\{(n-2)L_{ji} - Kg_{ji}(2(n-1))^{-1}\}, \\ \nabla_m K &= (a_m + b_m)K. \end{aligned}$$

If $b_m = -a_m$, we can obtain by virtue of equation (1.7) and (2.10), we get

$$(2.11) \quad \nabla_m K_{kjih} = a_m C_{kjih}$$

which leads to

Theorem 2.3: *A given Riemannian manifold (M^n, g) , which satisfies equations (2.10) and (2.11), for the vanishing conformal curvature tensor proves that the given manifold is a symmetric Riemannian manifold.*

Let us assume that

$$(2.12) \quad \nabla_m K_{kjih} = a_m K_{kjih} + b_m \zeta_{(j,k)}\{g_{jh}K_{ki} + g_{ki}K_{jh}\},$$

which leads to

$$\begin{aligned} (2.13) \quad \nabla_m K_{ji} &= a_m K_{ji} + b_m\{(2-n)K_{ji} - Kg_{ji}\}, \\ \nabla_m K &= a_m K + b_m 2K(1-n) \end{aligned}$$

Assuming $b_m = a_m/2(n-1)$, we can write with the help of equation (1.8).

$$(2.14) \quad \nabla_m K_{kjih} = a_m\{nK_{kjih} + (n-2)H_{kjih}\}/2(n-1),$$

which leads to

Theorem 2.4: *A given Riemannian manifold (M^n, g) , will be recurrent if and only if the con-harmonic curvature tensor vanishes.*

Let us assume that

$$(2.15) \quad \nabla_m K_{kjih} = a_m K_{kjih} + b_m \zeta_{(j,k)} \{g_{kh} T_{ji} + g_{ji} T_{kh}\},$$

which leads to

$$(2.16) \quad \begin{aligned} \nabla_m K_{ji} &= a_m K_{ji} + b_m \{(n-2)T_{ji} + T g_{ji}\}, \\ \nabla_m K &= \{a_m + (a + nb)b_m\}K \end{aligned}$$

where $T = K[b + \{a - (n-2)b\}n/2(n-1)]$.

Assuming $b_m = -\frac{a_m}{a+nb}$, we can observe that $\nabla_m K = 0$.

Now using equation (1.11), we can obtain

$$(2.17) \quad \nabla_m K_{kjih} = a_m \{(2a + nb)K_{kjih} - W_{kjih}\} / (a + nb),$$

which leads to

Theorem 2.5: *A given Riemannian manifold (M^n, g) , satisfying equation (2.15) will be recurrent if and only if the Yano and Sawaki curvature tensor defined by equation (1.11) vanishes.*

Let us assume that

$$(2.18) \quad \nabla_m K_{kjih} = a_m K_{kjih} + b_m \zeta_{(j,k)} \{\alpha g_{kh} K_{ji} + \beta g_{ji} K_{kh} + \gamma K g_{ji} g_{kh}\},$$

which leads to

$$(2.19) \quad \nabla_m K_{ji} = a_m K_{ji} + b_m [\{(n-1)\alpha - \beta\}K_{ji} + \{\beta + (n-1)\gamma\}K g_{ji}]$$

and

$$(2.20) \quad \nabla_m K = a_m K + b_m (n-1)K(\alpha + \beta + n\gamma)$$

Assuming $b_m = -a_m / (n-1)(\alpha + \beta + n\gamma)$,

we can get $\nabla_m K = 0$, which leads to

$$(2.21) \quad \nabla_m K_{kjih} = a_m K_{kjih} \{1 + (n-1)^{-1}(\alpha + \beta + n\gamma)^{-1} - (\alpha + \beta + n\gamma)^{-1} S_{kjih}\}$$

Hence:

Theorem 2.6: *A given Riemannian manifold (M^n, g) , satisfying equation (2.18) will be recurrent if and only if the tensor S_{kjih} defined by equation (1.12) vanishes.*

3. Semi- Symmetric Riemannian (SSR)-Manifold

Let us assume that we are given a bi-recurrent Riemannian manifold, i.e. it satisfies⁶

$$(3.1) \quad \nabla_p \nabla_m K_{kji h} = a_{mp} K_{kji h}$$

then we can obtain

$$(3.2) \quad \nabla_p \nabla_m K_{kji h} - \nabla_m \nabla_p K_{kji h} = (a_{mp} - a_{pm}) K_{kji h},$$

which leads to

Theorem 3.1: A bi-recurrent Riemannian manifold (M^n, g) shall be a SSR-manifold if and only if $a_{mp} = a_{pm}$.

From equation (1.9), we can obtain

$$(3.3) \quad (\nabla_p \nabla_m - \nabla_m \nabla_p) Z_{kji h} = (\nabla_p \nabla_m - \nabla_m \nabla_p) [K_{kji h} - K n^{-1} (n-1)^{-1} \zeta_{(j,k)} \{g_{hk} g_{ji}\}]$$

$$(3.4) \quad (\nabla_p \nabla_m - \nabla_m \nabla_p) Z_{kji h} = (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{kji h}$$

Hence, we have:

Theorem 3.2: A SSR-manifold (M^n, g) also satisfies $(\nabla_p \nabla_m - \nabla_m \nabla_p) Z_{kji h} = 0$.

From equation (1.6), we can obtain

$$\begin{aligned} (\nabla_p \nabla_m - \nabla_m \nabla_p) P_{kji h} &= (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{kji h} \\ &\quad - (n-1)^{-1} (\nabla_p \nabla_m - \nabla_m \nabla_p) [\zeta_{(j,k)} \{g_{hk} K_{ji}\}], \end{aligned}$$

which leads to

Theorem 3.3: The projective curvature tensor $P_{kji h}$ of a SSR-manifold (M^n, g) satisfies $(\nabla_p \nabla_m - \nabla_m \nabla_p) P_{kji h} = 0$.

From equation (1.7), we can obtain

$$\begin{aligned} (3.5) \quad (\nabla_p \nabla_m - \nabla_m \nabla_p) C_{kji h} &= (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{kji h} \\ &\quad + (n-2)^{-1} [\zeta_{(j,k)} \{g_{hj} (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{ki} + g_{ki} (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{hj}\}], \end{aligned}$$

which leads to

Theorem 3.4: The necessary and sufficient condition for the conformal curvature tensor to satisfy $(\nabla_p \nabla_m - \nabla_m \nabla_p) C_{kji h} = 0$ is that the Riemannian manifold (M^n, g) be an SSR-manifold.

From equation (1.8), we can obtain

$$(3.6) \quad (\nabla_p \nabla_m - \nabla_m \nabla_p) H_{kji h} = (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{kji h} \\ - (n-2)^{-1} [\zeta_{(j,k)} \{g_{hk} (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{ji} + g_{ki} (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{hk}\}],$$

which leads to

Theorem 3.5: *The necessary and sufficient condition for the con-harmonic curvature tensor to satisfy $(\nabla_p \nabla_m - \nabla_m \nabla_p) H_{kji h} = 0$, is that the Riemannian manifold (M^n, g) be an SSR-manifold.*

From equation (1.11), we can obtain

$$(3.7) \quad (\nabla_p \nabla_m - \nabla_m \nabla_p) W_{kji h} = a (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{kji h} \\ + b \{2(n-1)\}^{-1} \zeta_{(j,k)} [g_{kh} (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{ji} + g_{ji} (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{kh}],$$

which leads to

Theorem 3.6: *In case Ricci-tensor satisfies $(\nabla_p \nabla_m - \nabla_m \nabla_p) K_{ji} = 0$, SSR-manifold will also satisfy $(\nabla_p \nabla_m - \nabla_m \nabla_p) W_{kji h} = 0$. Conversely, if $W_{kji h}$ satisfies $(\nabla_p \nabla_m - \nabla_m \nabla_p) W_{kji h} = 0$, then either $(\nabla_p \nabla_m - \nabla_m \nabla_p) K_{ji} = 0$ or $b = 2(1-n)\alpha/(n-2)$.*

From equation (1.12), we can obtain

$$(3.8) \quad (\nabla_p \nabla_m - \nabla_m \nabla_p) S_{kji h} = (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{kji h} \\ + \alpha [g_{kh} (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{ji} - g_{jh} (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{ki}] \\ + \beta [g_{ji} (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{kh} - g_{ki} (\nabla_p \nabla_m - \nabla_m \nabla_p) K_{jh}],$$

which leads to

Theorem (3.7): *In case Ricci tensor satisfies $(\nabla_p \nabla_m - \nabla_m \nabla_p) K_{ji} = 0$, SSR-manifold will also satisfy $(\nabla_p \nabla_m - \nabla_m \nabla_p) S_{kji h} = 0$. Conversely, if $S_{kji h}$ satisfies $(\nabla_p \nabla_m - \nabla_m \nabla_p) S_{kji h} = 0$, then either $(\nabla_p \nabla_m - \nabla_m \nabla_p) K_{ji} = 0$ or $\beta = 1 + (n-1)\alpha$.*

4. Quarter-Symmetric Riemannian Manifold of First Kind (QSRM-I)

In this section we shall give following:

Definition 4.1. A Riemannian manifold M^n , for which the curvature tensor $K_{kji h}$ is non-zero and does not satisfy equations (1.1), (1.2) and (1.3) shall be called quarter-symmetric Riemannian manifold of first kind (QSRM-I), if it satisfies

$$(4.1) \quad (\nabla_m \nabla_p - \nabla_p \nabla_m) (\nabla_r K_{kji h}) = 0.$$

With the help of equation (2.1), we can obtain on simplifying, equation (4.1), we get

$$(4.2) \quad (\nabla_a K_{kjih})K_{mpr}^a + (\nabla_r K_{jih}^a)K_{mpk}^a + (\nabla_r K_{kaih})K_{mpj}^a + (\nabla_r K_{kjah})K_{mpi}^a + (\nabla_r K_{kjia})K_{mph}^a = 0.$$

Hence:

Theorem 4.1: *The necessary and sufficient condition for a Riemannian manifold (M^n, g) to be a quarter-symmetric Riemannian manifold of first kind (QSRM-I), is given by equation (4.2).*

Let us assume that the given Riemannian manifold be recurrent Ruse¹², i.e., it satisfies

$$(4.3) \quad \nabla_m K_{kjih} = a_m K_{kjih},$$

then we can obtain

$$(4.4) \quad (\nabla_m \nabla_p - \nabla_p \nabla_m)(\nabla_r K_{kjih}) = K_{kjih}(\nabla_m \nabla_p - \nabla_p \nabla_m)a_r + a_r(\nabla_m \nabla_p - \nabla_p \nabla_m)K_{kjih},$$

which leads to

Theorem 4.2: *If a Riemannian manifold (M^n, g) is recurrent and satisfies*

$$a_t K_{kjih} K_{mpr}^t + a_r \{K_{tjih} K_{mpk}^t + K_{ktih} K_{mpj}^t + K_{kjth} K_{mpi}^t + K_{kjit} K_{mph}^t\} = 0$$

then the Riemannian manifold is QSRM-I.

Let us assume that the given Riemannian manifold is generalised recurrent Riemannian manifold, then from equation (2.1), for the vector $b_m = -Kn^{-1}(n-1)^{-1}a_m$, we get $\nabla_m K_{kjih} = a_m Z_{kjih}$. Substituting this in the left-hand side of equation (4.1), we can obtain

$$(4.5) \quad (\nabla_m \nabla_p - \nabla_p \nabla_m)(\nabla_r K_{kjih}) = K_{kjih}(\nabla_m \nabla_p - \nabla_p \nabla_m)a_r + a_r(\nabla_m \nabla_p - \nabla_p \nabla_m)K_{kjih},$$

which leads to

Theorem 4.3: *If a Riemannian manifold (M^n, g) is generalised recurrent and it satisfies,*

$$a_t Z_{kjih} K_{mpr}^t + a_r \{Z_{tjih} K_{mpk}^t + Z_{ktih} K_{mpj}^t + Z_{kjth} K_{mpi}^t + Z_{kjit} K_{mph}^t\} = 0,$$

then it is a QSRM-I also.

Let us assume that a Riemannian manifold (M^n, g) is a UV-recurrent manifold⁸, then we have $\nabla_m K_{kjih} = a_m P_{kjih}$, which leads to

Theorem 4.4: *If a Riemannian manifold (M_n, g) is UV-recurrent and satisfies*

$a_t P_{kjih} K_{mpr}^t + a_r \{P_{tjih} K_{mpk}^t + P_{ktih} K_{mpj}^t + P_{kjth} K_{mpi}^t + P_{kjit} K_{mph}^t\} = 0$, then it will be a QSRM-I.

Let us assume that a Riemannian manifold (M^n, g) is satisfying equation (2.10), then we can obtain

Theorem 4.5: *If a Riemannian manifold (M^n, g) is satisfying equation (2.10) as well as*

$a_t C_{kjih} K_{mpr}^t + a_r \{C_{tjih} K_{mpk}^t + C_{ktih} K_{mpj}^t + C_{kjth} K_{mpi}^t + C_{kjit} K_{mph}^t\} = 0$ it will be a QSRM-I.

5. Quarter-Symmetric Riemannian Manifold of Second kind (QSRM-II)

In this section, we shall give the following:

Definition 5.1. A Riemannian manifold M^n for which the curvature tensor K_{kjih} is non-zero, shall be called quarter-symmetric Riemannian manifold of second kind (QSRM-II), if it satisfies

$$(5.1) \quad \nabla_r (\nabla_m \nabla_p - \nabla_p \nabla_m) K_{kjih} = 0.$$

Equation (5.1) can alternatively be expressed as

$$(5.2) \quad K_{mpk}^a \nabla_r K_{ajih} + K_{mpj}^a \nabla_r K_{kaih} + K_{mpi}^a \nabla_r K_{kjah} + K_{mph}^a \nabla_r K_{kjia} \\ + K_{ajih} \nabla_r K_{mpk}^a + K_{kaih} \nabla_r K_{mpj}^a + K_{kjah} \nabla_r K_{mpi}^a + K_{kjia} \nabla_r K_{mph}^a = 0$$

Hence:

Theorem 5.1: *The necessary and sufficient condition for a Riemannian manifold (M^n, g) to be a quarter-symmetric Riemannian manifold of second kind (QSRM-II), is given by equation (5.2).*

If a Riemannian manifold (M^n, g) is both quarter-symmetric Riemannian manifold of first and second kind from equations (4.2) and (5.2), we can obtain

$$(5.3) \quad (\nabla_a K_{kjih}) K_{mpr}^a = K_{ajih} \nabla_r K_{mpk}^a + K_{kaih} \nabla_r K_{mpj}^a + K_{kjah} \nabla_r K_{mpi}^a + \\ K_{kjia} \nabla_r K_{mph}^a$$

Hence:

Theorem 5.2: *A Riemannian manifold (M^n, g) satisfying equation (5.3), shall be both QSRM-I and QSRM-II.*

Remark. Some more interesting properties of QSRM-I and QSRM-II shall be studied in a subsequent paper.

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