

Investigating the Role of Fear and Top Predator Interference on the Spatio-Temporal Dynamics of Food Chain Model

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Abstract: In this paper, attempt has been made to grasp the idea about the role of fear effect of intermediate specialist predator on prey and the effect of the top predator interference in the spatial framework of a three species food chain model. By introducing a hybrid model and assuming that the interaction between prey and intermediate predator obeys the Volterra scheme, whereas Beddington-DeAngelis (BD) type functional response is taken in between the top predator and its favorite food. The stability properties of each equilibrium point of the temporal model system, blow up in finite time for both temporal and spatial systems corresponding to a large initial data are established. We pay attention to illustrate different types of patterns around the neighbourhood of a critical parameter. Choosing appropriate control parameter from the Turing space and using the existence criteria, stable patterns are obtained. The result of theoretical analysis well agrees with the result exhibited through numerical simulations around the critical parameter. Finally, numerical simulation illustrates the role of diffusion in the spatial domain by formations of different patterns.

Keywords: Fear effect, predator interference, Turing instability, food chain model, bifurcation.

1. Introduction

For a given prey-predator ecological system, fear of predator to prey is the key factor to generate different dynamical outcomes. Based on the choice of food, available amount of resource and growth structure, the food consuming process of higher trophic level species are complicated enough and yields interesting dynamical impact on community food-chain/food-web. The effect of fear on prey population leads to the cause of the reduction in the birth rate

of prey population and starvation^{1,2}. For a transient period of time, prey populations change their habitat from high predation zone to low predation zone and return back once the habitat becomes comparatively safe from the predators. This phenomenon can be exemplified as birds make an anti-predatory defence strategy against the sounds of the predators, which shows that the survival rate of birds is increasing while the number of reproduction rate is decreasing^{1,2}. Experimental results have been substantiated that the change of behaviour of the prey in the presence of predator has been noticed and it suggests that the effect is more powerful in comparison to the impact of direct predation³⁻⁷. Besides anti-predator responses, including habitat changes, foraging, vigilance, etc. shown by the prey, they can react to the predation risk, which may enhance their chance of endurance in the ecosystem^{1, 5-8}. The impact of fear of prey populations have directly shown in the reproduction of the prey. The psychological condition of the juvenile preys be affected by the fear of the preys and this leads to a questionable survival of life span as adult preys^{2,5,6,8}. To incorporate the fear effect, we have multiplied the growth term of prey (intrinsic growth rate, a_1) by the fear effect termed $f(l, Y) = 1/(1 + lY)$, where l is predator's fear effect on prey and Y is intermediate specialist predator density and the model system is designed/obtained.

The dynamical study reveals the incorporation of structural dynamics into topographical regions in spatial domain. This type of dynamics plays an important role to sustain the environmental biodiversity. In ecology, irregularities have been perceived over various scales in space and time. These irregularities improve the formation of patterns¹⁰ and demonstrate an environmental patchiness showing influence on the distribution of population. This scenario may be illustrated as species are assembling around water sources than anywhere else during dry season. Diffusion plays an important role in maintaining the stability of the considered system in spatiotemporal domain. Diffusion phenomena is exemplified as a number of population accumulate around an initial point in space and over the time is running out, the population expand over a large area near about the starting point. Reaction-diffusion models^{9,11} are investigated and studied elaborately. Diffusive instability of environmental systems have been arise. Spatial variability of the population carries out a momentous role under the restriction of mathematical conditions before or nearby the finite time of blow up. Population density along with time is the primary observation in the spatial domain. Nature of predator interference and population explosion of invasive species give a clear imagination in a certain paradox in ecology. In the past, it has been shown that many researchers have studied on infinitesimal

perturbation focusing the uniform state to saturation, stationary, spatially periodic solutions etc¹²⁻¹⁴. Employing analytical and numerical methods, the properties of nonlinear dynamical system is derived around the threshold point under some parametric conditions. Solutions that are associated with real world problems are fitted over finite domain and those are not excellent possibilities such as in long length scales, spatial domains etc. Furthermore, the limitations are completely dependent over time. Perturbation, characteristics of nonlinearity and dispersion are essential ingredients to form patterns in a spatial domain. As a matter of fact, we consider a small domain in spatial region and explore the patterns of a dynamical system over the domain. A major slow down has been observed near bifurcation threshold in the dynamical system.

From the ecological point of view, we describe the blow up phenomenon as explosive growth rate of species. Mathematically, we demonstrate blow-up at finite time of a nonlinear process via ordinary differential equation as

$$\lim_{t \rightarrow T^* < \infty} \|z\| \rightarrow \infty,$$

where the standard sup norm on $\mathbb{R}^n$ is denoted as $\|\cdot\|$, the variable z relies on time (t), and the finite time blow-up is T^* . When $T^* = \infty$, blow-up does not take place and global existence of the solution exists for all the time. Blow-up at finite time T^* is considered as inversion of the ecological system. Species that grow rapidly and diffuse in a way so that they have potential to harm the environment, are termed as *invasive* species. Invasive species have the tendency to outbreak its maximum level to cause environmental damage which has direct effect on agriculture, forestry, fisheries, human health and economy etc. Sometimes environmental conditions such as variation in weather temperatures, alternates the rain and diversity in snow patterns help to amplify the invasive populations. To control these species, pesticide residue and mycotoxin are spread over the coffee berries and it directly affects the exportation of the coffee market. It has been recorded that every year 200 millions dollars are spent to control these invasive species¹⁵.

In this manuscript, the formulation of model system and their attributes are elaborately discussed in section 2. The existence criteria and stability of the equilibrium points of the temporal model are discussed in section 2.1. Blow-up phenomenon of the designed model system at finite time in temporal and spatial domains are illustrated in section 3. Existence/non-existence criteria of Turing stability/instability are presented in sections 4. Numerical experimental results are presented in section 5. Conclusions and discussions are briefly presented in section 6.

2. Design of the Model System

Firstly, we take prey as X , specialist middle and generalist top predator as Y and Z respectively. The food source of specialist intermediate predator population (Y) is considered as prey population (X) and the food source of generalist top predator population (Z) is considered as intermediate predator population (Y). Here, the predation process of intermediate predator (Y) on prey (X) is Holling type III response and generalist conforms Beddington–DeAngelis (BD) functional response on its prey (Y). Thus, the formulation of the model system is as follows:

$$\begin{aligned}
 (2.1) \quad & \frac{\partial X}{\partial t} = \frac{a_1 X}{1 + lY} - b_1 X^2 - \frac{wX^2 Y}{X^2 + D} + d_1 \nabla^2 X, \\
 & \frac{\partial Y}{\partial t} = -a_2 Y + \frac{w_1 X^2 Y}{X^2 + D_1} - \frac{w_2 Y Z}{Y + \beta Z + D_2} + d_2 \nabla^2 Y, \\
 & \frac{\partial Z}{\partial t} = cZ^2 - \frac{w_3 Z^2}{Y + D_3} + d_3 \nabla^2 Z,
 \end{aligned}$$

where all parameters are positive constants and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. d_1, d_2, d_3 are diffusive constants of prey, middle predator and top predator respectively and $X(x, y, 0) > 0$, $Y(x, y, 0) > 0$, $Z(x, y, 0) > 0$. The elementary properties of differential equations¹⁶ yields a unique solution of the form $(X(t), Y(t), Z(t))$ of the temporal system (2.1). According to the Volterra scheme, the governing equation exhibiting the rate of change of population size for prey and intermediate predator has been written, implying that while prey population is not present, predator species dies out in exponential way. The intermediate predator Y and the top predator Z interact following the BD type functional response. Here a_1 be the reproduction rate of the prey (X), the environmental carrying capacity of X is K and we use the notation b_1 as $b_1 = \frac{a_1}{K}$. The fear effect of prey is represented by l . The rate of death and the efficiency of food conversion Y are termed as a_2 and w_1 respectively. The residual loss of the top predator Z for the major deficiency of its favourite food Y is termed as D_3 . The functional response of Y consuming X is represented by Holling type III, which is a prey dependent. The functional response of Z population upon Y is illustrated by Beddington-DeAngelis type functional response where the factor of mutual interference of Z is being considered. w, w_2 are the per capita reduction rate of the maximum values of X and Y may accomplish respectively, whereas w_1 denotes the rate of growth per capita, Y can achieve. w_3 be the rate of death of the top predator. β appears in the functional response of BD type and represent the intensity of interference between individuals of

predators. The growth of the generalist top predator follows sexually reproductive ratio where the probability of mating is in proportion to the male and female individuals within the populations ($Z \times Z = Z^2$)¹⁷⁻¹⁸. The decay of the predator population is mainly because of intra-species competition as $(-\frac{w_3}{Y+D_3})Z^2$. When the intermediate predator (Y) is absent, the top predator (Z) no longer exist if $cD_3 < w_3$ and grows without limit if the inequality reverses, which is not biologically acceptable situation¹⁹. c denote the growth rate of a generalist top predator and Z^2 signifies the fact that mating frequency is directly proportional to the number of male as well as female individuals. D_3 normalizes the residual reduction in the predator population because of severe scarcity of the favorite food. The characteristics of the top predator having sexual reproductive rate of growth has been studied by the author^{18, 20}.

2.1 Stability Analysis of the model system

Biologically, stability of a population often signifies the non-extinction of that individual population. Linear stability analysis demonstrates about the stability domain for a model system on introduction of small disturbances. Now, we perform the stability analysis of the temporal system.

Six possible non-negative equilibrium points $E_0(0,0,0)$, $E_X(K, 0, 0)$, $E_Z(0,0,\tilde{Z})$, $E_{XY}(\hat{X}, \hat{Y}, 0)$, $E_{XZ}(K, 0, \tilde{Z})$ and $E_{XYZ}(X^*, Y^*, Z^*)$ have been derived from the temporal system (2.1). The stability conditions and feasibility of five boundary equilibrium points of the system (2.1), namely $E_0(0,0,0)$, $E_X(K, 0, 0)$, $E_Z(0,0,\tilde{Z})$, $E_{XY}(\hat{X}, \hat{Y}, 0)$, $E_{XZ}(K, 0, \tilde{Z})$ are enlisted in Table I.

Table I. The analysis of the stability of the boundary equilibria of system (3.1)

	Equilibrium Points	The conditions of feasibility	Jacobian matrix and its correspondig eigenvalues	The status of stability
(i)	$E_0(0,0,0)$	always	$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = a_1, \lambda_2 = -a_2, \lambda_3 = 0$	Neutral
(ii)	$E_X(K, 0, 0)$	always	$\begin{pmatrix} -a_1 & la_1K - \frac{wK^2}{K^2 + D} & 0 \\ 0 & -a_2 + \frac{w_1K^2}{K^2 + D_1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = -a_1, \lambda_2 = -a_2 + \frac{w_1K^2}{K^2 + D_1}, \lambda_3 = 0$	Neutral

(iii)	$E_Z(0,0,\tilde{Z})$, $\tilde{Z}$ is any +ve constant	$c = \frac{w_3}{D_3}$	$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & -a_2 - \frac{w_2 \tilde{Z}}{\beta \tilde{Z} + D_2} & 0 \\ 0 & \frac{w_3 \tilde{Z}^2}{D_3^2} & 2\tilde{Z} \left(c - \frac{w_3}{D_3} \right) \end{pmatrix}$ $\lambda_1 = a_1, \lambda_2 = -a_2 - \frac{w_2 \tilde{Z}}{\beta \tilde{Z} + D_2}, \lambda_3 = 2\tilde{Z} \left(c - \frac{w_3}{D_3} \right) = 0$	Neutral
(iv)	$E_{XY}(\hat{X}, \hat{Y}, 0)$ $\hat{X} = \sqrt{\frac{a_2 D_1}{a_2 - w_1}}$ $\hat{Y} = \frac{-A_2 + \sqrt{A_2^2 - 4A_1 A_3}}{2A_1}$ $A_1 = Kw l \hat{X}$, $A_2 = Kw \hat{X} + a_1 l \hat{X}(\hat{X}^2 + D)$, $A_3 = a_1(\hat{X}^2 + D)(\hat{X} - K)$	$a_2 > w_1$, $\hat{X} < K$	$\begin{pmatrix} \frac{a_1}{1+l\hat{Y}} - \frac{2a_1 \hat{X}}{K} - \frac{2wD\hat{X}\hat{Y}}{(\hat{X}^2 + D)^2} & \frac{la_1 \hat{X}}{(1+l\hat{Y})^2} - \frac{w\hat{X}^2}{\hat{X}^2 + D} & 0 \\ \frac{2w_1 D_1 \hat{X} \hat{Y}}{\hat{X}^2 + D_1} & -a_2 + \frac{w_1 \hat{X}^2}{\hat{X}^2 + D_1} & -\frac{w_2 \hat{Y}}{(\hat{Y} + D_2)^2} \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_1 = - \left(-\frac{a_1}{1+l\hat{Y}} + \frac{2a_1 \hat{X}}{K} + \frac{2wD\hat{X}\hat{Y}}{(\hat{X}^2 + D)^2} \right), \lambda_2 = - \left(a_2 - \frac{w_1 \hat{X}^2}{\hat{X}^2 + D_1} \right), \lambda_3 = 0$	Neutral
(v)	$E_{XZ}(K, 0, \tilde{Z})$ $\tilde{Z}$ is any +ve constant	$c = \frac{w_3}{D}$	$\begin{pmatrix} -a_1 & ka_1 K - \frac{wK^2}{K^2 + D} & 0 \\ 0 & -a_2 - \frac{w_2 \tilde{Z}}{\beta \tilde{Z} + D_2} + \frac{w_1 K^2}{K^2 + D_1} & 0 \\ 0 & \frac{w_3 \tilde{Z}^2}{D_3^2} & 2\tilde{Z} \left(c - \frac{w_3}{D_3} \right) \end{pmatrix}$ $\lambda_1 = -a_1, \lambda_2 = - \left(a_2 + \frac{w_2 \tilde{Z}}{\beta \tilde{Z} + D_2} - \frac{w_1 K^2}{K^2 + D_1} \right), \lambda_3 = 2\tilde{Z} \left(c - \frac{w_3}{D_3} \right) = 0$	Neutral

The non-trivial equilibrium point of the model system (2.1) are termed as $E_{XYZ}(X^*, Y^*, Z^*)$ which are calculated in the following:

$E_{XYZ}(X^*, Y^*, Z^*)$ be the coexistence equilibrium point of system (2.1), where

$$Y^* = \frac{w_3}{c} - D_3, Z^* = \frac{(Y^* + D_2)[a_1 w_1 \{K - (1 + lY^*)X^*\}X^* - a_2 wK(1 + lY^*)Y^*]}{wK(1 + lY^*)(w_2 + a_2 \beta)Y^* - a_1 w_1 \beta \{K - (1 + lY^*)X^*\}X^*}.$$

Let us consider $1 + l \left(\frac{w_3}{c} - D_3 \right) = q$ (say) and assume that $0 < X^* < \frac{K}{q}$ and $w_3 > cD_3$ and X^* denote the solution of the following equation as

$$f(X^*) = X^{*3} - \frac{K}{q}X^{*2} + \left(b + \frac{Kw}{a_1} \left(\frac{w_3}{c} - D_3 \right) \right) X^* - \frac{Kb}{q}.$$

From above we obtain, $f(0) = -\frac{Kb}{q} < 0$ and $f\left(\frac{K}{q}\right) = \frac{K^2 w}{a_1 q} \left(\frac{w_3}{c} - D_3 \right) > 0$ as $w_3 > cD_3$. Since $f(0)f\left(\frac{K}{q}\right) < 0$, the positive root continues to exist

in between $\left(0, \frac{K}{q}\right)$. The existence of non-trivial equilibrium point $E_{XYZ}(X^*, Y^*, Z^*)$ satisfies.

Now, we derive the criteria of the stability of the coexistence equilibrium point $E_{XYZ}(X^*, Y^*, Z^*)$ of the non-spatial system (2.1). The characteristic equation at $E_{XYZ}(X^*, Y^*, Z^*)$ is given as follows:

$$(2.2) \quad P(\lambda) = \lambda^3 + \mu_2 \lambda^2 + \mu_1 \lambda + \mu_0 = 0,$$

having coefficients as

$$\mu_2 = -(J_{11} + J_{22} + J_{33}),$$

$$\mu_1 = J_{11}J_{33} + J_{11}J_{22} + J_{22}J_{33} - J_{32}J_{23} - J_{12}J_{21} - J_{13}J_{31},$$

$$\mu_0 = J_{11}J_{32}J_{23} + J_{22}J_{13}J_{31} + J_{33}J_{12}J_{21} - J_{11}J_{22}J_{33} - J_{31}J_{23}J_{12} - J_{13}J_{21}J_{32},$$

where $J = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix},$

describes as Jacobian matrix, which have been obtained at the non-trivial equilibrium point $E_{XYZ}(X^*, Y^*, Z^*)$ of the system (2.1). The entries of the Jacobian matrix are enlisted as follows:

$$\begin{aligned} J_{11} &= \frac{a_1}{1+lY} - 2b_1X - \frac{2wDXY}{(X^2+D)^2}, J_{12} = -\frac{a_1lX}{(1+lY)^2} - \frac{wX^2}{X^2+D}, \\ J_{13} &= 0, J_{21} = \frac{2w_1D_1XY}{(X^2+D_1)^2}, J_{22} = -a_2 + \frac{w_1X^2}{X^2+D_1} - \frac{(\beta Z+D_2)w_2Z}{(Y+\beta Z+D_2)^2}, \\ J_{23} &= \frac{-(Y+D_2)w_2Y}{(Y+\beta Z+D_2)^2}, J_{31} = 0, J_{32} = \frac{w_3Z^2}{(Y+D_3)^2}, J_{33} = 2Z \left(c - \frac{w_3}{Y+D_3} \right). \end{aligned}$$

Routh-Hurwitz criterion yields $E_{XYZ}(X^*, Y^*, Z^*)$ is locally asymptotically stable for the real part of the eigenvalue i.e., $\text{Re}(\lambda) < 0$ of temporal model (2.1), iff the following properties are satisfied:

$$(2.3) \quad \mu_2 > 0, \mu_1 > 0 \text{ and } [\mu_2\mu_1 - \mu_0] > 0.$$

3. Finite Time Blow-Up

The phenomena of exhibiting blow-up at finite time, from the biological view-point has been explored in the literatures²³⁻²⁶. Recently, some research

works have been started with the help of mathematical property termed as finite time blow up²¹⁻²². One can formulate the finite time blow-up as uncontrollable populations level and that is surely disaster for the ecosystem.

3.1 Finite Time Blow-Up of temporal System

Theorem 3.1: Consider the model system (2.1) for non-spatial cases and choose the non-spatial model system as follows:

$$(3.1) \quad \begin{aligned} \frac{dX}{dt} &= \frac{a_1 X}{1+lY} - b_1 X^2 - \frac{wX^2 Y}{X^2 + D}, \\ \frac{dY}{dt} &= -a_2 Y + \frac{w_1 X^2 Y}{X^2 + D_1} - \frac{w_2 YZ}{Y + \beta Z + D_2}, \\ \frac{dZ}{dt} &= cZ^2 - \frac{w_3 Z^2}{Y + D_3}. \end{aligned}$$

$Z(t)$ be the solution of (3.1), which has blow-up phenomena at finite time for non-spatial case, that is $\lim_{t \rightarrow T^* < \infty} \|Z(t)\|_\infty \rightarrow \infty$, for sufficient large values of initial data (X_0, Y_0, Z_0) .

Proof: Following the process given in Parshad *et al.*²² and on integrating the last equation of (3.1), we get

$$\frac{-1}{Z} + \frac{1}{Z_0} = ct - \omega_3 \int_0^t \frac{ds}{Y(s) + D_3},$$

which gives

$$Z = \frac{1}{\frac{1}{Z_0} - ct + \omega_3 \int_0^t \frac{ds}{Y(s) + D_3}}.$$

Our aim is to establish that the function

$$(3.2) \quad \phi(t) = \frac{1}{Z_0} - ct + \omega_3 \int_0^t \frac{ds}{Y(s) + D_3} \rightarrow 0 \text{ as } t \rightarrow T^* > 0.$$

Then Z be the solution that can has finite time blow-up property at $t = T^*$, by following the same process of Parshad *et al.*²². Essentially, for Y_0 taken large enough, there exists a $\delta > 0$ such that for all $t \in (0, \delta)$,

$$\frac{1}{Z_0} - ct + \omega_3 \int_0^t \frac{ds}{Y(s) + D_3} = \frac{1}{Z_0} + \left[-c + \omega_3 \frac{1}{t} \int_0^t \frac{ds}{Y(s) + D_3} \right] t < \frac{1}{Z_0} - \frac{c}{2} t.$$

If now Z_0 is taken sufficiently large, then one can obtain $T^* \in (0, \delta)$ so that $\frac{1}{Z_0} - \frac{c}{2}T^* = 0$, which proves the required result. By applying the mean value theorem, there exists some $T^* \in (0, \delta)$, $T^* < T^{**}$, satisfying $\emptyset(T^*)=0$. This indicates that the solution of (3.1) is $Z(t)$, which performs finite time blow-up phenomena at $t = T^*$. It follows the required result.

3.2. Sufficient Conditions for Blow-Up of Non-spatial System

We now discuss the sufficient conditions required for the appearance of the finite time blow-up, depending on (X_0, Y_0, Z_0) , initially small or large. For the model system (3.1), we explore the finite time blow-up in the temporal domain.

Theorem 3.2: *For the model system (3.1), we can select any parameter from the system (3.1) satisfying the criteria $\delta_1 > 0$ so that $c > \delta_1$. Then, the solution Z (state variable) of the model system (3.1) blows up in finite time for large enough initial data satisfying the condition as follows*

$$(3.3) \quad |Z_0| \ln \left(\frac{|Y_0|}{\frac{w_3}{c-\delta_1} - D_3} \right) > \frac{a_2\beta + w_2}{\delta_1\beta}$$

The blow-up time is defined as $T^* \leq \frac{1}{\delta_1|Y_0|}$.

Proof: Assume that the equation of the top predator

$$\frac{dZ}{dt} = \left(c - \frac{w_3}{Y + D_3} \right) Z^2,$$

of which the finite time blow-up phenomena is trivial, if $c > \frac{w_3}{D_3}$. However if $c < \frac{w_3}{D_3}$, to make sure the finite time blow-up, we must guarantee that

$$(3.4) \quad Y > \left(\frac{w_3}{c-\delta_1} - D_3 \right).$$

To get the result, we use middle predator equation for non-spatial case i.e.,

$$\frac{dY}{dt} = -a_2Y + \frac{w_1X^2Y}{X^2+D_1} - \frac{w_2YZ}{Y+\beta Z+D_2},$$

and we have via positivity

$$\frac{dY}{dt} \geq -a_2Y - \frac{w_2YZ}{Y+\beta Z+D_2} \geq -a_2Y - w_2 \frac{YZ}{\beta Z} \geq -\left(\frac{a_2\beta + w_2}{\beta} \right) Y.$$

Using (3.4), we estimate the lower bound of Y and the condition needs to satisfy as follows:

$$|Y| > |Y_0| e^{-\frac{(a_2\beta+w_2)}{\beta}t} > \left(\frac{w_3}{c-\delta_1} - D_3 \right).$$

From above we take the following inequality into consideration:

$$|Y_0| e^{-\frac{(a_2\beta+w_2)}{\beta}t} > \left(\frac{w_3}{c-\delta_1} - D_3 \right).$$

Now, after a simple calculation we obtain

$$\left(\frac{\beta}{a_2\beta+w_2} \right) \ln \left(\frac{|Y_0|}{\frac{w_3}{c-\delta_1}-D_3} \right) > t.$$

Note that

$$\frac{dz}{dt} = \delta_1 Z^2,$$

blows-up at time $T^* = \frac{1}{\delta_1|Z_0|}$. Thus, if we select data that follows

$$\frac{\beta}{(a_2\beta+w_2)} \ln \left(\frac{|Y_0|}{\frac{w_3}{c-\delta_1}-D_3} \right) > t > T^* = \frac{1}{\delta_1|Z_0|}.$$

Therefore, Y should remain above of the critical level $(\frac{w_3}{c-\delta_1} - D_3)$, due to the sufficient large time to blow-up Z at finite time. We conclude from the following condition

$$|Z_0| \ln \left(\frac{|Y_0|}{\frac{w_3}{c-\delta_1}-D_3} \right) > \frac{a_2\beta+w_2}{\delta_1\beta},$$

Z should have finite time blow-up, with the selection of the parameters are purely independent. This proves the theorem.

3.3. Blow-Up Phenomena of Spatial System at Finite Time

For large values of initial data, the finite time blow up phenomenon is discussed. Because of a certain similarity, the equation governing the generalist predator corresponding to the model system (2.1) is considered and we construct a theorem using First Eigenvalue Method (FEM) and Jensen's inequality with a probability measure, implying Z blows up under certain parametric restrictions. Jensen's inequality with a probability measure is defined as follows:

Lemma 3.3²⁷: Assume $p(x)$ be a probability measure on $\mathbb{R}$ and consider v be a non-negative function. When f is a convex function, then the following condition holds:

$$(3.5) \quad \int_{\mathbb{R}} f(v(x))p(x) dx \geq f\left(\int_{\mathbb{R}} v(x)p(x) dx\right).$$

Theorem 3.4: Consider the model system (2.1) which have been mentioned as a spatial tri-trophic food chain model. With the parametric criteria $c > \frac{w_3}{D_3}$ together with large enough initial value Z_0 such as

$$(3.6) \quad F(0) \ln\left[\frac{(cD_3 - w_3) \cdot F(0)}{d_3 \lambda_s D_3}\right] > \frac{d_3 \lambda_s}{c},$$

where $F(0) = \int_{\Omega} Z_0(u)W_1(u) du$. Then there exists finite time T^* such that

$$\lim_{t \rightarrow T^*} \int_{\Omega} Z(u, t)W_1(u) du = \infty.$$

$W_1(u)$ be a function such that $\nabla^2 W_1(u) = -\lambda_s W_1(u)$ on the domain Ω , $W_1(u) = 0$ on $\partial\Omega$. Also $W_1(u) > 0$ and $\int_{\Omega} W_1(u) du = 1$.

Proof: It follows from the spatial model system (2.1), the equation governing the rate of change of generalist predator as follows:

$$\frac{\partial Z}{\partial t} = cZ^2 - \frac{w_3 Z^2}{Y + D_3} + d_3 \nabla^2 Z.$$

On re-writing the equation yields

$$\frac{\partial Z}{\partial t} + \frac{w_3 Z^2}{Y + D_3} = cZ^2 + d_3 \nabla^2 Z.$$

Since $Y > 0$,

$$\frac{\partial Z}{\partial t} + \frac{w_3 Z^2}{D_3} \geq cZ^2 + d_3 \nabla^2 Z.$$

Therefore,

$$(3.7) \quad \frac{\partial Z}{\partial t} \geq d_3 \nabla^2 Z + \left(c - \frac{w_3}{D_3}\right) Z^2.$$

Let us assume that $F(t) = \int_{\Omega} Z(u, t)W_1(u) du$. On multiplying both sides of (3.7) by $W_1(u)$ and integrating by parts, we obtain,

$$\begin{aligned}
 F'(t) &= \int_{\Omega} \frac{\partial Z}{\partial t} W_1(u) du \\
 &\geq \int_{\Omega} \left(d_3 \nabla^2 Z + \left(c - \frac{w_3}{D_3} \right) Z^2 \right) W_1(u) du.
 \end{aligned}$$

Therefore, from above we have

$$(3.8) \quad F'(t) \geq -d_3 \lambda_s F(t) + \left(c - \frac{w_3}{D_3} \right) \int_{\Omega} Z^2 W_1(u) du.$$

Assume that the probability measure $p(x)$, defined on eigenfunction $W_1(x)$, over the domain Ω be such that

$$p(x) = \begin{cases} W_1, & \text{on } \Omega \\ 0, & \text{on } R \setminus \Omega \end{cases}$$

Here $f(Z) = Z^2$ represents a convex function and with the help of Jensen's inequality, one can obtain

$$\begin{aligned}
 \left(c - \frac{w_3}{D_3} \right) \int_{\Omega} Z^2 W_1(u) du &\geq \left(c - \frac{w_3}{D_3} \right) \left(\int_{\Omega} Z(u, t) W_1(u) du \right)^2 \\
 &= \left(c - \frac{w_3}{D_3} \right) (F(u))^2.
 \end{aligned}$$

Thus, replacing the above inequality in equation (3.8), we obtain

$$(3.9) \quad F'(t) \geq -d_3 \lambda_s F(t) + \left(c - \frac{w_3}{D_3} \right) (F(u))^2.$$

Algebraic calculations leads us to the following criteria required to satisfy for the occurrence of blow-up phenomenon

$$F(t) \geq \frac{d_3 \lambda_s D_3}{(c D_3 - w_3)},$$

such that

$$F(0) e^{-d_3 \lambda_s t} > \frac{d_3 \lambda_s D_3}{(c D_3 - w_3)}.$$

Thus, we obtain

$$\frac{1}{d_3 \lambda_s} \ln \frac{(c D_3 - w_3) F(0)}{d_3 \lambda_s D_3} > t.$$

Assuming that $\frac{dF}{dt} = c(F(t))^2$. At $T^* =$ (blow up time), from the above we obtain

$$\frac{1}{d_3 \lambda_s} \ln \frac{(cD_3 - w_3)F(0)}{d_3 \lambda_s D_3} > t > T^*.$$

Now, the blow up criteria satisfies

$$F(0) \ln \left[\frac{(cD_3 - w_3) \cdot F(0)}{d_3 \lambda_s D_3} \right] > \frac{d_3 \lambda_s}{c}.$$

This proves the theorem.

Table II. Conditions of blow-up phenomena at finite time in non-spatial and spatial domains of the system (2.1)

Model System	Blow-up condition of non-spatial case for Z		Figure	Blow-up condition of spatial case for Z	Figure
(3.1)	$c > \frac{w_3}{D_3}$	$c < \frac{w_3}{D_3}$		$c > \frac{w_3}{D_3}$	
	Trivial	$ Z_0 \ln \left(\frac{ Y_0 }{\frac{w_3}{c - \delta_1} - D_3} \right) > \frac{a_2 \beta + w_2}{\delta_1 \beta}$	(2a)	$F(0) \ln \left[\frac{(cD_3 - w_3) \cdot F(0)}{d_3 \lambda_s D_3} \right] > \frac{d_3 \lambda_s}{c}$	(3a)

4. SpatioTemporal Dynamics

This section discusses about Turing instability of our present model system (2.1). The calculations for the temporal as well as spatial Turing patterns are presented. The stability criteria around the non-trivial equilibrium state (X^*, Y^*, Z^*) are derived. Our main objective is to linearize our model system (2.1) near the homogeneous steady state (X^*, Y^*, Z^*) , depending upon time and space. Hence, we get the following set of equations²⁹:

$$(4.1 \text{ a}) \quad X = X^* + \hat{X}(\xi, t),$$

$$(4.1 \text{ b}) \quad Y = Y^* + \hat{Y}(\xi, t),$$

$$(4.1 \text{ c}) \quad Z = Z^* + \hat{Z}(\xi, t),$$

where $|\hat{X}(\xi, t)| \ll X^*$, $|\hat{Y}(\xi, t)| \ll Y^*$ and $|\hat{Z}(\xi, t)| \ll Z^*$. Usually, we consider

$$\begin{bmatrix} \hat{X}(\xi, t) \\ \hat{Y}(\xi, t) \\ \hat{Z}(\xi, t) \end{bmatrix} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} e^{\lambda t + i k \xi},$$

where ϵ_i for $i = 1, 2, 3$ denote the associated amplitudes, k represents the wave number, the perturbation growth rate is described by λ at time t and ξ is the spatial coordinate. On substituting the values of (4.1) into model system (2.1) and neglecting higher order terms, yields the characteristic equation as

$$(4.2) \quad (\mathbf{J} - \lambda \mathbf{I} - k^2 \mathbf{D}) \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = 0,$$

whereas

$$(4.3) \quad \mathbf{J} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}, \quad \mathbf{D} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}.$$

Here, the community matrix $\mathbf{J}$ enlisted in section 3 and $\mathbf{I}$ represents the identity matrix. Now, From Equation (4.2), we get

$$\begin{vmatrix} J_{11} - \lambda - k^2 d_1 & J_{12} & J_{13} \\ J_{21} & J_{22} - \lambda - k^2 d_2 & J_{23} \\ J_{31} & J_{32} & J_{33} - \lambda - k^2 d_3 \end{vmatrix} = 0,$$

which shows the relation of dispersion²⁸. Now, we compute the stability domain near the non-trivial steady state (X^*, Y^*, Z^*) and on writing the dispersion relation a cubic polynomial function is found as below

$$(4.4) \quad p(\lambda(k^2)) = \lambda^3 + \mu_2(k^2)\lambda^2 + \mu_1(k^2)\lambda + \mu_0(k^2)$$

with coefficients

$$\mu_2(k^2) = k^2(d_1 + d_2 + d_3) - (J_{11} + J_{22} + J_{33}),$$

$$\mu_1(k^2) = J_{11}J_{33} + J_{11}J_{22} + J_{22}J_{33} - J_{32}J_{23} - J_{12}J_{21} - J_{13}J_{31} - k^2((d_1 + d_3)J_{22} + (d_1 + d_2)J_{33} + (d_2 + d_3)J_{11}) + k^4(d_2d_3 + d_2d_1 + d_1d_3),$$

$$\mu_0(k^2) = J_{11}J_{32}J_{23} + J_{22}J_{13}J_{31} + J_{33}J_{12}J_{21} - J_{11}J_{22}J_{33} - J_{31}J_{23}J_{12} - J_{13}J_{21}J_{32} + k^2(d_1(J_{22}J_{33} - J_{32}J_{23}) + d_2(J_{11}J_{33} + J_{13}J_{31}) + d_3(J_{11}J_{22} - J_{12}J_{21})) - k^4(d_2d_1J_{33} + d_1d_3J_{22} + d_2d_3J_{11}) + k^6d_1d_2d_3.$$

According to the Routh-Hurwitz criteria for stability of the steady state, the real part of the characteristic value should be negative i.e., $\text{Re}(\lambda) < 0$. This gives the stability domain around the steady state point $E_{XYZ}(X^*, Y^*, Z^*)$ which requires the following properties to be satisfied:

$$(4.5) \quad \mu_2(k^2) > 0, \mu_1(k^2) > 0, \mu_0(k^2) > 0 \text{ and } [\mu_2\mu_1 - \mu_0](k^2) > 0.$$

Violation of the conditions mentioned above indicates about the instability (i.e. $\text{Re}(\lambda) > 0$) of the system (2.1). Now, we slightly disturb the systems about the homogeneous steady state (X^*, Y^*, Z^*) and explore the stability criteria of the system (2.1) in presence or absence of diffusion coefficients within certain range of values of k . The following criteria are provided below:

$$\text{Re}(\lambda(k^2 > 0)) > 0, \text{ for some } k \text{ and } \text{Re}(\lambda(k^2 = 0)) < 0,$$

where k is assumed to take real and positive values nonetheless k may be complex valued. This is known as *diffusion driven* or *Turing instability*. White and Gillian²⁸ discussed that three species food chain model display this kind of characteristics with several pattern formations. The stability conditions in the absence of diffusion coefficients of the system (2.1) are:

$$\mu_2(k^2 = 0) > 0, \mu_1(k^2 = 0) > 0, \mu_0(k^2 = 0) > 0 \text{ and} \\ [\mu_2\mu_1 - \mu_0](k^2 = 0) > 0.$$

on the other hand, it follows from the coefficients of (4.4), that the presence of diffusion coefficients ($k^2 > 0$), implies instability near the homogeneous steady state. It requires that atleast one of the signs of (4.5) need to be reversed, to get the unstable condition of the model system (3.1). This requires elaborate investigations of $\mu_2(k^2)$. Regardless of the value of the value k^2 , the value of $\mu_2(k^2)$ have to be positive as $(J_{11} + J_{22} + J_{33})$ stays negative. Therefore, the diffusion driven instability conditions can be written as:

$$(4.6) \quad \mu_0(k^2) \text{ and } [\mu_2\mu_1 - \mu_0](k^2).$$

These two functions form a cubic equation in k^2 , given by

$$G(k^2) = H_H + k^2 D_D + (k^2)^2 C_C + (k^2)^3 B_B, \text{ with } B_B > 0 \text{ and } H_H > 0.$$

The general frame of the coefficients of $G(k^2)$ are mentioned in White and Gillian²⁸. We now compute the minimum value of Turing point (K_T^2) , which is obtained from the minimum value of k^2 , so that the value of $G(k^2 = K_T^2) < 0$ for the criteria either $\mu_0(k^2)$ or $[\mu_2\mu_1 - \mu_0](k^2)$ negative for some values of k . This is possible only when

$$k^2 = k_T^2 = \frac{-C_C + \sqrt{C_C^2 - 3B_B D_D}}{3B_B},$$

this validates that k^2 is takes positive real values such that the criteria $\partial^2 G / \partial^2(k^2) > 0$ occurs and from which we obtain either

$$(4.7) \quad D_D < 0 \text{ or } C_C < 0,$$

which assures

$$C_C^2 - 3B_B D_D > 0.$$

Therefore $G(k^2) < 0$, if at $k^2 = K_T^2$

$$(4.8) \quad G_{min}(k^2) = 2C_C^3 - 9B_B C_C D_D - 2(C_C^2 - 3B_B D_D)^{3/2} + 27B_B^2 H_H < 0.$$

Therefore the necessity and sufficiency for generating pattern formation is validated by the conditions (4.7)-(4.8) near the coexistence steady state (X^*, Y^*, Z^*) around the diffusion driven instability domain. Again, for each cases, H_H should remain positive, to show stability whenever $k = 0$.

Now, on taking into account the one dimensional diffusion, one may replace $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ by $\nabla \equiv \frac{\partial}{\partial x}$ or $\nabla \equiv \frac{\partial}{\partial y}$ in the model system (2.1) where d_1 , d_2 and d_3 are diffusion constants of $X(x,t)$, $Y(x,t)$ and $Z(x,t)$ respectively with $X(x,0) > 0$, $Y(x,0) > 0$, $Z(x,0) > 0$ with said Neumann boundary criteria $X_x = Z_z = Y_y = 0$ taking domain as $\Omega = [0, \pi]$.

Theorem 4.1: *Consider the food-chain system (2.1) with 1D diffusion, and the community matrix J of reaction terms. Consider a parameter set for which J is stable, whilst the conditions (4.7)-(4.8) are satisfied. Then there exist diffusion coefficients (d_1, d_2, d_3) for which the spatially homogeneous steady state (X^*, Y^*, Z^*) can be driven unstable due to diffusion-driven instability.*

5. Numerical Simulation Results

First, we explore the numerical simulation results of the system (3.1) in the temporal domain. Taking the parameter set as given in Table III and illustrate the stability of the coexistence equilibrium point $E_{XYZ}(X^*, Y^*, Z^*)$ and the possibility of finite time blow up phenomena in the designed model system.

Table III. Parameters sets for the stability of coexistence equilibrium, finite time blow-up and stability regions with varying initial values Y_0 and Z_0 of the model system (3.1)

Non-spatial model of	Parameters set of the stability of coexistence equilibriums	Figure	Parameters set of finite time blow-up	Figure	Parameters set of the stability regions	Figure
(3.1)	$a_1 = 1, l = 0.02, b_1 = 0.03, a_2 = 0.05, b = 60, w = 0.75, w_1 = 0.6, w_2 = 0.2, w_3 = 0.42, \gamma = 50, \beta = 0.21, c = 0.03, D = 1$ IC: (10,8,7)	1(a)	$a_1 = 1, l = 0.02, b_1 = 0.03, a_2 = 0.05, b = 60, w = 0.75, w_1 = 0.6, w_2 = 0.2, w_3 = 0.5, \gamma = 60, \beta = 0.21, c = 0.08, D = 10$ IC: (10,8,7)	1(b)	$a_2 = 0.05, w_2 = 0.2, w_3 = 0.95, \delta_1 = 0.03, \beta = 0.21, c = 0.08, D = 1$	2(a)

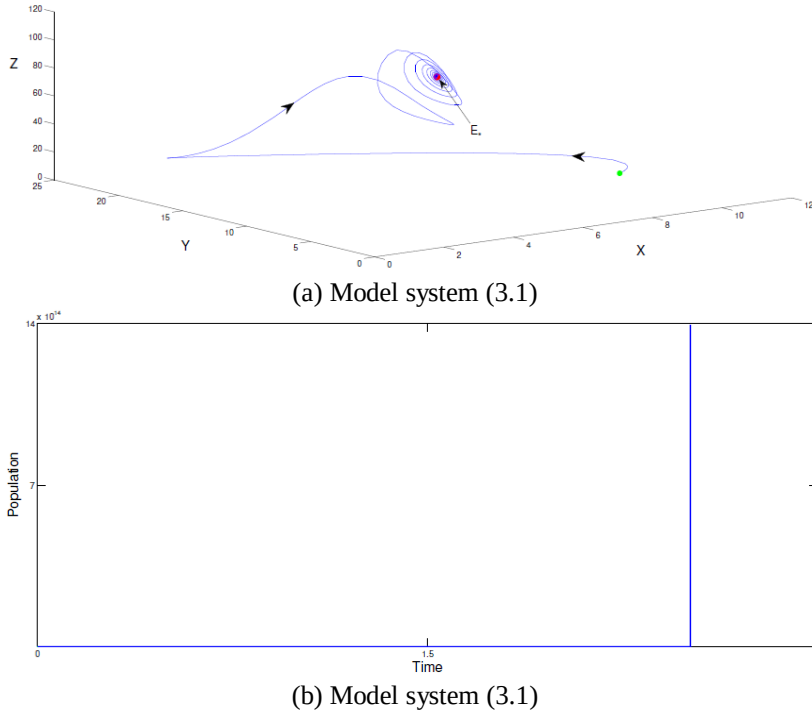


Figure 1. Phase portrait diagram of asymptotic stability of coexistence equilibrium and time series of the blow-up phenomenon at finite time of the non-spatial model system (3.1) is given in first and second panels respectively. Parameter sets are listed in Table III.

In the 1st panel of the Figure 1, we portrait the phase space diagrams of the systems (3.1) where the coexistence equilibrium point $E_{XYZ}(X^*, Y^*, Z^*)$ is asymptotically stable. In the 2nd panel of the Figure 1, the finite time blow-up phenomena of the top predator (Z) of the systems has been noticed with respect to the same parameter set as given in Table III.

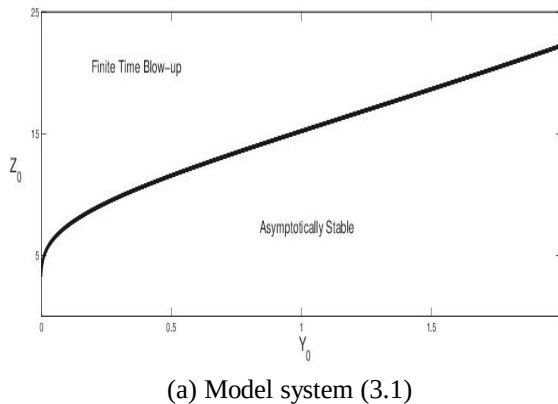


Figure 2. Stability region of the non-spatial model system (3.1) with varying initial values Y_0 and Z_0 . Parameters sets are given in Table III.

In Figure 2, we depict the asymptotic stability and the finite time blow-up regions in the Y_0Z_0 -phase plane of the model system (3.1) with respect to the parameter set given in Table III.

Figure 3 establishes the dynamics of blow-up scenario at finite time of the system (2.1) in the spatial domain.

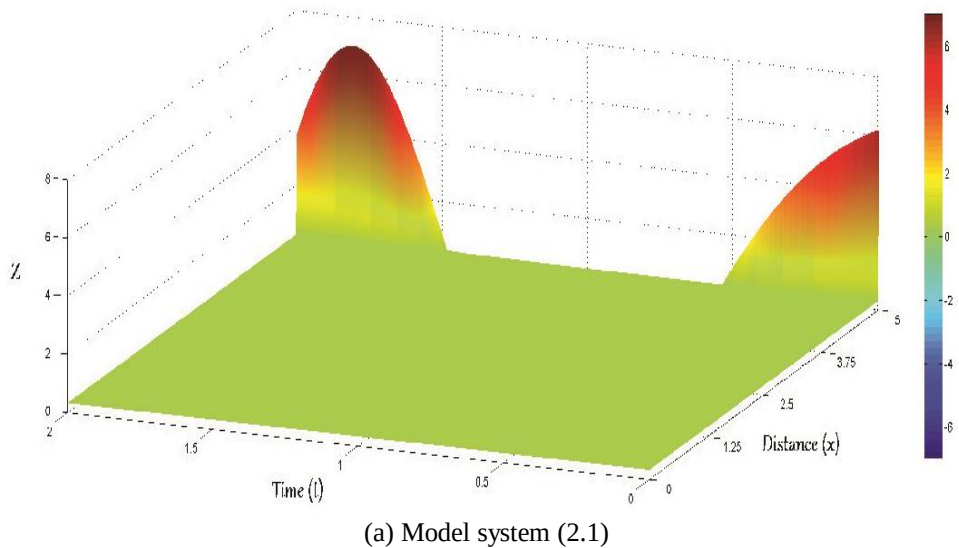


Figure 3. Blow-up phenomenon at finite time for spatial case of the model system (2.1). Parameters sets are given in Table IV.

Considering the parameter values as provided in Table IV and satisfying the no flux boundary condition, the theoretical results of blow-up scenario at finite time matches with Figure 3.

Table IV. Parameters sets for the finite time blow-up in the spatial domain of the model system (2.1) and one-dimensional patterns of the model system (2.1) respectively

Model	Parameters set of finite time blow-up in the special domain	Figure	Parameters set of one-dimensional patterns	Figure
(2.1)	$a_1 = 1, l = 0.02, b_1 = 0.03$ $a_2 = 0.05, b = 60, w = 0.75$ $w_1 = 0.6, w_2 = 0.2, w_3 = 0.0165$ $\gamma = 50, \beta = 0.21, c = 0.03,$ $D = 1,$ $d_1 = 10.1, d_2 = 20.1,$ $d_3 = 30.1$	3(a)	$a_1 = 1, l = 0.02, b_1 = 0.03$ $a_2 = 0.05, b = 60, w = 0.75$ $w_1 = 0.6, w_2 = 0.2, w_3 = 0.655$ $\gamma = 50, \beta = 0.21, c = 0.08,$ $D = 10,$ $d_1 = 0.00001, d_2 = 0.00002, d_3 = 0.00003$	4(a)

For the 1D diffusive system (2.1) and the parameter set provided in Table IV, the results are being investigated by numerical simulations. Figure 4 presents Turing patterns in the xt -plane.

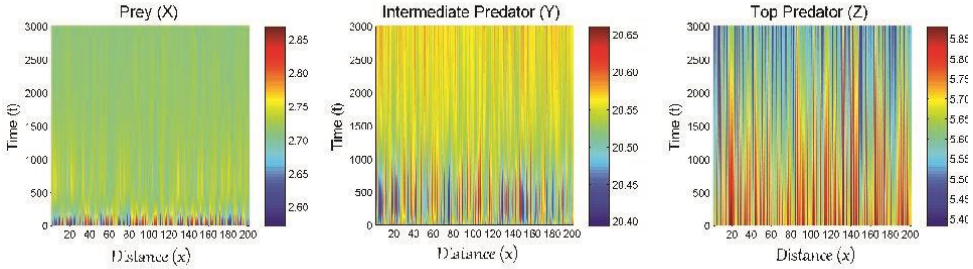
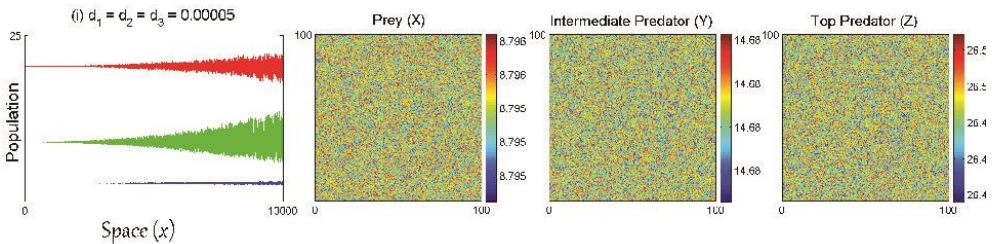
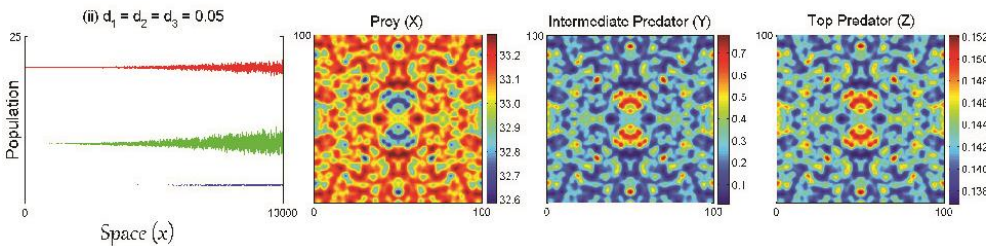


Figure 4. Snapshots of contour pictures of the time evolution of prey (X), intermediate predator (Y) and top predator (Z) in the xt -plane of the diffusive model (2.1) with incorporating one dimensional diffusion. Parameters sets are given in Table IV.

In the 1st panels of the Figure 5, solutions of the system (2.1) are presented with respect to space. For the homogeneous reaction-diffusion system, small spatial random perturbations are made around coexistence equilibrium points $E_{XYZ}(X^*, Y^*, Z^*)$. We observe four significant dynamical behaviours such as chaos, intermittent chaos, smooth oscillatory and stability covering almost over whole domain. Chaotic dynamics is observed in the system and presented in Figure 5a. Now, we observe the changes in the dynamics of the model system upon increasing the value of the diffusion parameters. The system (2.1) exhibits an intermittent chaos and are presented in Figure 5b.



(a)



(b)

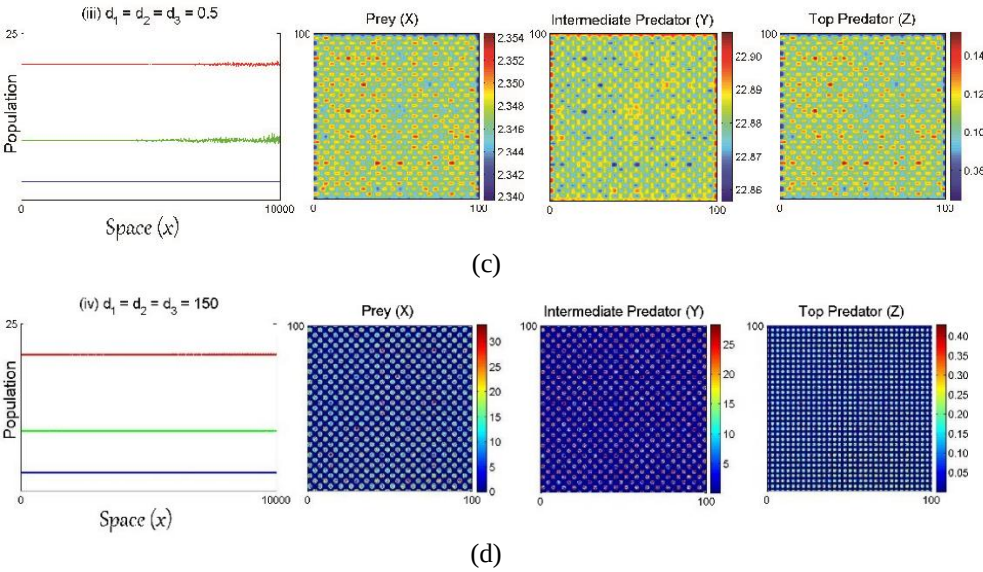


Figure 5. In the 1st panel we plot space(x)-series of the system (2.1) generated at the time $T=500$ and showing the dynamical effect of diffusivity constants d_1 , d_2 and d_3 . Here, *Blue line* represents prey (X), *Red line* indicates intermediate predator (Y) and *Green line* is for top predator (Z) populations. We consider space-step (h)=0.3, time-step (Δt)=0.2, ($X^* = 2.72, Y^* = 20.66, Z^* = 5.73$). In 2nd, 3rd and 4th panel, we depict the snapshots of Turing patterns of the evolution of the prey (X), intermediate predator (Y) and top predator (Z) in the xy-plane of the diffusive model (2.1) respectively. Parameters sets are given in Table V.

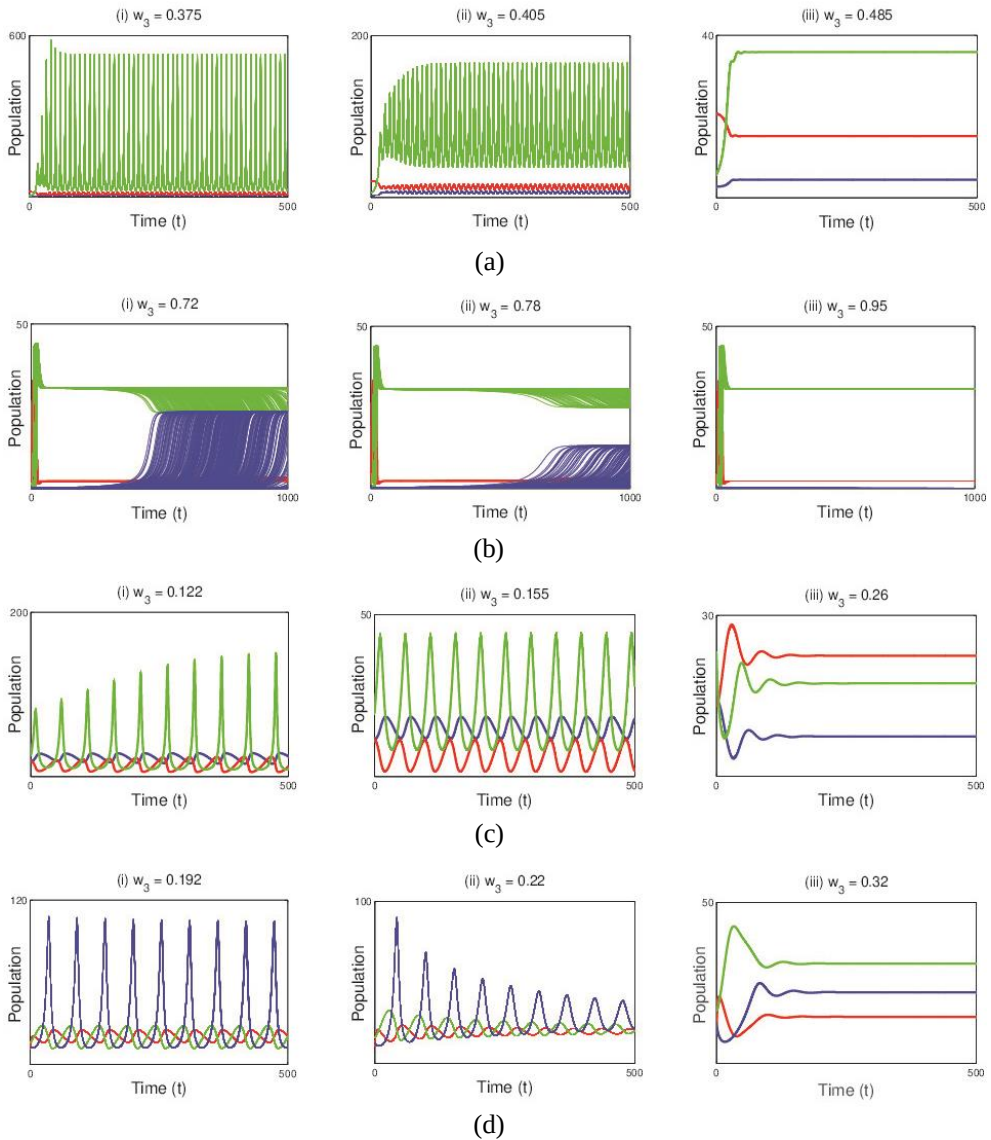
Furthermore, on increasing the values of diffusion parameters, one can observe that the dynamical irregularities get decreased and the emergence of smooth oscillatory dynamics is seen in the Figure 5c. Also, increasing values of the diffusion parameters leads us to the stability of the system, which validates our analytical findings that the chaotic nature of the dynamical system switches to stable one by increasing the diffusion parameters in Figure 5d. We also depict the the pattern formations of the prey (X), intermediate predator (Y) and top predator (Z) regarding each stages of the chaotic to stable nature of the system (2.1) in the 2nd, 3rd and 4th panels of the Figure 5 respectively.

Table V. Parameters set of the space series solutions and pattern formations corresponding their dynamical states of the diffusive model system (2.1).

Model	Parameters set	Diffusion parameters	Dynamical states	Figure
(2.1)	$a_1 = 1, l = 0.02, b_1 = 0.03$ $a_2 = 0.05, b = 60, w = 0.75$ $w_1 = 0.6, w_2 = 0.2, w_3 = 0.375$ $\gamma = 50, \beta = 0.21, c = 0.03, D = 1$	$d_1 = d_2 = d_3 = 0.00005$	Chaos	5
		$d_1 = d_2 = d_3 = 0.05$	Intermittent Chaos	
		$d_1 = d_2 = d_3 = 0.5$	Oscillatory	
		$d_1 = d_2 = d_3 = 150$	Stable	

In the 1st panel, we plot the space (x)-series diagram at time $T=500$, 2nd, 3rd and 4th panels show the corresponding pattern formations in xy -plane (in 100×100 xy -domain) at $T=500$. All the parameters sets of the system (2.1) are given in the Table V.

In Figure 6, the one-dimensional solutions of the system (2.1) are demonstrated in respect of time. We observe the change of the dynamical behaviours of the system (2.1) with changing the bifurcation parameter w_3 such as chaos, smooth oscillatory and stationary dynamics including over the whole domain.



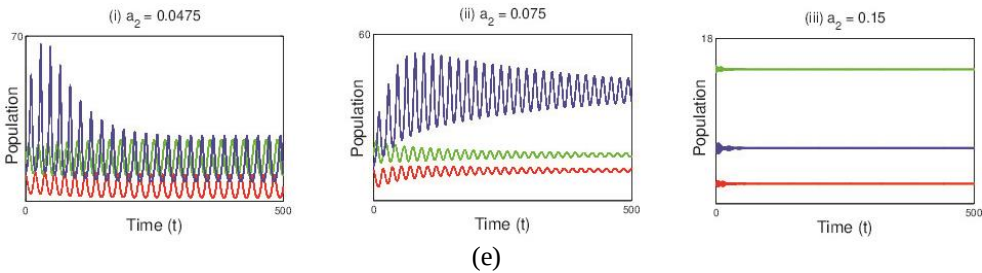


Figure 6. Time-series solutions of the system (2.1) are generated for $0 < t < T = 500$ with showing the dynamical effect of the bifurcation parameters w_3 and a_2 of the system respectively. Here, *Blue line* represents prey (X), *Red line* indicates intermediate predator (Y) and *Green line* is for top predator (Z) populations. We assume space-step (h)=0.3, time-step (Δt)=0.2. Parameters sets are given in Table VI.

Here, we examined the dynamical behaviour of the model system (2.1) near the non-trivial equilibrium point with incorporation of small perturbation along the boundary conditions by retaining the parameters set unchanged and altering diffusive parameters together with time. The parameter set of the system (2.1) is listed in the Table VI. Here, we describe the pattern formations to clarify the dynamical model system (2.1) in spatial domain and the considered initial conditions $X(x, y, 0) = X^* + \theta_1 \psi_1, Y(x, y, 0) = Y^* + \theta_2 \psi_2, Z(x, y, 0) = Z^* + \theta_3 \psi_3$, where $\theta_1 = \theta_2 = \theta_3 = 10^{-8}$ and ψ_i ($i = 1, 2, 3$) are uniformly distributed random numbers defined in $(-1.5, 1.5)$. This phenomenon evinces that Turing patterns exist for the reaction-diffusion model system (2.1).

Table VI. Parameters set of the time series solutions and pattern formations corresponding their dynamical states of the diffusive model system (2.1).

Model	Parameters set	Bifurcations parameters	Dynamical states	Figure
(2.1)	$a_1 = 1, l = 0.02, b_1 = 0.03, a_2 = 0.05, b = 60, w = 0.75, w_1 = 0.6, w_2 = 0.2, d_1 = d_2 = 0.00005, \gamma = 50, \beta = 0.21, c = 0.03, D = 1, d_3 = 0.00005$	$w_3 = 0.375$	Chaos	6(a)
		$w_3 = 0.405$	Oscillatory	
		$w_3 = 0.485$	Stable	

6. Discussions and Conclusions

In this manuscript, a comprehensive study on fear effect of prey and predator’s functional response, their blow-up properties on or before the blow-up time and the theoretical analysis and simulation results of on pattern formations have been studied extensively. The diffusive pattern formations of predator-preys interaction in spatial domain is observed and explored the

impact of dynamical behaviour of a diffusive model system (2.1). It is clear that the initial values Y_0 and Z_0 play important role on the asymptotic stability of the $E_{XYZ}(X^*, Y^*, Z^*)$ and the finite time blow-up of the top predator (Z) of the system (3.1) (see Figure 2). Again it implies from the numerical simulation results that switching in the stability domain is certain after introduction of the diffusive parameters in the model system. After incorporating the modest changes in the diffusion parameters d_1 , d_2 and d_3 of system (2.1) in one dimension, we observe that it can generate considerable changes in the qualitative behaviour of the outcomes of the system from chaotic to regular with increasing value of the diffusion parameters. The results obtained in one-dimensional dynamical system, shows a significant changes in the system (2.1) with varying the diffusion parameters. The nature of solution of the dynamical system exhibits a significant changes in the system (2.1) with varying the diffusion parameters. Similar to the eco-evolutionary process, the individual populations are dispersing between sub-populations over global populations. Based on the system parameters, initial conditions and on some occasions of fear effect, spatial model system exhibits complex dynamical behaviour.

The spatial pattern structures based on initial conditions, stays entirely on the limit cycle within the region and the solution remains in the this region forever. The elongated discussion on pattern formations of the model system (2.1) in spatial domain have been stated and also elucidate the dynamical effect of the diffusive model system (2.1). Turing pattern formations of the model system (2.1) have been investigated.

Small changes in the values of bifurcation parameter for the the model system (2.1) can give rise to notable dynamical changes³⁰. Numerical results show that the fear effect plays a significant role in the phenomena of pattern formation in spatial domain. w_3 represents the death rate of top predator (Z). Interestingly on reducing the value of w_3 suggests that the bifurcation parameter is going far away from the Turing bifurcation point and we witness the emergence of the Turing patterns of the model system (2.1) as stable $\rightarrow$ oscillatory $\rightarrow$ chaos. Similarly, if we follow the same process as above i.e., decreasing the parameter value a_2 , we observe that the Turing patterns emerge as it follows the same dynamical changes as above. Turing instabilities and the structural formations of Turing patterns have been illustrated extensively. Species spread into the space for food, shelter, mate etc. This phenomena lead to the formation of Turing patterns. In homogeneous environment, under the influence of structural behaviour, the Turing pattern may switch from stable to chaotic structure. The reduction in resource level under spatial heterogeneous environment and proportion of

spatially homogeneous species can be the cause of extinction. According to population biology, modifications in the values of control parameters in reaction-diffusion domain can generate the homogenous steady state through Turing patterns, which is not an acceptable phenomena in the ecological systems. Spatiotemporal dynamics is key to regulate the ecological biodiversity.

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