

A Study of Eilenberg-MacLane Space Through Infinite Symmetric Product Functor

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(Received May 6, 2022, Revised July 15, 2023, Accepted December 29, 2023)

Abstract: In this paper first we construct the Eilenberg-MacLane space through Moore space then we study this space through an infinite symmetric product functor. Finally, we study some applications.

Keywords: CW-complex, Infinite symmetric product, Suspension, Moore space Eilenberg-MacLane space.

1. Introduction

Dold-Thom¹ showed that for any CW-complex X , $\pi_i(SP^\infty(X)) \cong \widetilde{H}_i(X)$. E. H. Spanneir^{2,3} showed that if X is a connected CW-complex then there is a weak homotopy equivalence,

$$\rho : SP^\infty(X) \rightarrow \Omega SP^\infty(\Sigma X)$$

defined by $\rho(x_1, x_2, \dots, x_n)(t) = ((x_1, t), (x_2, t), \dots, (x_n, t))$, ρ is a continuous because its restriction to $SP^n(X)$ is continuous for every $n \geq 0$. Eckmonn-Hilton⁴ showed that for pointed CW-complexes X and Y , $[\Sigma X, Y] \cong [X, \Omega Y]$ where $[X, Y]$ denotes the set of all homotopy class of maps from X to Y . The functor SP^∞ has interesting properties that it can be used to define generalized Eilenberg-MacLane Spectrum given by P.K. Rana⁵.

Now we recall the following definitions and statements:

Definition 1.1: A pair (X, ε) consisting of a Hausdroff space X and a cell decomposition ε of X is called a CW-complex if the following axioms are satisfied:

- (i) **(characteristic map):** For each n -cell $e_\alpha^n \in \varepsilon$ there is a map $\varphi_\alpha : (D^n, S^{n-1}) \rightarrow (X, X^{n-1})$ restricting to a homeomorphism $\varphi_\alpha|_{(D^n, S^{n-1})} : D_\alpha^n - S^{n-1} \rightarrow e_\alpha^n$ and taking S^{n-1} into X_α^{n-1} .

- (ii) **(closure finite):** For any cell $e_\alpha \in \varepsilon$ the closure $\overline{e_\alpha}$ intersects only a finite number of other cells in ε .
- (iii) **(weak Topology):** A subset $A \subseteq X$ is closed if and only if $A \cap \overline{e_\alpha}$ is closed in X for each $e_\alpha \in \varepsilon$.

A pointed CW-complex is a CW-complex having X as a pointed topological space with a base point x_0 .

Definition 1.2: A pointed CW-complex is called an Eilenberg MacLane space if it has only one nontrivial homotopy group. If G is a group and n is a positive integer, the Eilenberg-MacLane space of type (G, n) is a pointed CW-complex X whose homotopy groups vanish in all dimensions except n , where $G = \pi_n(X)$ and G is to be abelian for $n > 1$.

We can write the notation $K(G, n)$ for a CW-complex which represents an Eilenberg-MacLane space of type (G, n) .

Definition 1.3: Let X be a topological space with base point $x_0 \in X$. Then the reduced suspension of X , denoted by ΣX is defined to be the quotient space $X \times I$ in which $(X \times \{1\}) \cup (\{x_0\} \times I) \cup (X \times \{0\})$ is identified to a single point. The identification map $f: X \times I \rightarrow \Sigma X$ be such that $[x, 0] = [x_0, t] = [x', 1]$, for all $x, x' \in X$ and for all $t \in I$, where $(x, t) \in X \times I$ and $[x, t] \in \Sigma X$. The point $[x_0, 0] \in \Sigma X$ is also denoted by x_0 . Thus ΣX is a pointed space with base point x_0 and $\Sigma X = X \wedge S^1$.

Definition 1.4: Let X be a topological space with base point $x_0 \in X$. For $n \geq 0$, we define the n -fold symmetric product^{6,7} of X , denoted by $SP^n X$ defined by $SP^0 X = x_0$, $SP^n X = X^n / S_n$ for $n \geq 1$, where X^n denotes the n -fold cartesian product of X with itself and S_n denotes the symmetric group on n objects regarding as acting on X^n by permuting the coordinates. We define $\lim_{n \rightarrow \infty} SP^n X = \bigcup_{n=1}^{\infty} SP^n X$ which is called an infinite symmetric product of X and is denoted by $SP^\infty X$.

P.K. Rana⁸ showed that SP^n and SP^∞ are covariant functor from the category of pointed topological spaces and base point preserving continuous maps to the category of pointed topological spaces and base point preserving continuous maps.

Definition 1.5: Given an abelian group G and $n \geq 1$ there is a CW-complex X such that $H_n(X) \cong G$ and $\widetilde{H}_i(X) = 0$ for $i \neq n$. This space X is called a Moore space of type (G, n) and is denoted by $M(G, n)$.

Definition 1.6: Let X be a pointed topological space with base point x_0 . Then the Loop space of X denoted by ΩX , defined to be the space of all continuous pointed map $\alpha: S^1 \rightarrow (X, x_0)$ equipped with compact open topology.

Definition 1.7: The spectrum $\underline{X} = \{X_n, \alpha_n\}$ given by $X_n = K(Z, n)$ and $\alpha_n: K(Z, n) \rightarrow \Omega K(Z, n+1)$, a base point preserving weak homotopy equivalence, is called an Eilenberg-MacLane spectrum.

Definition 1.8: The process of construction to obtain an Eilenberg-MacLane space $K(G, n)$ from a Moore space $M(G, n)$ by trivializing all homotopy groups beyond the n -th is called killing homotopy groups.

Lemma 1.9: Given an integer $n > 1$ and an abelian group G , then there is Moore space $M(G, n)$ which is a CW-complex with one 0-cell and all other cells in dimensions n and $n+1$ are such that $\pi_n(M(G, n)) \cong G$.

Proof: Using the definition 1.1 and definition 1.5, it follows.

Theorem 1.10: For any CW-complex X , $\pi_i(SP^\infty(X)) \cong \widetilde{H}_i(X)$.

Proof: Let C be the category of finite pointed CW-complexes and SP^∞ be the infinite symmetric product functor. Using Dold-Thom Theorem, $\pi_i(SP^\infty(X)) \cong \widetilde{H}_i(X)$ on C .

Theorem 1.11: If X is a connected CW-complex then there is a weak homotopy equivalence $\rho: SP^\infty(X) \rightarrow \Omega SP^\infty(\Sigma X)$ defined by $\rho(x_1, x_2, \dots, x_n)(t) = ((x_1, t), (x_2, t), \dots, (x_n, t))$.

Proof: Since $\rho(x_1, x_2, \dots, x_n)(t) = ((x_1, t), (x_2, t), \dots, (x_n, t))$, ρ is a continuous function because its restriction to $SP^n(X)$ is continuous for every $n \geq 0$ and its induced homomorphism $\rho_*: \pi_n(SP^\infty(X)) \rightarrow \pi_n(\Omega SP^\infty(\Sigma X))$ is an isomorphism.

Theorem 1.12: For pointed CW-complexes X and Y , $[\Sigma X, Y] \cong [X, \Omega Y]$ where $[X, Y]$ denotes the set of all homotopy class of maps from X to Y .

Proof: Let $\Sigma: C \rightarrow C$ be the suspension functor and Ω be the loop functor, for any pointed CW-Complexes X and Y , using Eckmann-Hilton⁵, it follows.

2. Existing Constructions of Eilenberg-MacLane space

Now we construct the Eilenberg-MacLane space in the following ways:

A Construction of $K(G, 1)$: Let us take an arbitrary group G not necessarily abelian. Let Δ_q be the q -simplex with ordered vertices $(g_0, g_1, \dots, g_q)$ of elements of G and $C(G)$ be the complex obtained as a quotient space of the collection of disjoint simplices Δ_q by identifying their certain faces by canonical linear homeomorphism, preserving the ordering of the vertices.

This attaches the q -simplex Δ_q to the $(q - 1)$ -simplexes $(g, \dots, \widehat{g}_i, \dots, g_q)$ where $\widehat{g}_i$ indicates that this vertex is omitted.

The group G acts on $C(G)$ by the left multiplication $\cdot (g_0, g_1, \dots, g_q) = (g \cdot g_0, g \cdot g_1, \dots, g \cdot g_q)$. Let $C(G) \bmod G$ be its quotient space. This action of G on $C(G)$ is a covering space action where $C(G)$ is simply connected and hence the quotient map $p: C(G) \rightarrow C(G) \bmod G$ is the universal covering of the orbit space $C(G) \bmod$, which is a $K(G, 1)$ space.

A Construction of $K(G, n)$ for $n > 1$:

This construction process is done in the following ways;

- (i) Construction of Moore space $M(G, n)$

We claim that the Eilenberg-MacLane spaces $K(G, n)$ exist for all proper (G, n) and each of them is unique up to homotopy type and the existence of Moore spaces follows from definition 1.8.

- (ii) Using killing homotopy;

A process of construction for obtaining an Eilenberg-MacLane space $K(G, n)$ by killing all homotopy groups above the n -th.

To do this we use the following theorems:

Theorem 2.1: *Given a CW-complex X and an integer $n > 0$, there exists a relative CW-complex (X', X) with cells in dimension $(n+1)$ only such that*

- a) $\pi_n(X') = 0$
- b) $\pi_m(X') \cong \pi_m(X)$, for $m < n$, where $X' = (X \cup \bigcup_{\alpha \in \Lambda} D^{n+1}) / \sim$, where x is identified to $f(x)$ for $x \in \partial D^{n+1}$.

Proof: Let $\{f_\alpha: S^n \rightarrow X, \alpha \in \Lambda\}$ represent a set of generators of $\pi_n(X)$. Then for each $\alpha \in \Lambda$, take an $(n + 1)$ -ball D_α^{n+1} and attach it by f_α to X to obtain $X' = (X \cup \bigcup_{\alpha \in \Lambda} D_\alpha^{n+1}) / \sim$, where x is identified to $f(x)$ for $x \in \partial D_\alpha^{n+1}$. This shows that the relative CW-complex (X', X) has only these D_α^{n+1} as $(n+1)$ -cells which precisely make the generators of $\pi_{n+1}(X)$ inessential.

Also below dimension n , the space X' has the same homotopy groups as the space X by inclusion $i: X \hookrightarrow X'$.

Using this theorem and the definition 1.8, we have the following:

- (iii) Applying Postnikov decomposition;

Postnikov system gives a way of constructing a Eilenberg-MacLane space. To do this we use the following theorem:

Theorem 2.2: *Any CW-complex X admits a decomposition into a tower such that*

- a) Cells in dimension n and above only;
- b) $\pi_m(X^{[n]}) = 0$ for all $m > n$;
- c) $i_n * \pi_m(X) \cong \pi_m(X^{[n]})$ for $m \leq n$, where $i_n: X \hookrightarrow X^{[n]}$ is the inclusion.

Proof: Apply the killing homotopy theorem 2.1 to kill $\pi_{n+1}(X)$ for $n \geq 0$, we have $X_n^{(1)} = (X \cup_{\alpha \in \Lambda} D_{\alpha}^{n+2})/\sim$ with $\pi_{n+1}(X_n^{(1)}) = 0$. Now apply the same procedure to $X_n^{(1)}$ to get $X_n^{(2)}$ with $\pi_{n+1}(X_n^{(2)}) = \pi_{n+2}(X_n^{(2)}) = 0$. Repeating this process we obtain $X_n^{(r)}$, now we write $X^{[n]} = \bigcup_{r \geq 1} X_n^{(r)}$ with weak topology. This gives a), b) and c) follows from the preservation of π_m by direct limits for CW-complexes.

(iv) Construction $K(G, n)$ uniquely determined up to homotopy equivalence by an abelian group G and $n > 1$. The construction of $K(G, n)$ is completed by setting $K(G, n) = M(G, n)[n]$, where $M(G, n)[n]$ can be viewed as successive approximations to $M(G, n)$. The CW-complex X of type (G, n) having all homotopy groups equal to 0 except for $\pi_n(X) = G$ is uniquely determined up to homotopy equivalence because if A and B are both Eilenberg-MacLane spaces of type (G, n) , then the identity homomorphism $\text{Id}: G \rightarrow G$ induces a map $h: A \rightarrow B$. Then h is a homotopy equivalence.

3. New Construction of Eilenberg-MacLane space

To do this we use the following results:

Lemma 3.1: For any CW-complex X , $H_k(X) = \widetilde{H}_k(X)$ if $k \neq 0$ and $H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}$.

Proposition 3.2: Given any abelian group G and an integer $n \geq 1$ there exists a CW-complex X such that $X = M(G, n)$.

Proof: Using the Definition 1.2 and Definition 1.5, it follows.

Proposition 3.3: X is a Moore space of type (G, n) if and only if $SP^{\infty}(X)$ is $K(G, n)$ for $n \geq 1$.

Proof: Let X is $M(G, n)$ for $n \geq 1$. Then $H_n(X) \cong G$ and $\widetilde{H}_i(X) = 0$ for $i \neq n$. As $n \geq 1$, so by lemma 3.1, $\widetilde{H}_i(X) \cong G$ if $i = n$ and $\widetilde{H}_i(X) = 0$ if $i \neq n$. Now by Dold-Thom Theorem, we have $\widetilde{H}_i(X) = \pi_i(SP^{\infty}(X))$. So $\pi_i(SP^{\infty}(X)) \cong G$, if $i = n$ and 0, if $i \neq n$. Thus $SP^{\infty}(X)$ is $K(G, n)$ for $n \geq 1$.

Conversely, let $SP^{\infty}(X)$ is $K(G, n)$ for $n \geq 1$. Now $\widetilde{H}_i(X) = \pi_i(SP^{\infty}(X)) \cong G$, if $i = n$ and 0, if $i \neq n$. Hence, $\widetilde{H}_i(X) \cong G$, if $i = n$ and 0, if $i \neq n$. Therefore $H_n(X) \cong G$ and $\widetilde{H}_i(X) = 0$ for $i \neq n$ as $n \geq 1$. Hence, X is $M(G, n)$. Let G be any abelian group and $n \geq 1$ be an integer, then by Proposition 3.2, there is a CW-complex X such that X is a $M(G, n)$. Now by Proposition 3.3, $SP^{\infty}(X) \cong SP^{\infty}(M(G, n)) = K(G, n)$.

Proposition 3.4: *For any given abelian group G and $n > 1$, $SP^\infty(\Sigma^k M(G, n))$ is $K(G, n + k)$.*

Proof: For $n > 1$, $M(G, n)$ is simply connected and hence connected. Now by Theorem 1.10, there is a weak homotopy equivalence $\rho : SP^\infty(M(G, n)) \rightarrow \Omega SP^\infty(\Sigma M(G, n))$. Using the composite map we can get a weak homotopy equivalence $\rho_k : SP^\infty(M(G, n)) \rightarrow \Omega^k SP^\infty(\Sigma^k M(G, n))$. Thus $SP^\infty(\Sigma^k M(G, n))$ is $K(G, n + k)$.

Proposition 3.5: *Let G be an abelian group and $M(G, n)$ be a Moore space. Then there is a weak homotopy equivalence $\beta_n : SP^\infty(M(G, n)) \rightarrow \Omega SP^\infty(M(G, n + 1))$*

Proof: Since $\pi_{n+1}(SP^\infty(M(G, n + 1))) \cong \pi_n(\Omega SP^\infty(M(G, n + 1))) \cong G$. Again $\pi_n(SP^\infty(M(G, n))) \cong \pi_{n+1}(SP^\infty(M(G, n + 1)))$, and hence $\pi_n(SP^\infty(M(G, n))) \cong \pi_n(\Omega SP^\infty(M(G, n + 1)))$ for each $n \geq 1$. Hence there is a continuous function $\beta_n : SP^\infty(M(G, n)) \rightarrow \Omega SP^\infty(M(G, n + 1))$, whose induced homomorphism $\beta_n^* : \pi_n(SP^\infty(M(G, n))) \rightarrow \pi_n(\Omega SP^\infty(M(G, n + 1)))$ is an isomorphism. Since all other groups are trivial and so β_n is a weak homotopy equivalence.

Again since $\Omega SP^\infty(M(G, n + 1))$ has the homotopy type of a CW-complex and hence β_n is a homotopy equivalence.

Proposition 3.6: *If $\alpha_n : A_n \rightarrow \Omega A_{n+1}$ be a homotopy equivalence, where $A_n = SP^\infty(M(G, n))$ then $\{A_n, \alpha_n\}$ define an Eilenberg-MacLane Spectrum.*

Proof: Since $SP^\infty(M(G, n)) = K(G, n)$, using Propositions 3.2, 3.3 and from P.K.Rana⁹, it follows.

A pointed topological space X with base point x_0 is called an H-space if there exists a continuous multiplication $\mu : X \times X \rightarrow X$ for which the constant map $c : X \rightarrow x_0 \in X$ is a homotopy identity.

An H-space (X, μ) is said to be homotopy associative if $\mu \circ (\mu \times I_p) \simeq \mu \circ (I_p \times \mu)$.

A continuous map $\varphi : X \rightarrow X$ is said to be homotopy inverse for X and μ if each of the composites $X \xrightarrow{(I_p, \varphi)} X \times X \xrightarrow{\mu} X$ and $X \xrightarrow{(\varphi, I_p)} X \times X \xrightarrow{\mu} X$ is homotopic to the constant map $c : X \rightarrow x_0$.

An associative H-space X with an inverse is called an H-group.

We know every loop space is a H-group and the set of homotopy classes of maps from any pointed space to an H-group admits a group structure and so there is a natural group structure on $[X, K(G, n)]$.

Proposition 3.7: *For all path connected n -dimensional CW-complex X , $[X, M(G, n)] \cong [X, SP^\infty(M(G, n))]$, $n > 1$.*

Proof: Since $M(G, n) \subset SP^\infty(M(G, n))$ and so $i_*: [X, M(G, n)] \rightarrow [X, SP^\infty(M(G, n))]$ is injective as $M(G, n)$ is simply connected. Surjectivity follows from the fact that X is n -dimensional CW-complex and cellular approximation.

Corollary 3.8: *Given an abelian group G an integer $n \geq 1$ and a path-connected n -dimensional CW-complex X , then $H^n(X, G) = [X, K(G, n)] = [X, SP^\infty(M(G, n))]$ is the cohomology group.*

Proof: Using Proposition 3.7 and from Albrecht Dold⁹, it follows.

4. Comparison between existing constructions and new Construction of Eilenberg-MacLane space

In the existing construction, for an abelian group G , an integer $n \geq 1$ and a path connected n -dimensional CW-complex X , then the cohomology group $H^n(X, G) = [X, K(G, n)]$, where $K(G, n)$ is an Eilenberg-MacLane space; but in the new construction this can be represented as $H^n(X, G) = [X, SP^\infty(\Sigma^{n-1}M(G, 1))] = [X, K(G, n)]$. This proof is laborious under existing construction of Eilenberg-MacLane space but in the new construction the proof is very easy i.e., it follows using Proposition 3.3 and corollary 3.8.

Acknowledgement: The authors are grateful to the anonymous referee for his/her valuable suggestions which considerably improved the presentation of the paper. Second and third authors are very much thankful to THE GOVERNMENT OF WEST BENGAL, DEPARTMENT OF SCIENCE AND TECHNOLOGY AND BIOTECHNOLOGY that encouraged them to research on this topic with sufficient grant. The first author is thankful to the Council of Scientific and Industrial Research, HRDG, India for grant Junior Research Fellowship (File No.:08/525(0006)/2019-EMR-I) during the tenure of which this work was done.

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