

On Ricci Semi-Symmetric Mixed Super Quasi-Einstein Nearly Kaehler Manifold

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Abstract: In this paper, we have studied some curvature properties of projective curvature tensor in projective Ricci semi-symmetric mixed super quasi-Einstein nearly Kaehler manifold and projective flat Ricci semi-symmetric mixed super quasi-Einstein nearly Kaehler manifold.

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1. Introduction

An even dimensional differentiable manifold M^n is said to be a nearly Kaehler manifold, if a complex structure F of type (1,1) and a pseudo-Riemannian metric g of the manifold satisfy

$$(1.1) \quad F^2 = -I,$$

$$(1.2) \quad g(FX, FY) = g(X, Y),$$

and

$$(1.3) \quad (\nabla_X F)Y + (\nabla_Y F)X = 0,$$

where $X, Y \in \mathfrak{X}(M)$ and $\mathfrak{X}(M)$ is Lie algebra of vector fields on the manifold. A Riemannian manifold or pseudo-Riemannian manifold (M^n, g) ($n \geq 2$) in which Ricci tensor be scalar multiple of Riemannian metric i.e.

$$(1.4) \quad S(X, Y) = \alpha g(X, Y),$$

is called Einstein manifold. The concept of Einstein manifold was given by Albert Einstein in differential geometry and mathematical physics. In equation (1.4), S represents the Ricci tensor of the manifold and α is a non-

zero scalar. The concept of Einstein manifold plays very important role in the study of Riemannian geometry and general theory of relativity.

Contracting equation (1.4), we get the expression for scalar curvature, which is given by

$$(1.5) \quad r = n\alpha,$$

where r denotes scalar curvature and n is dimension of the manifold.

A non-flat Riemannian manifold is said to be quasi-Einstein manifold if the non-zero Ricci tensor satisfies

$$(1.6) \quad S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y),$$

where α, β are scalars such that $\beta \neq 0$ and $A(X)$ is non-zero 1-form associated with a unit vector field ρ defined by $g(X, \rho) = A(X)$, for every vector field X , ρ is called the generator of the manifold. The concept of quasi-Einstein manifold was given by M. C. Chaki and R. K. Maity¹ in 2000. An n -dimensional quasi-Einstein manifold is denoted by $(QE)_n$. Taking contraction of equation (1.6), we get the expression for scalar curvature in quasi-Einstein manifold, which is given by

$$(1.7) \quad r = \alpha n + \beta,$$

with the help of equations (1.2) and (1.6), we get

$$(1.8) \quad \begin{cases} S(X, \rho) = (\alpha + \beta)A(X), \\ S(\rho, \rho) = (\alpha + \beta), \\ g(F\rho, \rho) = 0, \\ S(F\rho, \rho) = 0. \end{cases}$$

The Walker-space time is an example of a quasi-Einstein manifold. The concept of quasi-Einstein manifold came out during the study of exact solutions of Einstein fields equations as well as considerations of a quasi-umbilical hypersurfaces of semi-Euclidean space. Also, a quasi-Einstein manifolds can be taken as a model of the perfect fluid space time in general theory of relativity².

The notion of generalized quasi-Einstein manifold was introduced and studied by M. C. Chaki³ in 2001. Later on, in 2004, U. C. De and G. C. Ghosh⁴ cited an example of a generalised quasi-Einstein manifold and studied its geometrical properties.

If a non-zero Ricci tensor of type $(0,2)$ of a Riemannian manifold (M^n, g) ($n \geq 2$) satisfies

$$(1.9) \quad S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma C(X)C(Y),$$

then manifold is called generalized quasi-Einstein manifold,

where α, β and γ are scalars such that $\beta \neq 0, \gamma \neq 0$ and A, C are non-vanishing 1-forms associated with two orthogonal unit vectors ρ and μ by

$$(1.10) \quad g(X, \rho) = A(X), \quad g(X, \mu) = C(X), \quad g(\rho, \rho) = g(\mu, \mu) = 1,$$

an n -dimensional generalized quasi-Einstein manifold is denoted by $G(QE)_n$. After contraction of equation (1.9), we get

$$(1.11) \quad r = n\alpha + \beta + \gamma,$$

also, in 2004, M. C. Chaki⁵ introduced the notion of super quasi-Einstein manifolds. A Riemannian manifold (M^n, g) ($n \geq 2$) is said to be a super quasi-Einstein manifold if the non-zero Ricci tensor of type (0,2) satisfies

$$(1.12) \quad S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma[A(X)C(Y) + C(X)A(Y)] + \delta D(X, Y),$$

where α, β, γ and δ are non-zero scalars and A, C are non-vanishing 1-forms defined as (1.10) and ρ, μ are orthogonal unit vector fields, D is a non-zero symmetric tensor of type (0,2) with zero trace, which satisfies the condition

$$(1.13) \quad D(X, \rho) = 0, \quad \forall X.$$

also, α, β, γ and δ are called the associated scalars. A and C are associated 1-forms of the manifold and D is called the structure tensor of the manifold.

From the equations (1.2), (1.10) and (1.12) we can easily write

$$(1.14) \quad \begin{cases} S(X, \rho) = (\alpha + \beta)A(X) + \gamma C(X), \\ S(X, \mu) = \alpha C(X) + \gamma A(X), \\ S(\mu, \mu) = \alpha + \delta D(\mu, \mu), \\ S(\rho, \rho) = \alpha + \beta + \delta D(\rho, \rho), \\ g(F\rho, \rho) = g(F\mu, \mu) = 0, \\ S(F\mu, \mu) = \gamma A(F\mu) + \delta D(F\mu, \mu), \\ S(F\rho, \rho) = \gamma C(F\rho) + \delta D(F\rho, \rho). \end{cases}$$

In 2009, A.A. Shaikh⁶ introduced the notion of pseudo quasi-Einstein manifold. A semi-Riemannian manifold (M^n, g) ($n \geq 2$) is said to be a pseudo quasi-Einstein manifold if the non-zero Ricci tensor of type (0,2) satisfies the condition

$$(1.15) \quad S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \delta D(X, Y),$$

where α, β and δ are non-zero scalars and A is non-zero 1-forms defined as (1.10) and ρ denotes the unit vector called the generator of the manifold and D is a symmetric tensor of type (0,2) with zero trace defined as in (1.13).

From equation (1.2), (1.10), (1.13) and (1.15), we get

$$(1.16) \quad \begin{cases} S(X, \rho) = (\alpha + \beta)A(X), \\ S(\rho, \rho) = (\alpha + \beta), \\ g(F\rho, \rho) = 0, \\ S(F\rho, \rho) = \delta D(F\rho, \rho). \end{cases}$$

In 2008, A. Bhattacharya, M. Tarafdar and D. Debnath⁷ introduced the notion of mixed super quasi-Einstein manifolds. A non-flat Riemannian manifold (M^n, g) ($n \geq 2$) is said to be mixed super quasi-Einstein manifold if the non-zero Ricci tensor of type (0,2) satisfies

$$(1.17) \quad S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma C(X)C(Y) \\ + \delta [A(X)C(Y) + C(X)A(Y)] + \eta D(X, Y),$$

where $\alpha, \beta, \delta, \gamma$ and η are non-zero scalars and A, C are non-vanishing 1-forms. An n -dimensional mixed super quasi-Einstein manifold is denoted by $MS(QE)_n$.

From equation (1.2), (1.10), (1.13) and (1.17) we get

$$(1.18) \quad \begin{cases} S(X, \rho) = (\alpha + \beta)A(X) + \delta C(X), \\ S(X, \mu) = (\alpha + \gamma)C(X) + \delta A(X), \\ S(\mu, \mu) = \alpha + \gamma + \eta D(\mu, \mu), \\ S(\rho, \rho) = \alpha + \beta + \eta D(\rho, \rho), \\ g(F\rho, \rho) = g(F\mu, \mu) = 0, \\ S(F\mu, \mu) = \delta A(F\mu) + \eta D(F\mu, \mu), \\ S(F\rho, \rho) = \delta C(F\rho) + \eta D(F\rho, \rho). \end{cases}$$

In 2008, A.A. Shaikh and S.K. Jana⁸ developed the concept of pseudo generalised quasi-Einstein manifolds. A Riemannian manifold (M^n, g) ($n \geq 2$) is called a pseudo generalized quasi-Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies

$$(1.19) \quad S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) \\ + \gamma C(X)C(Y) + \delta D(X, Y),$$

after contraction of equation (1.19), we get

$$r = n\alpha + \beta + \gamma + \delta D,$$

now from (1.10), (1.13) and (1.19), we infer

$$(1.20) \quad \begin{cases} S(X, \rho) = (\alpha + \beta)A(X), \\ S(X, \mu) = (\alpha + \gamma)C(X), \\ S(\mu, \mu) = \alpha + \gamma + \delta D(\mu, \mu), \\ S(\rho, \rho) = \alpha + \beta + \delta D(\rho, \rho), \\ g(F\rho, \rho) = g(F\mu, \mu) = 0, \\ S(F\mu, \mu) = \delta D(F\mu, \mu) \\ S(F\rho, \rho) = \delta D(F\rho, \rho). \end{cases}$$

In 2012, S. K. Hui and R. S. Lemence⁹ studied generalized quasi-Einstein manifold and admitting a W_2 -curvature tensor and they have been shown that if a W_2 -curvature tensor satisfies $W_2 \cdot S = 0$ then either the associated scalars β and γ are equal or the curvature tensor R satisfies a definite condition. Some results on generalized quasi-Einstein manifolds were studied by D. G. Prakasha and H. Venkatesha¹⁰. During their studied they have proved that if a conharmonic curvature tensor $L \cdot S = 0$ in generalised quasi-Einstein manifold, then either M is a nearly quasi-Einstein manifold $N(QE)_n$ or the curvature tensor R satisfies a definite condition.

2. Semi-symmetric and Ricci semi-symmetric manifolds

A Riemannian manifold is said to be locally symmetric if $\nabla R = 0$, where R is a Riemannian curvature tensor of (M^n, g) and ∇ is a Levi-Civita connection on (M^n, g) . Different differential geometers studied the concept of locally symmetric properties of the manifold through different approaches. In 1982, Z. I. Szabò studied structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$. In 1950, walker¹¹ studied Ruse's spaces of recurrent curvature and a Riemannian space with recurrent conformal curvature tensor was studied by T. Adati and T. Miyazawa¹².

According to Z. I. Szabò¹³, if the manifold M satisfies the condition

$$(2.1) \quad (R(X, Y) \cdot R)(U, V)W = 0, \quad X, Y, U, V, W \in \mathfrak{X}(M)$$

for all vector fields X and Y , then the manifold is called a semi-symmetric manifold. For a $(0, k)$ -tensor field T on M , $k \geq 1$ and a symmetric $(0, 2)$ -tensor field A on M , the $(0, k+2)$ -tensor fields $R \cdot T$ and $Q(A, T)$ are defined by

$$(2.2) \quad \begin{aligned} (R \cdot T)(X_1, \dots, X_k; X, Y) &= -T(R(X, Y)X_1, \dots, X_k) \\ &\quad - \dots - T(X_1, \dots, X_{k-1}, R(X, Y)X_k), \end{aligned}$$

and

$$(2.3) \quad Q(A, T)(X_1, \dots, X_k; X, Y) = -T((X \wedge_A Y)X_1, \dots, X_k) \\ - \dots - T(X_1, \dots, X_{k-1}, (X \wedge_A Y)X_k),$$

where $X \wedge_A Y$ is the endomorphism given by

$$(2.4) \quad (X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y.$$

Definition 2.1: A semi-Riemannian manifold is said to be Ricci semi-symmetric if the following condition satisfied¹⁴

$$(2.5) \quad (R(X, Y) \cdot S)(Z, U) = -S(R(X, Y)Z, U) - S(Z, R(X, Y)U) = 0,$$

for all $X, Y, Z, U \in \mathfrak{X}(M^n)$.

In the continuation of above studied B. B. Chaturvedi and B. K. Gupta¹⁵⁻¹⁹ studied Bochner Ricci semi-symmetric Hermitian manifold, Ricci pseudo-symmetric mixed generalized quasi-Einstein Hermitian manifolds, Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold, Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold and Bochner Ricci pseudo-symmetric Hermitian manifolds. Also, B. B. Chaturvedi and Pankaj Pandey²⁰ studied the flatness of weak symmetric manifolds and Riemannian manifold with a special type of semi-symmetric non-metric connection was studied by B. B. Chaturvedi and P. N. Pandey²¹.

The projective curvature tensor W in nearly Kaehler manifold is defined²² by

$$(2.6) \quad W(\bar{X}, \bar{Y}, Z, U) = R(X, Y, Z, U) + g((\nabla_X F)Y, (\nabla_Z F)U) \\ + \frac{1}{n-1} \{F(X, U)S(Y, Z) - F(Y, U)S(X, Z)\},$$

where g is the positive definite metric of the manifold, S is a Ricci tensor of type (0,2) and $'F(X, Y) = g(FX, Y)$.

In a nearly Kaehler manifold, a projective curvature tensor satisfies the condition

$$(2.7) \quad W(\bar{X}, \bar{Y}, Z, U) = W(X, Y, \bar{Z}, \bar{U}).$$

Now we define:

Definition 2.2: An even dimensional nearly Kaehler manifold (M^n, g) is said to be projective Ricci semi-symmetric mixed super quasi-Einstein nearly Kaehler manifold if the projective curvature tensor satisfies the condition $W \cdot S = 0$, i.e.,

$$(2.8) \quad (W(X, Y) \cdot S)(Z, U) = -S(W(X, Y)Z, U) - S(Z, W(X, Y)U) = 0,$$

For all $X, Y, Z, U \in \mathfrak{X}(M^n)$.

3. Projective Ricci semi-symmetric mixed super quasi-Einstein nearly Kaehler manifold

In this section, we will define and study projective Ricci semi-symmetric mixed super quasi-Einstein nearly Kaehler manifold.

Definition 3.1: A nearly kaehler manifold is said to be a mixed super quasi-Einstein nearly Kaehler manifold if it satisfies the equation (1.17) and in this paper it is denoted by $MS(QENK)_n$.

Definition 3.2: A quasi-Einstein nearly kaehler manifold is said to be a projective Ricci semi-symmetric mixed super quasi-Einstein nearly Kaehler manifold, if it satisfies the equation (2.8).

By using equations (1.17) and (2.8), we can write

$$(3.1) \quad \begin{aligned} & \alpha[W(X, Y, Z, U) + W(X, Y, U, Z)] \\ & + \beta[A(W(X, Y)Z)A(U) + A(Z)A(W(X, Y)U)] \\ & + \gamma[C(W(X, Y)Z)C(U) + C(Z)C(W(X, Y)U)] \\ & + \delta \left[\begin{aligned} & A(W(X, Y)Z)C(U) + C(W(X, Y)Z)A(U) \\ & + A(Z)C(W(X, Y)U) + C(Z)A(W(X, Y)U) \end{aligned} \right] \\ & + \eta[D(W(X, Y)Z, U) + D(Z, W(X, Y)U)] = 0, \end{aligned}$$

replacing $\bar{X}, \bar{Y}$ in place of X, Y in (3.1) respectively, we get

$$(3.2) \quad \begin{aligned} & \alpha[W(\bar{X}, \bar{Y}, Z, U) + W(\bar{X}, \bar{Y}, U, Z)] \\ & + \beta[A(W(\bar{X}, \bar{Y})Z)A(U) + A(Z)A(W(\bar{X}, \bar{Y})U)] \\ & + \gamma[C(W(\bar{X}, \bar{Y})Z)C(U) + C(Z)C(W(\bar{X}, \bar{Y})U)] \\ & + \delta \left[\begin{aligned} & A(W(\bar{X}, \bar{Y})Z)C(U) + C(W(\bar{X}, \bar{Y})Z)A(U) \\ & + A(Z)C(W(\bar{X}, \bar{Y})U) + C(Z)A(W(\bar{X}, \bar{Y})U) \end{aligned} \right] \\ & + \eta[D(W(\bar{X}, \bar{Y})Z, U) + D(Z, W(\bar{X}, \bar{Y})U)] = 0, \end{aligned}$$

again replacing $\bar{Z}, \bar{U}$ in place of Z, U in (3.1) respectively, we get

$$(3.3) \quad \begin{aligned} & \alpha[W(X, Y, \bar{Z}, \bar{U}) + W(X, Y, \bar{U}, \bar{Z})] \\ & + \beta[A(W(X, Y)\bar{Z})A(\bar{U}) + A(\bar{Z})A(W(X, Y)\bar{U})] \\ & + \gamma[C(W(X, Y)\bar{Z})C(\bar{U}) + C(\bar{Z})C(W(X, Y)\bar{U})] \\ & + \delta \left[\begin{aligned} & A(W(X, Y)\bar{Z})C(\bar{U}) + C(W(X, Y)\bar{Z})A(\bar{U}) \\ & + A(\bar{Z})C(W(X, Y)\bar{U}) + C(\bar{Z})A(W(X, Y)\bar{U}) \end{aligned} \right] \\ & + \eta[D(W(X, Y)\bar{Z}, \bar{U}) + D(\bar{Z}, W(X, Y)\bar{U})] = 0, \end{aligned}$$

now subtracting equation (3.2) from (3.3) also using (2.7), we have

$$\begin{aligned}
 (3.4) \quad & \beta[A(W(\bar{X}, \bar{Y})Z)A(U) + A(Z)A(W(\bar{X}, \bar{Y})U)] \\
 & -\beta[A(W(X, Y)\bar{Z})A(\bar{U}) + A(\bar{Z})A(W(X, Y)\bar{U})] \\
 & +\gamma[C(W(\bar{X}, \bar{Y})Z)C(U) + C(Z)C(W(\bar{X}, \bar{Y})U)] \\
 & -\gamma[C(W(X, Y)\bar{Z})C(\bar{U}) + C(\bar{Z})C(W(X, Y)\bar{U})] \\
 & +\delta \left[\begin{aligned} & A(W(\bar{X}, \bar{Y})Z)C(U) + C(W(\bar{X}, \bar{Y})Z)A(U) \\ & +A(Z)C(W(\bar{X}, \bar{Y})U) + C(Z)A(W(\bar{X}, \bar{Y})U) \end{aligned} \right] \\
 & -\delta \left[\begin{aligned} & A(W(X, Y)\bar{Z})C(\bar{U}) + C(W(X, Y)\bar{Z})A(\bar{U}) \\ & +A(\bar{Z})C(W(X, Y)\bar{U}) + C(\bar{Z})A(W(X, Y)\bar{U}) \end{aligned} \right] \\
 & +\eta[D(W(\bar{X}, \bar{Y})Z, U) + D(Z, W(\bar{X}, \bar{Y})U)] \\
 & -\eta[D(W(X, Y)\bar{Z}, \bar{U}) + D(\bar{Z}, W(X, Y)\bar{U})] = 0,
 \end{aligned}$$

putting $Z = U = \rho$ we get

$$(3.5) \quad 2\beta W(\bar{X}, \bar{Y}, \rho, \rho) + 2\delta W(\bar{X}, \bar{Y}, \rho, \mu) = 0.$$

Now, in equation (3.5), if we take $\beta = 0$, then we get

$$(3.6) \quad \delta W(\bar{X}, \bar{Y}, \rho, \mu) = 0,$$

which implies either $\delta = 0$ or $W(\bar{X}, \bar{Y}, \rho, \mu) = 0$.

Now if $\delta = 0$, then from equation (1.17), we get

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma C(X)C(Y) + \eta D(X, Y).$$

Then manifold becomes a pseudo-generalized quasi-Einstein manifold.

Thus, we conclude:

Theorem 3.1: *In a projective Ricci semi-symmetric mixed super quasi-Einstein nearly Kaehler manifold if $\beta = 0$ then the manifold is either a projective Ricci semi-symmetric pseudo generalized quasi-Einstein nearly Kaehler manifold or $W(\bar{X}, \bar{Y}, \rho, \mu) = 0$.*

In equation (3.5), if we take $\delta = 0$, then we get

$$(3.7) \quad \beta W(\bar{X}, \bar{Y}, \rho, \rho) = 0,$$

which implies either $\beta = 0$ or $W(\bar{X}, \bar{Y}, \rho, \rho) = 0$.

Thus, we conclude:

Theorem 3.2: *In a projective Ricci semi-symmetric mixed super quasi-Einstein nearly Kaehler manifold if $\delta = 0$ then either $\beta = 0$ or $W(\bar{X}, \bar{Y}, \rho, \rho) = 0$.*

If we take $\rho = \mu$, in equation (3.5), we can write

$$(3.8) \quad (\beta + \delta)W(\bar{X}, \bar{Y}, \mu, \mu) = 0,$$

which implies either $\beta = -\delta$ or $W(\bar{X}, \bar{Y}, \mu, \mu) = 0$.

Thus, we conclude:

Theorem 3.3: *In a projective Ricci semi-symmetric mixed super quasi-Einstein nearly Kaehler manifold if $\rho = \mu$ then either $\beta = -\delta$ or $W(\bar{X}, \bar{Y}, \mu, \mu) = 0$.*

4. Projectively flat Ricci semi-symmetric mixed super quasi-Einstein nearly Kaehler manifold

For a projective flat nearly Kaehler manifold, from equation (2.6), we can write

$$(4.1) \quad R(X, Y, Z, U) = -g((\nabla_X F)Y, (\nabla_Z F)U) \\ - \frac{1}{n-1} \{ 'F(X, U)S(Y, Z) - 'F(Y, U)S(X, Z) \}.$$

In a nearly Kaehler manifold²⁰, we have, $R(\bar{X}, \bar{Y}, Z, U) - R(X, Y, Z, U) = g((\nabla_X F)Y, (\nabla_Z F)U)$ and $'F(X, Y) = g(FX, Y)$, using these relations in (4.1), we get

$$(4.2) \quad R(\bar{X}, \bar{Y}, Z, U) = \frac{1}{n-1} \{ g(\bar{X}, U)S(Y, Z) - g(\bar{Y}, U)S(X, Z) \},$$

from equation (2.5) and (4.2), we have

$$(4.3) \quad \frac{1}{n-1} \{ g(\bar{X}, U)S(QY, Z) - g(\bar{Y}, U)S(QX, Z) \} = 0,$$

if we take λ is an eigen value of Q corresponding to eigen vector X , then $QX = \lambda X$ i.e., $S(X, Z) = \lambda g(X, Z)$ (where the manifold is not Einstein) and hence $S(QX, Z) = \lambda S(X, Z)$.

Then from equation (4.3), we have

$$(4.4) \quad \frac{\lambda}{n-1} \{ g(\bar{X}, U)S(Y, Z) - g(\bar{Y}, U)S(X, Z) \} = 0,$$

if we take $\frac{\lambda}{n-1} \neq 0$, then from equation (4.4), we have

$$g(\bar{X}, U)S(Y, Z) - g(\bar{Y}, U)S(X, Z) = 0,$$

now putting $Z = U = \rho$, we get

$$(4.5) \quad (\alpha + \beta)[A(\bar{X})A(Y) - A(\bar{Y})A(X)] \\ + \delta[A(\bar{X})C(Y) - A(\bar{Y})C(X)] = 0.$$

Now if we take $(\alpha + \beta) = 0$, then from equation (4.5), we get

$$\delta[A(\bar{X})C(Y) - A(\bar{Y})C(X)] = 0,$$

which implies either $\delta = 0$ or $A(\bar{X})C(Y) = A(\bar{Y})C(X)$.

Thus, we conclude:

Theorem 4.1: *In a projective flat Ricci semi-symmetric mixed super quasi-Einstein nearly Kaehler manifold if $(\alpha + \beta) = 0$ and λ is not an eigen value of Ricci operator Q then either $\delta = 0$ or the vector fields ρ and μ corresponding to 1-forms A and C such that $A(\bar{X})C(Y) = A(\bar{Y})C(X)$.*

Now in equation (4.5), if we take $\delta = 0$, then we get

$$(\alpha + \beta)[A(\bar{X})A(Y) - A(\bar{Y})A(X)] = 0,$$

which implies either $\alpha + \beta = 0$ or $A(\bar{X})A(Y) = A(\bar{Y})A(X)$.

Thus, we conclude:

Theorem 4.2: *In a projective flat Ricci semi-symmetric mixed super quasi-Einstein nearly Kaehler manifold if $\delta = 0$ and λ is not an eigen value of Ricci operator Q then either $\alpha + \beta = 0$ or $A(\bar{X})A(Y) = A(\bar{Y})A(X)$.*

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