

Characterization of Kenmotsu Manifolds Admitting a Non-Symmetric Non-Metric Connection

Abhishek Singh, C. K. Mishra, Lalit Kumar and Shraddha Patel

Department of Mathematics and Statistics,

Dr RML Avadh University, Ayodhya, UP, India

Email: abhi.rmlau@gmail.com ; chayankumarmishra@gmail.com ;
lalitkumargautamjrf@gmail.com ; shraddhapatelbbk@gmail.com

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Abstract: In this paper we have studied some properties of the curvature tensor, the conformal curvature tensor, the con-harmonic curvature tensor of Kenmotsu manifolds admitting a non-symmetric non-metric connection. Further, it has been proved that a Kenmotsu manifold endowed with a non-symmetric non-metric connection is regular. It has been also proved that a Ricci semi-symmetric Kenmotsu manifold admitting a non-symmetric non-metric connection $\tilde{\nabla}$, the Ricci soliton of data (g, ξ, λ) is expanding.

Keywords: Kenmotsu manifolds, non-symmetric non-metric connection, Curvature tensor, Ricci semi-symmetric manifolds, Einstein manifolds, Ricci soliton.

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1. Introduction

The notion of Kenmotsu manifolds was defined by Kenmotsu¹ and he proved that a locally Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kaehler manifold N with warping function $f(t) = se^t$, where s is a non-zero constant. Kenmotsu¹ introduced a class of almost contact metric manifold and named as Kenmotsu manifold. Since the properties of Kenmotsu manifolds have studied by several authors such as Sinha and Srivastava², Cihan and De³ and many others. Friedmann and Schouten⁴ describes that the main invariants of an affine connection are its torsion and curvature. A connection ∇ is called torsion-free or symmetric if its torsion tensor $T(X, Y)$ becomes zero and if not zero then it is known as non-symmetric. A linear connection $\tilde{\nabla}$ on M^n is called non-symmetric if

$$(1.1) \quad \tilde{T}(X, Y) = 2g(\phi X, Y)$$

holds $\forall X, Y$ on M^n . Further, a connection ∇ on a manifold M is called a metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, and if $\nabla g \neq 0$ then it is known as non-metric. The study of semi-symmetric connection in a Riemannian manifold was introduced by Yano.⁵ Golab⁶ defined and studied quarter-symmetric connection in a differentiable manifold with affine connection. Further various properties of quarter-symmetric metric connection have been studied by many geometers like Rastogi⁷, Mishra and Pandey⁸, Yano and Imai⁹, Kumar et al.¹⁰ and many others. Later on some other authors such as Chaubey et al.¹¹, M. M. Tripathi¹² proved the existence of a new connection and given some particular cases. Recently Chaubey¹³ studied the semi-symmetric non-metric connection. Also Bagewadi and Ingalahalli¹⁴ studied the Ricci soliton and Lorentzian α -Sasakian manifolds. Pankaj et al.¹⁵ studied the Sasakian manifolds admitting a non-symmetric non-metric connection.

Above studies motivated us to study the properties of Kenmotsu manifold admitting a non-symmetric non-metric connection. The aim of this paper is to study the properties of Kenmotsu manifolds admitting a non-symmetric non-metric connection. Section 1 is introductory. Section 2 consists of some basic results of Kenmotsu manifolds. The necessary results of a non-symmetric non-metric connection are given in the section 3. Basic properties of Riemannian curvature tensor, conformal curvature tensor, concircular curvature tensor, conharmonic curvature tensor, a non-symmetric non-metric connection were studied in section 4. Ricci semi-symmetric manifolds with respect to a non-symmetric non-metric connection has been studied in section-5.

2. Preliminaries

Let M be a $(2n + 1)$ -dimensional almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ , a vector field ξ , a 1-form η , and the Riemannian metric g on M satisfying

$$(2.1) \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi(X)) = 0, \quad g(X, \xi) = \eta(X),$$

$$(2.2) \quad \phi^2 X = -X + \eta(X)\xi, \quad g(X, \phi Y) = -g(\phi X, Y),$$

$$(2.3) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

An almost contact metric structure (ϕ, ξ, η, g) is a Kenmotsu manifolds iff

$$(2.4) \quad (\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X.$$

Hereafter we denote the Kenmotsu manifolds of dimensional $(2n + 1)$ by M . From the above relation, it follows that

$$(2.5) \quad \nabla_X \xi = X - \eta(X)\xi,$$

$$(2.6) \quad (\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y) = g(\phi X, \phi Y),$$

$$(2.7) \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$(2.8) \quad R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi,$$

$$(2.9) \quad R(\xi, X)\xi = X - \eta(X)\xi,$$

$$(2.10) \quad \eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X),$$

$$(2.11) \quad S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y),$$

$$(2.12) \quad S(X, \xi) = -2n\eta(X),$$

$$(2.13) \quad S(X, Y) = g(QX, Y),$$

where R , S and Q denote the curvature tensor, Ricci tensor and Ricci operator of M , respectively, with respect to the Levi-Civita connection.

Definition 2.1. An almost contact metric manifold M is said to be an η -Einstein manifold if there exists the real valued functions λ_1, λ_2 such that

$$(2.14) \quad S(X, Y) = \lambda_1 g(X, Y) + \lambda_2 \eta(X)\eta(Y).$$

For $\lambda_2 = 0$, the manifold M is an Einstein manifold.

Definition 2.2. A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by

$$(2.15) \quad \mathcal{L}_V g + 2S + 2\lambda g = 0,$$

where $\mathcal{L}_V g$ is a Lie-derivative of Riemannian metric g with respect to vector field V and λ is a real constant. It is said to be shrinking, steady and expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.

3. A non-symmetric non-metric connection

Let us define, a linear connection $\tilde{\nabla}$ as

$$(3.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + g(\phi X, Y)\xi,$$

satisfying

$$(3.2) \quad \tilde{T}(X, Y) = 2g(\phi X, Y)\xi,$$

and

$$(3.3) \quad (\tilde{\nabla}_X g)(Y, Z) = -\eta(Z)g(\phi X, Y) - \eta(Y)g(\phi X, Z).$$

for arbitrary vector fields X, Y and Z is said to be a non-symmetric non-metric connection. Also, we have

$$(3.4) \quad (\tilde{\nabla}_X \phi)(Y) = (\nabla_X \phi)(Y) + g(\phi X, \phi Y)\xi,$$

$$(3.5) \quad (\tilde{\nabla}_X \eta)(Y) = (\nabla_X \eta)(Y) - g(\phi X, Y),$$

$$(3.6) \quad (\tilde{\nabla}_X g)(\phi Y, Z) = -\eta(Z)g(\phi X, \phi Y).$$

On replacing Y by ξ in the equation (3.1), we have

$$(3.7) \quad (\tilde{\nabla}_X \xi) = \nabla_X \xi.$$

On replacing $X = \xi$ in the equation (3.3), we have

$$(3.8) \quad (\tilde{\nabla}_\xi g)(Y, Z) = 0.$$

Hence we have,

Proposition 3.1. *The vector field ξ is invariant with respect to Levi-Civita connection ∇ and non-symmetric non-metric connection $\tilde{\nabla}$.*

Proposition 3.2. *Co-variant differentiation of Riemannian metric g with respect to contra-variant vector field ξ vanish identically in a contact metric manifold admitting a non-symmetric non-metric connection $\tilde{\nabla}$.*

The curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ defined as follows

$$(3.9) \quad \tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X, Y]}Z,$$

where $X, Y, Z \in TM$, using equation (3.1), we have

$$(3.10) \quad \begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + g((\nabla_X \phi)Y, Z)\xi - g((\nabla_Y \phi)X, Z)\xi \\ &\quad + g(\phi Y, Z)\nabla_X \xi - g(\phi X, Z)\nabla_Y \xi, \end{aligned}$$

where

$$(3.11) \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$$

is the Riemannian curvature tensor of Levi-Civita connection.

Proposition 3.3. *The relation between Riemannian curvature tensor admitting a non-symmetric non-metric connection $\tilde{\nabla}$ and Levi-Civita connection ∇ is defined by the equation (3.10).*

4. Some curvature tensor of Kenmotsu manifolds with a non-symmetric non-metric connection

Using equations (2.1), (2.2), (2.4) and (2.5), we have

$$(4.1) \quad \tilde{R}(X, Y)Z = R(X, Y)Z + 2\eta(Z)g(\phi X, Y)\xi - g(\phi X, Z)Y + g(\phi Y, Z)X.$$

Contracting equation (4.1) with respect to X , we have

$$(4.2) \quad \tilde{S}(Y, Z) = S(Y, Z) + 2ng(\phi Y, Z).$$

Using equation (2.13) in equation (4.2), we have

$$(4.3) \quad \tilde{Q}(Y) = Q(Y) + 2n(\phi Y).$$

Again contracting equation (4.2), we have

$$(4.4) \quad \tilde{r} = r.$$

Where $\tilde{S}(Y, Z)$; $S(Y, Z)$, $\tilde{Q}$; Q and $\tilde{r}$; r are the Ricci tensors, Ricci operators and scalar curvatures of a non-symmetric non-metric connection $\tilde{\nabla}$ and Levi-Civita connection ∇ .

On replacing $X = \xi$ in the equation (4.1) and using equations (2.1) and (2.2), we have

$$(4.5) \quad \tilde{R}(\xi, Y)Z = R(\xi, Y)Z + g(\phi Y, Z)\xi.$$

In view of equations (2.8) and (4.5), we have

$$(4.6) \quad \tilde{R}(\xi, Y)Z = \eta(Z)Y - g(Y, Z)\xi + g(\phi Y, Z)\xi.$$

Again on replacing $Z = \xi$ in the equation (4.1) and using equations (2.1), (2.3) and (2.7), we have

$$(4.7) \quad \begin{aligned} \tilde{R}(X, Y)\xi &= R(X, Y)\xi + 2g(\phi Y, Z)\xi \\ &= \eta(X)Y - \eta(Y)X + 2g(\phi Y, Z)\xi \neq 0. \end{aligned}$$

Thus we have the following theorem:

Theorem 4.1. *Every $(2n + 1)$ -dimensional Kenmotsu manifold equipped with $\tilde{\nabla}$ is regular with respect to $\tilde{\nabla}$.*

Now operating η on both side of equation (4.1) and using equation (2.1), we have

$$(4.8) \quad \begin{aligned} \eta(\tilde{R}(X, Y)Z) &= 2g(\phi X, Y)\eta(Z) + \eta(Y)g(X, Z) - \eta(X)g(Y, Z) \\ &\quad + g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y). \end{aligned}$$

On contracting equation (4.7) with respect to X , we have

$$(4.9) \quad \tilde{S}(Y, \xi) = -2n \eta(Y).$$

On taking $\tilde{R}(X, Y)Z = 0$ in the equation (4.1), we have

$$(4.10) \quad R(X, Y)Z = g(\phi X, Z)Y - g(\phi Y, Z)X - 2g(\phi X, Y)\eta(Z)\xi.$$

In view of $\eta(R(X, Y, Z, W)) = g(R(X, Y)Z, W)$ and equation (4.10), we have

$$(4.11) \quad \eta(R(X, Y, Z, W)) = g(\phi X, Z)g(Y, W) - g(\phi Y, Z)g(X, W) \\ - 2g(\phi X, Y)\eta(Z)\eta(W).$$

Contracting the above equation with respect to vector X , we have

$$(4.12) \quad S(Y, Z) = -2ng(\phi Y, Z).$$

Using equation (2.13) in equation (4.12), we have

$$(4.13) \quad Q(Y) = -2n(\phi Y).$$

Again, contracting equation (4.12), we have

$$(4.14) \quad r = 0.$$

The conformal curvature tensor of $\tilde{V}$ is defined as

$$(4.15) \quad \tilde{C}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{2n-1} [\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y \\ + g(Y, Z)\tilde{Q}X - g(X, Z)\tilde{Q}Y] \\ + \frac{\tilde{r}}{2n(2n-1)} [g(Y, Z)X - g(X, Z)Y].$$

Now, using equations (4.1), (4.2), (4.3) and (4.4) in the equation (4.15), we have

$$(4.16) \quad \tilde{C}(X, Y)Z - C(X, Y)Z = 2g(\phi X, Y)\eta(Z)\xi + g(\phi Y, Z)X - g(\phi X, Z)Y \\ - \frac{2n}{(2n-1)} [g(\phi Y, Z)X - g(\phi X, Z)Y \\ + g(Y, Z)(\phi X) - g(X, Z)(\phi Y)],$$

where

$$(4.17) \quad C(X, Y)Z = R(X, Y)Z - \frac{1}{(2n-1)} [S(Y, Z)X - S(X, Z)Y \\ + g(Y, Z)QX - g(X, Z)QY] \\ + \frac{r}{2n(2n-1)} [g(Y, Z)X - g(X, Z)Y].$$

Again, we define conharmonic curvature tensor of $\tilde{\nabla}$ as

$$(4.18) \quad \tilde{L}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{2n-1}[\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y \\ + g(Y, Z)\tilde{Q}X - g(X, Z)\tilde{Q}Y].$$

Using equations (4.1), (4.2), (4.3) and (4.4) in the equation (4.18), we have

$$(4.19) \quad \tilde{L}(X, Y)Z - L(X, Y)Z = 2g(\phi X, Y)\eta(Z)\xi + g(\phi Y, Z)X - g(\phi X, Z)Y \\ - \frac{2n}{(2n-1)}[g(\phi Y, Z)X - g(\phi X, Z)Y \\ + g(Y, Z)(\phi X) - g(X, Z)(\phi Y)],$$

where

$$(4.20) \quad L(X, Y)Z = R(X, Y)Z - \frac{1}{(2n-1)}[S(Y, Z)X - S(X, Z)Y \\ + g(Y, Z)QX - g(X, Z)QY].$$

Thus, we have the following theorem:

Theorem 4.2. *In a Kenmotsu manifold admitting a non-symmetric non-metric connection $\tilde{\nabla}$, the necessary and sufficient condition for the conformal curvature tensor of $\tilde{\nabla}$ coincides with that of ∇ is that the conharmonic curvature tensor of $\tilde{\nabla}$ is equal to that of ∇ .*

The concircular curvature tensor of a Riemannian manifold is defined as

$$(4.21) \quad V(X, Y)Z = R(X, Y)Z + \frac{r}{2n(2n-1)}[g(Y, Z)X - g(X, Z)Y].$$

In view of equations (4.4), (4.10), (4.14) and (4.21), we have the following theorem:

Theorem 4.3. *In a Kenmotsu manifold admitting the non-symmetric non-metric connection $\tilde{\nabla}$, the necessary and sufficient condition for the concircular curvature tensor coincides with curvature tensor is scalar curvature of $\tilde{\nabla}$ to be zero.*

5. Ricci semi-symmetric Kenmotsu manifold

A $(2n + 1)$ -dimensional contact metric manifold M with a non-symmetric non-metric connection is said to be Ricci semi-symmetric if

$$(\tilde{R}(X, Y) \cdot \tilde{S})(Z, U) = 0.$$

i.e.

$$(5.1) \quad \tilde{S}(\tilde{R}(X, Y)Z, U) + \tilde{S}(Z, \tilde{R}(X, Y)U) = 0.$$

Taking $X = \xi$ and using equation (4.6) in the above equation, we have

$$(5.2) \quad \tilde{S}(\tilde{R}(\xi, Y)Z, U) + \tilde{S}(Z, \tilde{R}(\xi, Y)U) = 0.$$

$$(5.3) \quad \eta(Z)\tilde{S}(Y, U) - g(Y, Z)\tilde{S}(\xi, U) + g(\phi Y, Z)\tilde{S}(\xi, U) \\ + \eta(U)\tilde{S}(Z, Y) - g(Y, U)\tilde{S}(Z, \xi) + g(\phi Y, U)\tilde{S}(Z, \xi) = 0.$$

In view of equation (4.9), the above equation yields

$$(5.4) \quad \eta(Z)\tilde{S}(Y, U) + 2ng(Y, Z)\eta(U) - 2ng(\phi Y, Z)\eta(U) \\ + \eta(U)\tilde{S}(Z, Y) + 2ng(Y, U)\eta(Z) - 2ng(\phi Y, U)\eta(Z) = 0.$$

Again replacing $U = \xi$ and using equation (4.9) in the above equation, we have

$$(5.5) \quad \tilde{S}(Y, Z) = 2ng(\phi Y, Z) - 2ng(Y, Z).$$

Using equation (4.2) in the above equation, we have

$$(5.6) \quad S(Y, Z) = -2ng(Y, Z).$$

On contracting equation (5.6), we have

$$(5.7) \quad r = -2n(2n + 1).$$

With the help of equation (5.7), equation (4.4) takes the form

$$(5.8) \quad \tilde{r} = -2n(2n + 1).$$

In view of equation (5.6), we can state the following theorem:

Theorem 5.1. *A Ricci semi-symmetric Kenmotsu manifold equipped with a non-symmetric non-metric connection $\tilde{\nabla}$ is an Einstein manifold.*

Ricci soliton of data (g, V, λ) is defined by the equation (2.15), where g, V, λ are Riemannian metric, a vector field and a real constant. Here we have two conditions with regard to the vector field $V : V \in \text{span}(\xi)$ and $V \perp \text{Span}(\xi)$. Let us discuss first case i. e. $V \in \text{Span}(\xi)$. Ricci soliton of data (g, ξ, λ) on a Kenmotsu manifold admitting a non-symmetric non-metric connection $\tilde{\nabla}$ which is defined as under:

$$(5.9) \quad (\tilde{\mathcal{L}}_{\xi}g)(X, Y) + 2\tilde{S}(X, Y) + 2\lambda g(X, Y) = 0.$$

$\forall X, Y \in TM$. Here $\tilde{\mathcal{L}}_{\xi}g$, the Lie-derivative of Riemannian metric g with respect to ξ admitting a non-symmetric non-metric connection $\tilde{\nabla}$, is defined as

$$\begin{aligned}
 (\tilde{E}_\xi g)(X, Y) &= \xi g(X, Y) + g(\tilde{\ell}_\xi X, Y) + g(X, \tilde{\ell}_\xi Y) \\
 (5.10) \quad &= \xi g(X, Y) + g(\tilde{\nabla}_\xi X - \tilde{\nabla}_X \xi, Y) + g(X, \tilde{\nabla}_\xi Y - \tilde{\nabla}_Y \xi) \\
 &= \xi g(X, Y) + g(\tilde{\nabla}_\xi X, Y) + g(X, \tilde{\nabla}_\xi Y) - g(\tilde{\nabla}_X \xi, Y) \\
 &\quad - g(X, \tilde{\nabla}_Y \xi).
 \end{aligned}$$

Thus

$$(5.11) \quad (\tilde{E}_\xi g)(X, Y) = (\tilde{\nabla}_\xi g)(X, Y) - g(\tilde{\nabla}_X \xi, Y) - g(X, \tilde{\nabla}_Y \xi).$$

Now using equations (2.1), (2.3), (2.5), (3.3), (3.7) and (5.11), we have

$$(5.12) \quad (\tilde{E}_\xi g)(X, Y) = -2g(\phi X, \phi Y).$$

Using equations (5.6) and (5.12) in the equation (5.9), we have

$$(5.13) \quad -g(\phi X, \phi Y) - 2ng(X, Y) + 2ng(\phi X, Y) + \lambda g(X, Y) = 0.$$

On taking $X = Y = \xi$ and using equation (2.1) in the above equation (5.13), we have

$$(5.14) \quad \lambda = 2n > 0.$$

Hence we have the following theorem:

Theorem 5.2. *On a Ricci semi-symmetric Kenmotsu manifold admitting a non-symmetric non-metric connection $\tilde{\nabla}$, the Ricci soliton of data (g, ξ, λ) is expanding.*

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