

Starlike Meromorphic Functions with Positive Coefficients Defined Using Convolution

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Abstract: In this paper we study a class of functions

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n \quad (f_n \geq 0), \text{ for certain meromorphic functions } g \text{ and } h$$

defined in the punctured unit disc $U^* = \{z : 0 < |z| < 1\}$, satisfying

$$\left| \frac{\frac{(f * g)(z) - 1}{(f * h)(z)}}{2\zeta \left[\frac{(f * g)(z) - \alpha}{(f * h)(z)} + \left[1 - \frac{(f * g)(z)}{(f * h)(z)} \right] \right]} \right| < \beta.$$

we obtain coefficient inequality, distortion theorem, closure theorems and an integral operator defined on the new class $M_p(g, h, \alpha, \beta, \zeta)$ also discussed.

Key Words and Phrases: Meromorphic function, starlike function, convolution, positive coefficients, distortion theorem, closure theorem, integral operator.

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1. Introduction

Let Σ denote the class of normalized meromorphic functions f of the form

$$(1.1) \quad f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n$$

defined on the punctured unit disk $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. A function $f \in \Sigma$ is meromorphic starlike of order α ($0 \leq \alpha < 1$) if

$$-\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad \left(z \in \Delta = \Delta^* \cup \{0\} \right).$$

The class of all such functions is denoted by $\Sigma^*(\alpha)$. Similarly the class of convex functions of order α is defined. Let Σ_p be the class of functions $f \in \Sigma$ with $f_n \geq 0$. The subclass of Σ_p consisting of starlike functions with positive real part:

$$MR_p(\alpha) = \left\{ f / \operatorname{Re} \left(-z^2 f'(z) \right) > \alpha \right\}, \quad (0 \leq \alpha < 1).$$

In Definition 1.1 below, we unify these classes by using convolution. The Hadmard product or convolution of two functions $f(z)$ given by (1.1) and

$$(1.2) \quad g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} g_n z^n$$

is defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n g_n z^n.$$

Definition 1.1: Let $f(z) \in \Sigma_p$ be given by (1.1) and $g(z) \in \Sigma_p$ be given by (1.2). The class $M_p(g, h, \alpha, \beta, \xi)$ is defined by

$$M_p(g, h, \alpha, \beta, \xi) = \left\{ f \in \Sigma_p : \left| \frac{\frac{(f * g)(z)}{(f * h)(z)} - 1}{2\zeta \left[\frac{(f * g)(z)}{(f * h)(z)} - \alpha \right] + \left[1 - \frac{(f * g)(z)}{(f * h)(z)} \right]} \right| < \beta \right\}$$

$$0 \leq \alpha < 1, \quad 0 < \beta \leq 1, \quad \frac{1}{2} < \xi \leq 1.$$

2. Main Results

Coefficient Inequalities: Our first result gives a necessary and sufficient condition for a function f to be in the class $M_p(g, h, \alpha, \beta, \xi)$.

Theorem 2.1: Let $f(z) \in \sum_p$ be given by (1.1). Then $f(z) \in M_p(g, h, \alpha, \beta, \xi)$ if and only if

$$(2.1) \quad \sum_{n=1}^{\infty} \left[g_n - h_n - \beta(2\xi g_n - 2\xi\alpha h_n - g_n + h_n) \right] f_n \leq 2\xi\beta(1-\alpha).$$

Proof: Suppose $f(z) \in M_p(g, h, \alpha, \beta, \xi)$. Then

$$\begin{aligned} & \left| \frac{\frac{(f^*g)(z)}{(f^*h)(z)} - 1}{2\xi \left[\frac{(f^*g)(z)}{(f^*h)(z)} - \alpha \right] + \left[1 - \frac{(f^*g)(z)}{(f^*h)(z)} \right]} \right| \\ &= \left| \frac{(f^*g)(z) - (f^*h)(z)}{2\xi [(f^*g)(z) - \alpha((f^*h)(z))] + (f^*h)(z) - (f^*g)(z)} \right| \\ &= \left| \frac{(f^*g)(z) - (f^*h)(z)}{(2\xi - 1)(f^*g)(z) + (1 - 2\xi\alpha)(f^*h)(z)} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty} f_n(g_n - h_n)z^n}{(2\xi - 1) \left(\frac{1}{z} + \sum_{n=1}^{\infty} f_n g_n z^n \right) + (1 - 2\xi\alpha) \left(\frac{1}{z} + \sum_{n=1}^{\infty} f_n h_n z^n \right)} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty} f_n(g_n - h_n)z^{n+1}}{(2\xi - 1) \left(1 + \sum_{n=1}^{\infty} f_n g_n z^{n+1} \right) + (1 - 2\xi\alpha) \left(1 + \sum_{n=1}^{\infty} f_n h_n z^{n+1} \right)} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty} f_n(g_n - h_n)z^{n+1}}{2\xi(1-\alpha) + \sum_{n=1}^{\infty} [(2\xi - 1)g_n + (1 - 2\xi\alpha)h_n] f_n z^{n+1}} \right| < \beta \end{aligned}$$

as $f(z) \in M_p(g, h, \alpha, \beta, \xi)$.

This implies that

$$\sum_{n=1}^{\infty} [g_n - h_n - \beta((2\xi - 1)g_n + (1 - 2\xi\alpha)h_n)] f_n \leq 2\xi\beta(1 - \alpha).$$

Therefore, we obtain

$$\sum_{n=1}^{\infty} [g_n - h_n - \beta(2\xi g_n - 2\xi\alpha h_n - g_n + h_n)] f_n \leq 2\xi\beta(1 - \alpha).$$

Conversely, we assume that (2.1) holds. Then we see that

$$\begin{aligned} & \left| \frac{\frac{(f * g)(z)}{(f * h)(z)} - 1}{2\xi \left[\frac{(f * g)(z)}{(f * h)(z)} - \alpha \right] + \left[1 - \frac{(f * g)(z)}{(f * h)(z)} \right]} \right| \\ &= \left| \frac{(f * g)(z) - (f * h)(z)}{(2\xi - 1)(f * g)(z) + (1 - 2\xi\alpha)(f * h)(z)} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty} f_n(g_n - h_n)z^{n+1}}{2\xi(1 - \alpha) + \sum_{n=1}^{\infty} [(2\xi - 1)g_n + (1 - 2\xi\alpha)h_n] f_n z^{n+1}} \right| \\ &\leq \frac{\sum_{n=1}^{\infty} f_n(g_n - h_n)z^{n+1}}{2\xi(1 - \alpha) - \sum_{n=1}^{\infty} [(2\xi - 1)g_n + (1 - 2\xi\alpha)h_n] f_n z^{n+1}} \leq \beta \end{aligned}$$

as (2.1) holds. Therefore we have $f(z) \in M_p(g, h, \alpha, \beta, \xi)$.

Theorem 2.2: If $f(z) \in M_p(g, h, \alpha, \beta, \xi)$ then

$$f_n \leq \frac{2\xi\beta(1 - \alpha)}{g_n - h_n - \beta(2\xi g_n - 2\xi\alpha h_n - g_n + h_n)}.$$

Proof: Since $f(z) \in M_p(g, h, \alpha, \beta, \xi)$, by Theorem 2.1, we have for each n

$$\begin{aligned} & \left[g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n) \right] f_n \\ & \leq \sum_{n=1}^{\infty} \left[g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n) \right] f_n \leq 2\xi\beta(1-\alpha). \end{aligned}$$

Therefore

$$f_n \leq \frac{2\xi\beta(1-\alpha)}{g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n)}.$$

These results are entirely new and not found anywhere in the literature.

Distortion Theorem

Theorem 2.3: Let $\alpha h_1 - g_1 \leq \alpha h_n - h_n$. If $f(z) \in M_p(g, h, \alpha, \beta, \xi)$, then

$$\frac{1}{r} - r \frac{2\xi\beta(1-\alpha)}{g_1 - \alpha h_1} \leq |f(z)| \leq \frac{1}{r} + r \frac{2\xi\beta(1-\alpha)}{g_1 - \alpha h_1} \quad (|z| = r).$$

Proof: Since $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n$, we have

$$\begin{aligned} |f(z)| & \leq \frac{1}{r} + \sum_{n=1}^{\infty} f_n r^n \leq \frac{1}{r} + \sum_{n=1}^{\infty} f_n \\ & \leq \frac{1}{r} + r \frac{2\xi\beta(1-\alpha)}{g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n)}, \quad n=1, 2, 3, \dots \\ & \leq \frac{1}{r} + r \frac{2\xi\beta(1-\alpha)}{g_1 - \alpha h_1}. \end{aligned}$$

Similarly we can show that

$$|f(z)| \geq \frac{1}{r} - r \frac{2\xi\beta(1-\alpha)}{g_1 - \alpha h_1}.$$

3. Closure Theorems

Let the functions $F_k(z)$ be given by

$$(3.1) \quad F_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_{n,k} z^n, \quad k = 1, 2, 3, \dots, m.$$

We prove the following closure theorems for the class $M_p(g, h, \alpha, \beta, \xi)$.

Theorem 3.1: *The class $M_p(g, h, \alpha, \beta, \xi)$ is closed under convex linear combination.*

Proof: Let the function $F_k(z)$ given by (3.1) be in the class $M_p(g, h, \alpha, \beta, \xi)$. Then it is enough to show that the function

$$H(z) = \lambda F_1(z) + (1 - \lambda) F_2(z), \quad (0 \leq \lambda \leq 1)$$

is also in the class $M_p(g, h, \alpha, \beta, \xi)$.

$$\text{Since } 0 \leq \lambda \leq 1, \quad H(z) = \frac{1}{z} + \sum_{n=1}^{\infty} [\lambda f_{n,1}(z) + (1 - \lambda) f_{n,2}(z)] z^n,$$

we observe that

$$\begin{aligned} & \sum_{n=1}^{\infty} (g_n - h_n - \beta(2g_n - 2\xi\alpha h_n - g_n + h_n)) [\lambda f_{n,1}(z) + (1 - \lambda) f_{n,2}(z)] \\ &= \lambda \sum_{n=1}^{\infty} (g_n - h_n - \beta(2g_n - 2\xi\alpha h_n - g_n + h_n)) f_{n,1}(z) \\ & \quad + (1 - \lambda) \sum_{n=1}^{\infty} (g_n - h_n - \beta(2g_n - 2\xi\alpha h_n - g_n + h_n)) f_{n,2}(z) \\ &\leq 2\beta\xi(1 - \alpha). \end{aligned}$$

Therefore by Theorem 2.1, $H(z) \in M_p(g, h, \alpha, \beta, \xi)$. This completes the proof.

Theorem 3.2: *Let $F_0(z) = \frac{1}{z}$ and*

$$F_n(z) = \frac{1}{z} + \frac{2\xi\beta(1 - \alpha)}{g_n - h_n - \beta(2\xi g_n - 2\xi\alpha h_n - g_n + h_n)} z^n.$$

Then $f(z) \in M_p(g, h, \alpha, \beta, \xi)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum_{n=0}^{\infty} \lambda_n F_n(z)$, where $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$.

Proof: Let

$$f(z) = \sum_{n=0}^{\infty} \lambda_n F_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\lambda_n 2\beta\xi(1-\alpha)}{g_n - h_n - \beta(2g_n - 2\xi\alpha h_n - g_n + h_n)} z^n.$$

Then

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\lambda_n 2\beta\xi(1-\alpha)}{g_n - h_n - \beta(2g_n - 2\xi\alpha h_n - g_n + h_n)} \cdot \frac{g_n - h_n - \beta(2g_n - 2\xi\alpha h_n - g_n + h_n)}{2\beta\xi(1-\alpha)} \\ &= \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_0 \leq 1. \end{aligned}$$

By Theorem 2.1, we have $f(z) \in M_p(g, h, \alpha, \beta, \xi)$.

Conversely, assume that $f(z) \in M_p(g, h, \alpha, \beta, \xi)$. From Theorem 2.2, we have

$$f_n \leq \frac{2\xi\beta(1-\alpha)}{g_n - h_n - \beta(2\xi g_n - 2\xi\alpha h_n - g_n + h_n)}, \text{ for } n = 1, 2, 3, \dots$$

We may take

$$\lambda_n = \frac{g_n - h_n - \beta(2\xi g_n - 2\xi\alpha h_n - g_n + h_n)}{2\xi\beta(1-\alpha)} f_n, \text{ for } n = 1, 2, 3, \dots$$

and $\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n$. Then $f(z) = \sum_{n=0}^{\infty} \lambda_n F_n(z)$.

4. Integral Operator

In this section, we consider integral transforms of functions in the class $M_p(g, h, \alpha, \beta, \xi)$.

Theorem 4.1: Let the function $f(z)$ given by (1.1) be in $M_p(g, h, \alpha, \beta, \xi)$. Then the integral operator

$$F(z) = c \int_0^1 u^c f(uz) dz \quad (0 < u \leq 1, 0 < c < \infty)$$

is in $M_p(g, h, \alpha, \beta, \xi)$ where

$$\delta \leq \frac{(c+n+1)[g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n)]}{2c\beta\xi h_n + (c+n+1)[g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n)]} \\ - \frac{c(1-\alpha)[g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n)]}{2c\beta\xi h_n + (c+n+1)[g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n)]}$$

Proof: Let $f(z) \in M_p(g, h, \alpha, \beta, \xi)$. Then

$$\begin{aligned} F(z) &= c \int_0^1 u^c f(uz) dz \\ &= c \int_0^1 u^c \left[\frac{1}{uz} + \sum_{n=1}^{\infty} f_n(uz)^n \right] du \\ &= c \int_0^1 \left[\frac{u^{c-1}}{z} + \sum_{n=1}^{\infty} f_n u^{n+c} z^n \right] du \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} f_n z^n. \end{aligned}$$

It is sufficient to show that

$$(4.1) \quad \sum_{n=1}^{\infty} \frac{c[g_n - h_n - \beta(2\xi g_n - 2\xi \delta h_n - g_n + h_n)]}{(c+n+1)2\beta\xi(1-\delta)} f_n \leq 1.$$

Since $M_p(g, h, \alpha, \beta, \xi)$, we have

$$\sum_{n=1}^{\infty} \frac{[g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n)]}{2\beta\xi(1-\alpha)} f_n \leq 1.$$

Note that (4.1) is satisfied if

$$\frac{c[g_n - h_n - \beta(2\xi g_n - 2\xi \delta h_n - g_n + h_n)]}{(c+n+1)2\beta\xi(1-\delta)} \leq \frac{[g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n)]}{2\beta\xi(1-\alpha)}$$

Solving for δ we obtain

$$\delta \leq \frac{(c+n+1)[g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n)]}{2c\beta\xi h_n + (c+n+1)[g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n)]} - \frac{c(1-\alpha)[g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n)]}{2c\beta\xi h_n + (c+n+1)[g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n)]} = F(n), \text{ say.}$$

It can be shown that $F(n+1)-F(n) > 0$ for all n . Therefore $F(n)$ is increasing and $F(n) \geq F(1)$. Thus the result follows.

Theorem 4.4: Let the function $f(z)$ given by (1.1) be in $M_p(g, h, \alpha, \beta, \xi)$.

$$F(z) = \frac{1}{c}[(c+1)f(z) + zf'(z)] = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c+n+1}{c} f_n z^n, \quad c > 0.$$

Then $F(z)$ is in $M_p(g, h, \alpha, \beta, \xi)$ for $|z| \leq r(\alpha, \beta, \xi,)$ where

$$r(\alpha, \beta, \xi) = \inf_n \left(\frac{2c\xi\beta(1-\alpha)}{[g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n)](c+n+1)} \right)^{\frac{1}{n+1}}.$$

Proof: Let $w = \frac{f * g}{f * h}$. Then it is sufficient to show that

$$\left| \frac{w-1}{2\xi(w-\alpha)+(1-w)} \right| < \beta.$$

A computation shows that this is satisfied if

$$(4.3) \quad \sum_{n=1}^{\infty} \frac{(c+n+1)[g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n)]}{2c\xi\beta(1-\alpha)} f_n |z|^{n+1} \leq 1.$$

Since $f \in M_p(g, h, \alpha, \beta, \xi)$ by Theorem 2.1, we have

$$\sum_{n=1}^{\infty} \frac{[g_n - h_n - \beta(2\xi g_n - 2\xi \alpha h_n - g_n + h_n)]}{2\xi\beta(1-\alpha)} f_n \leq 1.$$

The equation (4.3) is satisfied if for all n ,

$$\begin{aligned} & \frac{(c+n+1) \left[g_n - h_n - \beta (2\xi g_n - 2\xi \alpha h_n - g_n + h_n) \right]}{2c\xi\beta(1-\alpha)} |z|^{n+1} \\ & \leq \frac{g_n - h_n - \beta (2\xi g_n - 2\xi \alpha h_n - g_n + h_n)}{2\xi\beta(1-\alpha)} z^{n+1} \leq 1. \end{aligned}$$

Solving for $|z|$, we get

$$|z|^{n+1} \leq \frac{2c\xi\beta(1-\alpha)}{(c+n+1) \left[g_n - h_n - \beta (2\xi g_n - 2\xi \alpha h_n - g_n + h_n) \right] (c+n+1)}$$

Therefore we obtain

$$|z| \leq \left(\frac{2c\xi\beta(1-\alpha)}{\left[g_n - h_n - \beta (2\xi g_n - 2\xi \alpha h_n - g_n + h_n) \right] (c+n+1)} \right)^{\frac{1}{n+1}}.$$

The desired result follows by mere substitution of $|z| = r(\alpha, \beta, \xi)$.

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