

Scalar Curvature of Two Dimensional Special Quartic Finsler Space

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Abstract: In the present paper we have obtained scalar curvature R of a two dimensional Finsler space with special quartic metric given by $L^4 = c_0 \dot{x}^4 + 6c_2 \dot{x}^2 \dot{y}^2 + c_4 \dot{y}^4$. Some particular cases of special quartic metric have been considered and explicit expression for scalar curvature R has been found.

Key Words: Finsler space, special quartic metric, scalar curvature.

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1. Introduction

There are several papers¹⁻⁶ in which cubic Finsler spaces have been studied. This cubic metric mostly appears as a generalization of Euclidean or Riemannian metric. In 1995 M. Matsumoto and K. Okubo⁷ introduced the Christoffel symbols of m -th order. They obtained connection coefficients of Berwald and the differential equations of geodesic. They specially treated the two dimensional Finsler spaces with cubic metric and quartic metric, and obtained the main scalar. T. N. Pandey and V. K. Chaubey⁸ obtained the scalar curvature of a two dimensional cubic Finsler space.

In a two dimensional Finsler space F^2 , the main scalar I and the scalar curvature R have important roles. The purpose of the present paper is to obtain scalar curvature R of a two dimensional Finsler space with special quartic metric

$$(1.1) \quad L^4 = c_0 \dot{x}^4 + 6c_2 \dot{x}^2 \dot{y}^2 + c_4 \dot{y}^4,$$

where c_0, c_2, c_4 are functions of x and y only.

2. Preliminaries

The quartic Finsler metric $L(x, y)$ of an n -dimensional differentiable manifold M^n is defined as

$$(2.1) \quad L^4(x, y) = a_{ijkh}(x) y^i y^j y^k y^h,$$

where $a_{ijkh}(x)$ are components of a symmetric tensor field of (0,4)-type depending on the position $x(=x^i)$ alone. A Finsler space with this metric is called a quartic Finsler space. Putting

$$(2.2) \quad L^3 a_i(x, y) = a_{ijkh}(x) y^j y^k y^h, \quad L^2 a_{ij}(x, y) = a_{ijkh}(x) y^k y^h,$$

$$L a_{ijk}(x, y) = a_{ijkh}(x) y^h,$$

the normalized supporting element $l_i = \partial_i L$, the angular metric tensor $h_{ij} = L \partial_i \partial_j L$, the fundamental tensor $g_{ij} = \frac{1}{2} \partial_i \partial_j L^2 = h_{ij} + l_i l_j$ and the C -tensor $C_{ijk} = \frac{1}{4} \partial_i \partial_j \partial_k L^2$ are respectively given by the equations⁷

$$(2.3) \quad (a) \quad l_i = a_i, \quad (b) \quad h_{ij} = 3(a_{ij} - a_i a_j), \quad (c) \quad g_{ij} = 3a_{ij} - 2a_i a_j,$$

$$(d) \quad L C_{ijk} = 3(a_{ijk} - a_{ij} a_k - a_{jk} a_i - a_{ki} a_j + 2a_i a_j a_k).$$

Let us call $a_{ij}(x, y)$ the basic tensor. The metric L is regular if the basic tensor has the non-vanishing determinant. Throughout our theories of quartic root metrics, we should suppose the regularity of the metrics.

By $a^{ij}(x, y)$ we denote the reciprocal of $a_{ij}(x, y)$; the reciprocal $g^{ij}(x, y)$ of the fundamental tensor $g_{ij}(x, y)$ is given as

$$3g^{ij} = a^{ij} + 2a^i a^j, \quad l^i = a^i,$$

where $a_i a^i = 1$, $a^i = a^{ij} a_j$, $l^i = y^i / L = g^{ij} l_j$.

3. Scalar curvature of two dimensional special quartic Finsler space

In two dimensional Finsler space the Berwald frame (l_i, m_i) has important role⁹. Any tensor field can be expressed in terms of this frame.

We are concerned with a quartic Finsler space equipped with a Berwald connection $B\Gamma = (G'_{jk}, G'_j, 0)$, where $G'_j = \partial_j G^i$ and $G'_{jk} = \partial_j G'_k$,

$G_j = g_{ij}G^i = \frac{1}{4}(y^r \partial_j \partial_r L^2 - \partial_j L^2)$, ∂_j and ∂_j representing partial derivatives with respect to y^j and x^j respectively.

There are five torsion tensors and three curvature tensors in the theory of Finsler space equipped with any Finsler connection. If we are concerned with Berwald connection $B\Gamma$, then only non-vanishing torsion tensor is $(v)hv$ -torsion tensor R'_{jk} given by

$$(3.1) \quad R'_{jk} = \delta_k G'_j - \delta_j G'_k,$$

where $\delta_k = \partial_k - G'_k \partial_r$.

The $(v)hv$ -torsion tensor R'_{jk} in two dimensional Finsler space may be written as⁹

$$(3.2) \quad R'_{jk} = LRm'(l_j m_k - l_k m_j),$$

where R is h -scalar curvature and $m' = g^{ij} m_j$.

The quartic metric in two dimensional Finsler space is given by (2.1), where i, j, k, h have values 1 and 2. If we put

$$(a_{1111}, a_{1112}, a_{1122}, a_{1222}, a_{2222}) = (c_0, c_1, c_2, c_3, c_4), (x^1, x^2) = (x, y) \text{ and } (y^1, y^2) = (\dot{x}, \dot{y}),$$

then (2.1) may be written as

$$L^4 = c_0 \dot{x}^4 + 4c_1 \dot{x}^3 \dot{y} + 6c_2 \dot{x}^2 \dot{y}^2 + 4c_3 \dot{x} \dot{y}^3 + c_4 \dot{y}^4.$$

Now we consider $c_1 = c_3 = 0$, then the above metric becomes

$$(3.3) \quad L^4 = c_0 \dot{x}^4 + 6c_2 \dot{x}^2 \dot{y}^2 + c_4 \dot{y}^4.$$

We term it a special quartic metric and Finsler space equipped with metric (3.3) is termed as special quartic Finsler space.

The Christoffel symbols of fourth order for a quartic Finsler space have been defined by M. Matsumoto⁷ and are given by

$$(3.4) \quad \{ijkh, l\} = \frac{1}{6} \left\{ \partial_i a_{jklh} + \partial_j a_{ikhl} + \partial_k a_{ijhl} + \partial_h a_{ijkl} - \partial_l a_{ijkh} \right\}.$$

By putting $\frac{\partial c_a}{\partial x} = c_{a1}$, $\frac{\partial c_a}{\partial y} = c_{a2}$, $a = 0, 2, 4$ we get ten Christoffel symbols of fourth order for the metric (3.3):

$$\begin{aligned}\{1111, 1\} &= \frac{1}{2} c_{01}, & \{1111, 2\} &= -\frac{1}{6} c_{02}, & \{1112, 1\} &= \frac{1}{6} c_{02}, \\ \{2222, 2\} &= \frac{1}{2} c_{42}, & \{2222, 1\} &= -\frac{1}{6} c_{41}, & \{2221, 2\} &= \frac{1}{6} c_{41}, \\ \{1112, 2\} &= \frac{1}{2} c_{21}, & \{1122, 1\} &= \frac{1}{6} c_{21}, & \{2221, 1\} &= \frac{1}{2} c_{22}, \\ \{1122, 2\} &= \frac{1}{6} c_{22}.\end{aligned}$$

If we put $H = L^4 \det a_{ij}$, then from (2.2), we have

$$(3.5) \quad H = A\dot{x}^4 + B\dot{x}^2\dot{y}^2 + C\dot{y}^4,$$

where $A = c_0 c_2$, $B = c_0 c_4 - 3c_2^2$, $C = c_2 c_4$

Then the values of G^1, G^2 in two dimensional special quartic Finsler space are the solutions of linear equation $a_{ir} G^r = \frac{1}{4L^2} \{r j k h, i\} y^r y^j y^k y^h$ and are given by⁷

$$(3.6) \quad 4HG^1 = a\dot{x}^6 + b\dot{x}^5\dot{y} + c\dot{x}^4\dot{y}^2 + d\dot{x}^3\dot{y}^3 + e\dot{x}^2\dot{y}^4 + f\dot{x}\dot{y}^5 + g\dot{y}^6,$$

where $a = \frac{c_2 c_{01}}{2}$, $b = c_2 c_{02}$, $c = \frac{c_4 c_{01}}{2} - 3c_2 c_{21}$, $d = \frac{2}{3} c_4 c_{02}$,

$$e = c_4 c_{21} - \frac{3c_2 c_{41}}{2}, \quad f = 2c_4 c_{22} - c_2 c_{42}, \quad g = -\frac{c_4 c_{41}}{6},$$

and

$$(3.7) \quad 4HG^2 = a'\dot{x}^6 + b'\dot{x}^5\dot{y} + c'\dot{x}^4\dot{y}^2 + d'\dot{x}^3\dot{y}^3 + e'\dot{x}^2\dot{y}^4 + f'\dot{x}\dot{y}^5 + g'\dot{y}^6,$$

where $a' = -\frac{c_0 c_{02}}{2}$, $b' = 2c_0 c_{21} - c_2 c_{01}$, $c' = c_0 c_{22} - \frac{3c_2 c_{02}}{2}$, $d' = \frac{2c_0 c_{41}}{3}$,

$$e' = \frac{c_0 c_{42}}{2} - 3c_2 c_{22}, \quad f' = c_2 c_{41}, \quad g' = \frac{c_2 c_{42}}{2}.$$

Thus we have

Theorem 3.1: The Berwald connection coefficients G^1 and G^2 for two dimensional special quartic Finsler space with metric (3.3) are given by (3.6) and (3.7).

In two dimensional Finsler space, the non-vanishing components of R^1_{jk} are only $R^1_{12} = -R^1_{21}$ and $R^2_{12} = -R^2_{21}$. We are concerned with R^1_{12} . From (3.1), we have

$$(3.1)' \quad R^1_{12} = \frac{\partial G^1_1}{\partial y} - \left(\frac{\partial G^1_1}{\partial x} G^1_2 + \frac{\partial G^1_1}{\partial y} G^2_2 \right) - \frac{\partial G^1_2}{\partial x} + \left(\frac{\partial G^1_2}{\partial x} G^1_1 + \frac{\partial G^1_2}{\partial y} G^2_1 \right),$$

and from (3.2), we have

$$(3.2)' \quad R^1_{12} = L R m^1 (l_1 m_2 - l_2 m_1).$$

Firstly, we are concerned with equation (3.2)' for the metric (3.3). The equation (2.2) explicitly may be written as

$$(3.8) \quad \begin{cases} L^3 a_1 = c_0 \dot{x}^3 + 3c_2 \dot{x}\dot{y}^2, & L^3 a_2 = 3c_2 \dot{x}^2 \dot{y} + c_4 \dot{y}^3, \\ L^2 a_{11} = c_0 \dot{x}^2 + c_2 \dot{y}^2, & L^2 a_{12} = L^2 a_{21} = 2c_2 \dot{x}\dot{y}, \\ L^2 a_{22} = c_2 \dot{x}^2 + c_4 \dot{y}^2, & La_{111} = c_0 \dot{x}, \quad La_{112} = c_2 \dot{y}, \\ La_{122} = c_2 \dot{x}, & La_{222} = c_4 \dot{y}. \end{cases}$$

With the help of (3.8), the equation (2.3c) may be expressed as

$$(3.9) \quad \begin{cases} L^6 g_{11} = c_0 \dot{x}^2 (c_0 \dot{x}^4 + 3c_4 \dot{y}^4) + 3c_2 \dot{y}^2 (3c_0 \dot{x}^4 + c_4 \dot{y}^4), \\ L^6 g_{12} = 2(9c_2^2 - c_0 c_4) (\dot{x}\dot{y})^3, \\ L^6 g_{22} = 3c_2 \dot{x}^2 (c_0 \dot{x}^4 + 3c_4 \dot{y}^4) + c_4 \dot{y}^2 (3c_0 \dot{x}^4 + c_4 \dot{y}^4) \end{cases}$$

Since $l_i = \frac{\partial L}{\partial y^i}$, we have

$$(3.10) \quad L^3 l_1 = c_0 \dot{x}^3 + 3c_2 \dot{x}\dot{y}^2, \quad L^3 l_2 = 3c_2 \dot{x}^2 \dot{y} + c_4 \dot{y}^3.$$

The components m^i and $m_i (i=1,2)$ of Berwald frame are related with l_i and l^i by the relation⁹

$$(3.11) \quad (m^1, m^2) = \frac{1}{\sqrt{g}} (-l_2, l_1), \quad (m_1, m_2) = \sqrt{g} (-l^2, l^1),$$

where $g = \det(g_{ij})$. In view of (3.11) equation (3.2)' reduces to

$$(3.12) \quad R_{12}^1 = -R_{l_2}(l_1\dot{x} + l_2\dot{y}),$$

where l_1 and l_2 are given by (3.10).

Secondly, we are concerned with equation (3.1)'. From (3.5), we have

(3.13)

$$\left[\begin{aligned} \dot{H}_1 &= \frac{\partial H}{\partial \dot{x}} = 4A\dot{x}^3 + 2B\dot{x}\dot{y}^2, & \dot{H}_2 &= \frac{\partial H}{\partial \dot{y}} = 2B\dot{x}^2\dot{y} + 4C\dot{y}^3, \\ \dot{H}_{11} &= \frac{\partial^2 H}{\partial \dot{x}^2} = 12A\dot{x}^2 + 2B\dot{y}^2, & \dot{H}_{22} &= \frac{\partial^2 H}{\partial \dot{y}^2} = 2B\dot{x}^2 + 12C\dot{y}^2, \\ H_1 &= \frac{\partial H}{\partial x} = A_1\dot{x}^4 + B_1\dot{x}^2\dot{y}^2 + C_1\dot{y}^4, \\ H_2 &= \frac{\partial H}{\partial y} = A_2\dot{x}^4 + B_2\dot{x}^2\dot{y}^2 + C_2\dot{y}^4, \\ H_{12} &= \frac{\partial \dot{H}_1}{\partial y} = 4A_2\dot{x}^3 + 2B_2\dot{x}\dot{y}^2, \\ H_{21} &= \frac{\partial \dot{H}_2}{\partial x} = 4C_1\dot{x}^3 + 2B_1\dot{x}^2\dot{y}, \end{aligned} \right.$$

$$\text{where } A_1 = \frac{\partial A}{\partial x}, \quad B_1 = \frac{\partial B}{\partial x}, \quad C_1 = \frac{\partial C}{\partial x}, \quad A_2 = \frac{\partial A}{\partial y}, \quad B_2 = \frac{\partial B}{\partial y}, \quad C_2 = \frac{\partial C}{\partial y}.$$

Since $G_j^i = \partial_j G^i$ and $G_{jk}^i = \partial_j \partial_k G^i$, from (3.6) we have

$$(3.14a) \quad \left[\begin{aligned} 4HG_1^1 &= 6a\dot{x}^5 + 5b\dot{x}^4\dot{y} + 4c\dot{x}^3\dot{y}^2 + 3d\dot{x}^2\dot{y}^3 + 2e\dot{x}\dot{y}^4 + f\dot{y}^5 - 4\dot{H}_1G_1^1, \\ 4HG_{11}^1 &= 30a\dot{x}^4 + 20b\dot{x}^3\dot{y} + 12c\dot{x}^2\dot{y}^2 + 6d\dot{x}\dot{y}^3 + 2e\dot{y}^4 \\ &\quad - 4\dot{H}_{11}G_1^1 - 8\dot{H}_1G_1^1, \\ 4HG_{12}^1 &= 5b\dot{x}^4 + 8c\dot{x}^3\dot{y} + 9d\dot{x}^2\dot{y}^2 + 8e\dot{x}\dot{y}^3 + 5f\dot{y}^4 \\ &\quad - 4\dot{H}_2G_1^1 - 4\dot{H}_{12}G_1^1 - 4\dot{H}_1G_2^1, \end{aligned} \right.$$

$$(3.14b) \quad \begin{cases} 4HG_2^1 = b\dot{x}^5 + 2c\dot{x}^4\dot{y} + 3d\dot{x}^3\dot{y}^2 + 4e\dot{x}^2\dot{y}^3 \\ \quad + 5f\dot{x}\dot{y}^4 + 6g\dot{y}^5 - 4\dot{H}_2G^1, \\ 4HG_{22}^1 = 2c\dot{x}^4 + 6d\dot{x}^3\dot{y} + 12e\dot{x}^2\dot{y}^2 + 20f\dot{x}\dot{y}^3 \\ \quad + 30g\dot{y}^4 - 8\dot{H}_2G_2^1 - 4\dot{H}_{22}G^1, \end{cases}$$

and from (3.7) we have

$$(3.15) \quad \begin{cases} 4HG_1^2 = 6a'\dot{x}^5 + 5b'\dot{x}^4\dot{y} + 4c'\dot{x}^3\dot{y}^2 + 3d'\dot{x}^2\dot{y}^3 \\ \quad + 2e'\dot{x}\dot{y}^4 + f'\dot{y}^5 - 4\dot{H}_1G^2, \\ 4HG_2^2 = b'\dot{x}^5 + 2c'\dot{x}^4\dot{y} + 3d'\dot{x}^3\dot{y}^2 + 4e'\dot{x}^2\dot{y}^3 \\ \quad + 5f'\dot{x}\dot{y}^4 + 6g'\dot{y}^5 - 4\dot{H}_2G^2. \end{cases}$$

From (3.14a) and (3.14b) we have

$$(3.16) \quad \begin{cases} 4H \frac{\partial G_1^1}{\partial y} = 6a_2\dot{x}^5 + 5b_2\dot{x}^4\dot{y} + 4c_2\dot{x}^3\dot{y}^2 + 3d_2\dot{x}^2\dot{y}^3 + 2e_2\dot{x}\dot{y}^4 \\ \quad + f_2\dot{y}^5 - 4H_2G_1^1 - 4H_{12}G^1 - 4\dot{H}_1D_2, \\ 4H \frac{\partial G_2^1}{\partial x} = b_1\dot{x}^5 + 2c_1\dot{x}^4\dot{y} + 3d_1\dot{x}^3\dot{y}^2 + 4e_1\dot{x}^2\dot{y}^3 + 5f_1\dot{x}\dot{y}^4 \\ \quad + 6g_1\dot{y}^5 - 4H_1G_2^1 - 4H_{21}G^1 - 4\dot{H}_2D_1. \end{cases}$$

where

$$a_2 = \partial_2 a, \quad b_2 = \partial_2 b, \quad c_2 = \partial_2 c, \quad d_2 = \partial_2 d, \quad D_2 = \partial_2 G^1,$$

$$b_1 = \partial_1 b, \quad c_1 = \partial_1 c, \quad d_1 = \partial_1 d, \quad e_1 = \partial_1 e, \quad D_1 = \partial_1 G^1,$$

In view of equations (3.5), (3.13), (3.14a), (3.14b), (3.15), (3.16), the equation (3.1)' reduces to

$$(3.17) \quad R_{12}^1 = \frac{X_1}{16H^2},$$

where

$$\begin{aligned}
 (3.18) \quad X_1 = & [(24Ha_2\dot{x}^5 + 20Hb_2\dot{x}^4\dot{y} + 16Hc_2\dot{x}^3\dot{y}^2 + 12Hd_2\dot{x}^2\dot{y}^3 + 8He_2\dot{x}\dot{y}^4 \\
 & + 4f_2H\dot{y}^5 - 16HH_2G_1^1 - 48HH_{12}G_1^1 - 16HH_1D_2) \\
 & - (30a\dot{x}^4 + 20b\dot{x}^3\dot{y} + 12c\dot{x}^2\dot{y}^2 + 6d\dot{x}\dot{y}^3 + 2e\dot{y}^4 - 4\dot{H}_1G_1^1) \\
 & \times (b\dot{x}^5 + 2c\dot{x}^4\dot{y} + 3d\dot{x}^3\dot{y}^2 + 4e\dot{x}^2\dot{y}^3 + 5f\dot{x}\dot{y}^4 + 6g\dot{y}^5 - 4\dot{H}_2G_1^1) \\
 & - (5b\dot{x}^4 + 8c\dot{x}^3\dot{y} + 9d\dot{x}^2\dot{y}^2 + 8e\dot{x}\dot{y}^3 + 5f\dot{y}^4 - 4\dot{H}_{12}G_1^1 - 4\dot{H}_1G_2^1 - 4\dot{H}_2G_1^1) \\
 & \times (b'\dot{x}^5 + 2c'\dot{x}^4\dot{y} + 3d'\dot{x}^3\dot{y}^2 + 4e'\dot{x}^2\dot{y}^3 + 5f'\dot{x}\dot{y}^4 + 6g'\dot{y}^5 - 4\dot{H}_2G_2^2) \\
 & - (4b_1H\dot{x}^5 + 8c_1H\dot{x}^4\dot{y} + 12d_1\dot{x}^3\dot{y}^2 + 16e_1H\dot{x}^2\dot{y}^3 + 20f_1H\dot{x}\dot{y}^4 + 24g_1H\dot{y}^5 \\
 & - 16HH_{21}G_1^1 - 16HH_2D_1 - 16HH_1G_2^1) \\
 & + (5b\dot{x}^4 + 8c\dot{x}^3\dot{y} + 9d\dot{x}^2\dot{y}^2 + 8e\dot{x}\dot{y}^3 + 5f\dot{y}^4 - 4\dot{H}_2G_1^1 - 4\dot{H}_{12}G_1^1 - 4\dot{H}_1G_2^1) \\
 & \times (6a\dot{x}^5 + 5b\dot{x}^4\dot{y} + 4c\dot{x}^3\dot{y}^2 + 3d\dot{x}^2\dot{y}^3 + 2e\dot{x}\dot{y}^4 + f\dot{y}^5 - 4\dot{H}_1G_1^1) \\
 & + (2c\dot{x}^4 + 6d\dot{x}^3\dot{y} + 12e\dot{x}^2\dot{y}^2 + 20f\dot{x}\dot{y}^3 + 30g\dot{y}^4 - 8H_2G_2^1 - 4\dot{H}_{22}G_1^1) \\
 & \times (6a'\dot{x}^5 + 5b'\dot{x}^4\dot{y} + 4c'\dot{x}^3\dot{y}^2 + 3d'\dot{x}^2\dot{y}^3 + 2e'\dot{x}\dot{y}^4 + f'\dot{y}^5 - 4\dot{H}_1G_2^2)].
 \end{aligned}$$

Now in view of (3.12) and (3.17) we have

$$(3.19) \quad R = -\frac{X_1}{16H^2l_2(l_1\dot{x} + l_2\dot{y})},$$

where l_1 and l_2 are given by (3.10), H is given by (3.5) and X_1 is given by (3.18).

Theorem 3.2: *The scalar curvature R of two dimensional Finsler space with a special quartic metric (3.3) is given by (3.19).*

Although we have taken a special quartic metric (3.3) for two dimensional Finsler space, but the expression of R given by (3.19) is very lengthy and has not been written in terms of line-elements explicitly. Therefore in the next article we shall consider particular cases of special quartic metric and we shall find the value of R in terms of line-elements explicitly.

4. Some particular cases of two dimensional special quartic Finsler metric

We consider the two particular cases of metric (3.3).

Case 1: Let $c_0 = c_4$ and $c_2 = 0$. Then the metric becomes $L^4 = c_0(\dot{x}^4 + \dot{y}^4)$ and in this case A, B, C given in (3.5) will be such that $A = 0, B = c_0^2, C = 0$, so that (3.5) reduces to $H = c_0^2 \dot{x}^2 \dot{y}^2$. Furthermore a, b, c, d, e, f, g given in (3.6) and $a', b', c', d', e', f', g'$ given in (3.7) reduce to

$$a = 0, \quad b = 0, \quad c = \frac{1}{2} c_0 c_{01}, \quad d = \frac{2}{3} c_0 c_{02}, \quad e = 0, \quad f = 0, \quad g = -\frac{1}{6} c_0 c_{01},$$

$$a' = -\frac{1}{6} c_0 c_{02}, \quad b' = 0, \quad c' = 0, \quad d' = \frac{2}{3} c_0 c_{01},$$

$$e' = \frac{1}{2} c_0 c_{02}, \quad f' = 0, \quad g' = 0.$$

Hence (3.6), (3.7) and (3.13) reduce to

$$(4.1) \quad 4HG^1 = \frac{1}{2} c_0 c_{01} \dot{x}^4 \dot{y}^2 + \frac{2}{3} c_0 c_{02} \dot{x}^3 \dot{y}^3 - \frac{1}{6} c_0 c_{01} \dot{y}^6,$$

$$(4.2) \quad 4HG^2 = -\frac{1}{6} c_0 c_{02} \dot{x}^6 + \frac{2}{3} c_0 c_{01} \dot{x}^3 \dot{y}^3 + \frac{1}{2} c_0 c_{02} \dot{x}^2 \dot{y}^4,$$

$$(4.3) \quad \dot{H}_1 = 2c_0^2 \dot{x} \dot{y}^2, \quad \dot{H}_2 = 2c_0^2 \dot{x}^2 \dot{y}, \quad \dot{H}_{11} = 2c_0^2 \dot{y}^2, \quad \dot{H}_{22} = 2c_0^2 \dot{x}^2,$$

$$H_1 = 2c_0 c_{01} \dot{x}^2 \dot{y}^2, \quad H_2 = 2c_0 c_{02} \dot{x}^2 \dot{y}^2,$$

$$H_{12} = 4c_0 c_{02} \dot{x} \dot{y}^2, \quad H_{21} = 4c_0 c_{01} \dot{x}^2 \dot{y}.$$

Substituting these values in (3.14) and (3.15) we get

$$(4.4) \quad \begin{cases} 4c_0 G_1^1 = c_{01} \dot{x} + \frac{2}{3} c_{02} \dot{y} + \frac{1}{3} c_{01} \frac{\dot{y}^4}{\dot{x}^3}, & 4c_0 G_2^1 = \frac{2}{3} \left(c_{02} \dot{x} - c_{01} \frac{\dot{y}^3}{\dot{x}^2} \right), \\ 4c_0 G_1^2 = \frac{2}{3} \left(c_{01} \dot{y} - c_{02} \frac{\dot{x}^3}{\dot{y}^2} \right), & 4c_0 G_2^2 = \frac{2}{3} c_{01} \dot{x} + c_{02} \dot{y} + \frac{1}{3} c_{02} \frac{\dot{x}^4}{\dot{y}^3}, \end{cases}$$

$$(4.5) \quad \begin{cases} 4c_0 G_{11}^1 = c_{01} \left(1 - \frac{\dot{y}^4}{\dot{x}^4} \right), & 4c_0 G_{12}^1 = \frac{2}{3} c_{02} + \frac{4}{3} c_{01} \frac{\dot{y}^3}{\dot{x}^3}, \\ 4c_0 G_{22}^1 = -2c_{01} \frac{\dot{y}^2}{\dot{x}^2}. \end{cases}$$

Therefore (3.1)' gives

$$(4.6) \quad 16c_0^2 R_{12}^1 = \frac{X_2}{\dot{x}^3},$$

where

$$(4.7) \quad X_2 = \left[\left(\frac{4}{3} c_0 c_{012} - \frac{32}{9} c_{01} c_{02} \right) (\dot{x}^4 + \dot{y}^4) + \left(\frac{8}{3} c_0 c_{022} - \frac{26}{9} c_{02}^2 \right) \dot{x}^3 \dot{y} \right. \\ \left. + \left(\frac{8}{3} c_0 c_{011} - \frac{26}{9} c_{01}^2 \right) \dot{x} \dot{y}^3 - \frac{2}{9} \left(c_{01}^2 \frac{\dot{y}^7}{\dot{x}^3} + c_{02}^2 \frac{\dot{x}^7}{\dot{y}^3} \right) \right],$$

$$\text{and } c_{012} = \frac{\partial^2 c_0}{\partial x \partial y}, \quad c_{022} = \frac{\partial^2 c_0}{\partial y^2}, \quad c_{011} = \frac{\partial^2 c_0}{\partial x^2}.$$

Also in this particular case, from (3.10) and (3.12) we have

$$(4.8) \quad R_{12}^1 = -\frac{R c_0 \dot{y}^3}{\sqrt{c_0 (\dot{x}^4 + \dot{y}^4)}}.$$

From (4.6) and (4.8) we get the value of scalar curvature R as

$$(4.9) \quad R = -\frac{\sqrt{c_0 (\dot{x}^4 + \dot{y}^4)}}{16c_0^3 \dot{x}^3 \dot{y}^3} X_2.$$

Theorem 4.1: *The (h) scalar curvature R of two dimensional Finsler space with quartic metric $L^4 = c_0 (\dot{x}^4 + \dot{y}^4)$ is given by (4.9), where X_2 is given by (4.7).*

Furthermore if we take $c_0 = x$, then the metric becomes $L^4 = x(\dot{x}^4 + \dot{y}^4)$

and X_2 given by (4.7) reduces to $X_2 = -\frac{26}{9} \dot{x} \dot{y}^3 - \frac{2}{9} \frac{\dot{y}^7}{\dot{x}^3}$. Hence

$$(4.10) \quad R = \frac{(13+u)\sqrt{1+u}}{72x^2\sqrt{x}}, \text{ where } u = \frac{\dot{y}^4}{\dot{x}^4}.$$

Theorem 4.2: *The (h) scalar curvature R of two dimensional Finsler space with quartic metric $L^4 = x(\dot{x}^4 + \dot{y}^4)$ is given by (4.10).*

For $c_0 = y$, equation (4.9) reduces to

$$(4.11) \quad R = \frac{(13+v)\sqrt{1+v}}{72y^2\sqrt{y}}, \text{ where } v = \frac{\dot{x}^4}{\dot{y}^4}.$$

Theorem 4.3: The (h) scalar curvature R of two dimensional Finsler space with quartic metric $L^4 = y(\dot{x}^4 + \dot{y}^4)$ is given by (4.11).

If $c_0 = \text{constant}$ then from (4.7) we get $X_2 = 0$ so that $R = 0$. Actually for $c_0 = \text{constant}$ the metric becomes Minkowski metric for which the (v)hv-torsion tensor vanishes. Therefore it is natural that $R = 0$ for $c_0 = \text{constant}$.

Case 2: Let $c_0 = c_4 = 0$. Then the metric becomes $L^4 = 6c_2\dot{x}^2\dot{y}^2$. Also in this case

$$H = -3c_2^2\dot{x}^2\dot{y}^2, \quad G^1 = \frac{c_{21}}{4c_2}\dot{x}^2, \quad G^2 = \frac{c_{22}}{4c_2}\dot{y}^2, \quad G_1^1 = \frac{c_{21}}{2c_2}\dot{x},$$

$$G_{11}^1 = \frac{c_{21}}{c_2}, \quad G_2^2 = \frac{c_{22}}{2c_2}\dot{y}, \quad G_2^1 = G_1^2 = G_{12}^1 = G_{22}^1 = 0.$$

Therefore equation (3.1)' gives

$$(4.12) \quad R_{12}^1 = \left(c_{212} - \frac{c_{21}c_{22}}{c_2} \right) \frac{\dot{x}}{2c_2}, \text{ where } c_{212} = \frac{\partial^2 c_2}{\partial x \partial y}.$$

Also in this case (3.10) will give $L^3 l_1 = 3c_2\dot{x}\dot{y}^2$, $L^3 l_2 = 3c_2\dot{x}^2\dot{y}$. Therefore equation (3.12) will give

$$(4.13) \quad R_{12}^1 = -R\dot{x}\sqrt{\frac{3c_2}{2}}.$$

From (4.12) and (4.13) we conclude the following:

Theorem 4.4: The scalar curvature R of a two dimensional Finsler space with quartic metric $L^4 = 6c_2\dot{x}^2\dot{y}^2$ is independent of directional arguments $\dot{x}$, $\dot{y}$ and is given by $R = \frac{1}{c_2} \sqrt{\frac{2}{3c_2}} \left(\frac{c_{21}c_{22}}{c_2} - c_{212} \right)$.

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