

Geometric and Physical Properties of Curvature Tensors - A Review

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Abstract: Bernard Riemann was the first to define curvature tensor. Most of the curvature tensors are defined with the help of Riemann curvature tensor, Ricci tensor and metric tensor. It has been observed that different combinations of Ricci tensor and metric tensor in the defined tensors lead to some of the different geometrical and physical properties.

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1. Introduction

Bernard Riemann on 10 June 1854 delivered a lecture on geometry (chosen by Gauss) titled, “*Über die Hypothesen welche Geometrie Zu Grunde liegen*”. In the first part, he posed the problem of how to define an n -dimensional space and ended up giving a definition what now a days referred to as Riemannian space. In the lecture, the notion of the shortest line on the surface, now called geodesic was introduced. The main point of Riemann's lecture was the definition of the curvature tensor. The second part of Riemann's lecture posed a much deeper question about the relationship between real world we live in and geometry. Riemann asked what has been the dimension of real space and what geometry described the real space? Riemann's ideas were not fully understood by the scientists of that time (except Gauss who appreciated the depth of Riemann's thoughts), until sixty years later when the mathematical apparatus developed from his address, Einstein found the frame to fit his physical ideas, his cosmology and cosmogony. Hilbert also developed field equations at about the same time. The spirit of Riemann's address was just the physics needed: the metric structure determined by data¹. Gregorio Ricci-Curbastro, known as the inventor of tensor calculus, further expanded tensor calculus by 1900 with the

help of his student Tullio Levi-Civita, which was effectively used by Albert Einstein to develop his general theory of relativity.

2. Curvature Tensors

Hermann Weyl in 1918 introduced that are known as Weyl geometries and defined projective and conformal curvature tensors, using invariant properties². Weyl curvature tensor is considered a measure of the curvature of space-time or more generally a pseudo-Riemannian manifold. Yano in 1940 coined the idea of a concircular transformation, which preserves circles³. Under this transformation, the concircular curvature tensor remains invariant.

2.1 W^* and W_1 curvature tensors

On the line of the Weyl projective tensor Pokhariyal and Mishra⁴ defined W^* and W_1 curvature tensors to study their relativistic significance. Donia et.al.⁵ studied the W^* -curvature tensor and shown that the energy-momentum tensor T of a space-time having a semi-symmetric W^* curvature tensor is semi-symmetric, whereas the energy-momentum tensor having divergence free W^* curvature tensor is of Codazzi type. A space-time having a traceless W^* curvature tensor is Einstein and a W^* curvature flat space-time is also Einstein. Chaubey⁶ studied Generalized Robertson Walker (GRW) space-time with W_1 curvature tensor and showed that the energy-momentum tensor of the space-time is of Codazzi type if and only if the W_1 curvature tensor is divergence free. It was further shown that pseudo Ricci symmetric space-time with divergence-free W_1 curvature tensor is GRW and a necessary and sufficient condition for a perfect fluid space-time is also given.

2.2 W_2 Curvature tensor

The W_2 Curvature tensor defined by Pokhariyal and Mishra⁷ has been widely studied in various manifolds. This tensor has been studied in the Riemannian manifold⁸, and in the Sasakian manifold⁹. Matsumoto *et.al.*¹⁰ have studied W_2 -curvature tensor in the P-Sasakian manifold, which admits certain tensor fields. Pokhariyal¹¹ has studied this tensor on an Einstein manifold. Moindi¹² studied W_2 -curvature tensor in different manifolds. Shaikh *et.al.*¹³ have introduced the notion of Weakly W_2 -symmetric manifolds and studied some of their properties along with many non-trivial examples. Teleshian and Hosseinzadeh¹⁴ studied W_2 -curvature tensor on $N(k)$ -quasi Einstein manifolds. The role of W_2 -curvature tensor in the study of Kenmotsu manifolds was investigated by Yildiz and De¹⁵. Katende¹⁶ studied W_2 -curvature tensor in LP-Sasakian and Einstein Sasakian manifolds. Zengin¹⁷

studied Riemannian manifolds admitting W_2 -curvature tensor. Venkatesh *et.al.*¹⁸ have studied Lorentzian para-Sasakian manifolds satisfying certain conditions on W_2 -curvature tensor. Moindi *et.al.*¹⁹ have studied W_2 -Recurrent Sasakian manifold. Pokhariyal *et.al.*²⁰ have studied W_2 -recurrent LP-Sasakian manifold. Shenawy and Unal²¹ have studied W_2 -curvature tensor on warped product manifolds and provided some applications. Yadav, Chaubey and Suthar²² have studied W_2 -curvature tensor on (LCS) n -manifolds. Venkatesh and Shanmukha²³ studied W_2 -curvature tensor on generalized Sasakian space forms. Ruganzu *et.al.*²⁴ have studied Eta-Ricci soliton on Lorentzian Para-Sasakian manifold defined with W_2 -curvature tensor. Haseeb *et al.*²⁵ studied the properties of W_2 -curvature tensor on indefinite Kenmotsu manifolds. Mullick and De²⁶ have shown that a W_2 -flat space-time is conformally flat and hence it is of Petrov type 0. Further, it was shown that if the perfect fluid space-time with vanishing W_2 -curvature tensor obeys Einstein's field equations without cosmological constant, then space-time has vanishing acceleration vector and expansion scalar and the perfect fluid always behaves as a cosmological constant. It has also been shown that in a perfect fluid space-time of constant scalar curvature with divergence-free W_2 -curvature tensor the energy-momentum tensor is of Codazzi type and the possible local cosmological structure of such a space-time is of type I, D or 0. Ahsan and Ali²⁷ considered space-time satisfying Einstein's field equations with vanishing W_2 tensor and established the existence of Killing and conformal Killing vector fields. Studying the divergence of W_2 tensor it was shown that a perfect fluid space-time with conserved W_2 tensor represents either an Einstein space or a Friedmann- Robertson Walker cosmological model.

2.3 W_3 and W_4 curvature tensors

In 1973, Pokhariyal²⁸ defined W_3 and W_4 -curvature tensors to study geometrical and physical properties. It was shown by Pokhariyal²⁹ that a W_3 flat almost product and almost decomposable manifold is flat. Further, properties of W_3 tensor were studied in almost Tachibana space and almost Kahler space. Hui³⁰ studied weakly W_3 symmetric $(WW\ 3S)_n$ manifolds and its decomposability. It was shown that Einstein $(WW\ 3S)_n$ is $(WS)_n$ if and only if the scalar curvature of the manifold vanishes. It was also shown that in a decomposable $(WW\ 3S)_n$ one of the decompositions is Ricci symmetric as well as locally symmetric but other decomposition is a manifold of constant curvature. Moindi *et.al.*³¹ have studied the geometric properties of W_3 curvature tensor in the K-contact Riemannian manifold. Pokhariyal³² studied W_3 curvature tensor in the space-time of general relativity and showed that Bianchi-like differential identity is satisfied by this tensor if and only if the

Ricci tensor is of Codazzi type. The space-time satisfying Einstein's field equations with vanishing W_3 tensor has been considered and it has been shown that energy-momentum tensor is proportional to the metric tensor and the existence of Killing as well as conformal Killing vector fields have been established. Further, for a W_3 flat perfect fluid space-time satisfying Einstein's field equations, the isotropic pressure has been found to be a function of the cosmological constant and non-zero gravitational constant. Moindi *et.al.*³³ studied η -Ricci soliton on W_3 semi-symmetric LP-Sasakian manifolds. Isaiah *et.al.*³⁴ have studied W_3 curvature tensor in the LP-Sasakian manifold. Moindi *et.al.*³⁵ studied W_4 -curvature tensor on A-Einstein manifold while Ketende *et.al.*³⁶ studied W_4 -symmetric K-contact Riemannian manifold. Kim³⁷ introduced the notion of pseudo- W_4 -symmetric manifold and studied its various properties by imposing some geometric conditions. Singh and Mishra³⁸ studied generalized Sasakian space form with W_4 -curvature tensor and found Ricci tensor, Ricci operator and scalar curvature in W_4 -flat generalized Sasakian space form. Further, some relations between differentiable functions f_1 , f_2 and f_3 were found. Chagpar *et.al.*³⁹ have studied W_4 -curvature tensor in the space-time of general relativity and shown that it does not satisfy the Bianchi differential identity even with Ricci tensor being of Codazzi type. It is further shown that Einstein's field equations with cosmological constant and Rainich conditions for the existence of non-null electro-variance can be expressed with the help of contracted part of W_4 -curvature tensor. Finally, it was shown that W_4 -flat space-time signifies an empty field.

2.4 W_5 to W_{12} Curvature tensors

The geometric and physical properties of tensors $W_5, W_6, W_7, W_8, W_9, W_{10}, W_{11}$ and W_{12} defined by Pokhariyal⁴⁰⁻⁴² have been studied. Njui *et. al.*⁴³ have studied W_5 -Symmetric K-contact Riemannian manifold. Pokhariyal⁴⁴ has studied W_6 -curvature tensor and its symmetric and skew-symmetric parts in Z and U in Sasakian as well as LP-Sasakian manifolds. Njori *et.al.*⁴⁵ have studied W_6 and W_8 curvature tensors in Kenmotsu manifold admitting semi-symmetric metric connection. Njori *et.al.*⁴⁶⁻⁴⁷ have studied W_8 -curvature tensor in Sasakian manifold and LP-Sasakian manifold. Further, Njori *et.al.*⁴⁸ studied W_8 -curvature tensor in generalized Sasakian space form.

2.5 m -projective tensor

Breaking the curvature tensors W^*, W_1, W_2 and W_3 into skew-symmetric parts, the tensor so defined by Pokhariyal and Mishra^{4,7} and Pokhariyal²⁸,

referred to as m -curvature tensor has been studied in various manifolds as well as in space-time of general relativity. Ojha provided a note on m -projective curvature tensor⁴⁹ and studied m -projective flat Sasakian manifolds⁵⁰. Chaubey and Ojha⁵¹ studied m -projective tensor in a Kenmotsu manifold. Chaubey⁵²⁻⁵³ studied some properties of LP-Sasakian manifolds equipped with m -projective curvature tensor and in studied weakly m -projective symmetric manifolds. Further, Chaubey *et.al.*⁵⁴ studied some properties of m -curvature tensor in Kenmotsu manifolds. J.P. Singh⁵⁵ has studied m -projective recurrent Riemannian manifold. Singh and Pandey⁵⁶ studied m -projective tensor on $N(k)$ contact metric manifold. Prakasha *et.al.*⁵⁷ have studied m -projective tensor on Lorentzian para-Sasakian manifolds. Chaubey and his coauthors have studied the properties of m -projective curvature tensor on $N(k)$ -quasi-Einstein manifolds⁵⁸⁻⁶⁰, trans-Sasakian and f -Kenmotsu manifolds⁶⁶, cosymplectic manifolds⁵⁹, Riemannian manifolds⁶⁷, trans-Sasakian manifolds^{68,69}, generalized Sasakian space-forms⁶⁶. Chattopadhyay *et.al.*⁶⁷ studied application of m -projective tensor in general relativity. Pokhariyal and Mishra^{4,7} defined P -curvature tensor (popularly used as m -curvature tensor) and studied the relativistic significance. The P -curvature tensor satisfies Bianchi differential identity if and only if Ricci tensor is of Codazzi type^{68,70}. Pokhariyal and Chaubey⁶⁹ studied P -curvature tensor in the space-time of general relativity and showed that the divergence of the tensor $G_{\alpha\beta}$ defined with the help of $P_{\alpha\beta}$ and further properties with vanishing P -curvature tensor are investigated. The P_1 -curvature tensor has been defined by breaking W_3 -curvature tensor into skew-symmetric parts in X and Y to study the relativistic significance²⁸. It has been shown by Chagpar *et.al.*⁷¹ that Bianchi-like differential identity is satisfied by P_1 -tensor if and only if the Ricci tensor is Codazzi type. Einstein-like field equations can be expressed with the help of the contracted part of P_1 tensor, which is conserved if the energy-momentum tensor is Codazzi type and the scalar $P_1 = 2R$. Further properties of P_1 -flat space-time are investigated. Tripathi and Gupta⁷² defined τ -curvature tensor which is a generalization of curvature tensors: conformal, quasi-conformal, conharmonic, projective, pseudo-projective, m -projective and $W_0, W_0^*, W_i, i = 1, 2 \dots 9$. Several derivations on $(N(k), \xi)$ semi-Riemannian manifold, whose particular cases are: $N(k)$ contact metric, Sasakian, Kenmotsu, ϵ -Sasakian, para Sasakian and ϵ -para-Sasakian, were studied. The pseudo symmetric conditions of τ -curvature tensor on $N(k)$ contact metric manifolds using some special flatness conditions on τ -curvature tensor were investigated.

3. $A(X, Y, Z, T)$ Curvature Tensor

This tensor was defined by Pokhariyal and Mishra⁴ by breaking W_1 -curvature tensor into skew-symmetric parts in Z and T to study some of the properties. Further properties are being investigated in this presentation.

$$\begin{aligned}
 A(X, Y, Z, T) &= \frac{1}{2}[W_1(X, Y, Z, T) - W_1(X, Y, T, Z)] \\
 (3.1) \quad &= R(X, Y, Z, T) - 1/2(n-1)[g(Y, T)\text{Ric}(X, Z) \\
 &\quad - g(X, T)\text{Ric}(Y, Z) + g(X, Z)\text{Ric}(Y, T) - g(Y, Z)\text{Ric}(X, T)]
 \end{aligned}$$

Further

$$\begin{aligned}
 (3.2) \quad &A(X, Y, Z, T) = -A(Y, X, Z, T); \\
 &A(X, Y, Z, T) = -A(X, Y, T, Z); \\
 &A(X, Y, Z, T) = A(Z, T, X, Y).
 \end{aligned}$$

3.1. Bianchi Differential Identity for $A(X, Y, Z, T)$ tensor

The Bianchi differential identity is given by

$$(3.3) \quad (\nabla_U R)(X, Y, Z, T) + (\nabla_Z R)(X, Y, T, U) + (\nabla_T R)(X, Y, U, Z) = 0.$$

For V_4 the 4-dimensional space-time of general relativity from (3.1), we have

$$\begin{aligned}
 (3.4) \quad A(X, Y, Z, T) &= R(X, Y, Z, T) - (1/6)[g(Y, T)\text{Ric}(X, Z) - g(X, T)\text{Ric}(Y, Z) \\
 &\quad + g(X, Z)\text{Ric}(Y, T) - g(Y, Z)\text{Ric}(X, T)]
 \end{aligned}$$

$$\begin{aligned}
 (3.5) \quad &(\nabla_U A)(X, Y, Z, T) + (\nabla_Z A)(X, Y, T, U) + (\nabla_T A)(X, Y, U, Z) \\
 &= (\nabla_U R)(X, Y, Z, T) - (1/6)[g(Y, T)(\nabla_U \text{Ric})(X, Z) - g(X, T)(\nabla_U \text{Ric})(Y, Z) \\
 &\quad + g(X, Z)(\nabla_U \text{Ric})(Y, T) - g(Y, Z)(\nabla_U \text{Ric})(X, T)] + (\nabla_Z R)(X, Y, T, U) \\
 &\quad - \frac{1}{6}[g(Y, U)(\nabla_Z \text{Ric})(X, T) - g(X, U)(\nabla_Z \text{Ric})(Y, T) + g(X, T)(\nabla_Z \text{Ric})(Y, U) \\
 &\quad - g(Y, T)(\nabla_Z \text{Ric})(X, U) + (\nabla_T R)(X, Y, U, Z) - (1/6)[g(Y, Z)(\nabla_T \text{Ric})(X, U) \\
 &\quad - g(X, Z)(\nabla_T \text{Ric})(Y, U) + g(X, U)(\nabla_T \text{Ric})(Y, Z) - g(Y, U)(\nabla_T \text{Ric})(X, Z)].
 \end{aligned}$$

By virtue of (3.3), on simplification, we have

$$\begin{aligned}
 (3.6) \quad &(\nabla_U A)(X, Y, Z, T) + (\nabla_Z A)(X, Y, T, U) + (\nabla_T A)(X, Y, U, Z) \\
 &= (-1/6)[g(Y, T)\{\nabla_U \text{Ric}(X, Z) - (\nabla_Z \text{Ric})(X, U)\} \\
 &\quad - g(X, T)\{\nabla_U \text{Ric}(Y, Z) - \nabla_Z \text{Ric}(Y, U)\} \\
 &\quad - g(Y, Z)\{\nabla_U \text{Ric}(X, T) - (\nabla_T \text{Ric})(Y, U)\} \\
 &\quad + g(Y, U)\{(\nabla_Z \text{Ric})(X, T) - (\nabla_T \text{Ric})(X, Z)\} \\
 &\quad + g(X, U)\{\nabla_T \text{Ric}(Y, Z) - \nabla_Z \text{Ric}(Y, T)\} \\
 &\quad - g(X, Z)\{\nabla_T \text{Ric}(Y, U) - \nabla_U \text{Ric}(Y, T)\}].
 \end{aligned}$$

If the Ricci tensor is of Codazzi type⁷⁰, then

$$(3.7) \quad (\nabla_Z \text{Ric})(X, T) = (\nabla_T \text{Ric})(X, Z) = (\nabla_U \text{Ric})(X, Z).$$

Using (3.7) in (3.6), we get

$$(3.8) \quad (\nabla_U A)(X, Y, Z, T) + (\nabla_Z A)(X, Y, T, U) + (\nabla_T A)(X, Y, U, Z) = 0,$$

which is Bianchi like differential identity for tensor $A(X, Y, Z, T)$. The converse can be proved similarly. Thus, we have the following theorem.

Theorem 3.1. *For the 4-dimensional space-time, the A-curvature tensor satisfies Bianchi-like differential identity if and only if the Ricci tensor is of Codazzi type.*

Considering $A(X, Y, Z, T)$ curvature tensor in local coordinates, we have

$$(3.9) \quad A_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{2(n-1)} [g_{\beta\delta} R_{\alpha\gamma} - g_{\alpha\delta} R_{\beta\gamma} + g_{\alpha\gamma} R_{\beta\delta} - g_{\beta\delta} R_{\alpha\gamma}].$$

This can be written as

$$(3.10) \quad A_{\beta\gamma\delta}^{\alpha} = R_{\beta\gamma\delta}^{\alpha} - \left\{ \frac{1}{2(n-1)} \right\} [g_{\beta\delta} R_{\gamma}^{\alpha} - g_{\delta}^{\alpha} R_{\beta\gamma} + g_{\gamma}^{\alpha} R_{\beta\delta} - g_{\beta\delta} R_{\gamma}^{\alpha}].$$

Contracting α and δ , on simplification, we get

$$(3.11) \quad A_{\beta\gamma} = \{1/2(n-1)\} [(3n-4)R_{\beta\gamma} + Rg_{\beta\gamma}].$$

For $n = 4$, in V_4 we have,

$$(3.12) \quad A_{\beta\gamma} = (4/3)R_{\beta\gamma} + (R/6)g_{\beta\gamma}.$$

Hence, $A_{\beta\gamma}$ does not vanish in an Einstein space. Further, the scalar invariant $A = 2R$.

3.2 A-Flat Space-Time

The properties of A-flat space-time are now investigated.

Definition: The space-time is said to be A-flat, if the tensor $A_{\alpha\beta\gamma\delta}$ vanishes in it. Thus, for A-flat space-time, we have

$$(3.13) \quad R_{\beta\gamma\delta}^{\alpha} = \{1/2(n-1)\} [g_{\beta\delta} R_{\gamma}^{\alpha} - g_{\delta}^{\alpha} R_{\beta\gamma} + g_{\gamma}^{\alpha} R_{\beta\delta} - g_{\beta\delta} R_{\gamma}^{\alpha}].$$

Contacting α and δ , on simplification, we get

$$(3.14) \quad R_{\beta\gamma} = \{2 - n/2(n-1)\} R_{\beta\gamma} - \{R/2(n-1)\} g_{\beta\gamma}.$$

For V_4 , we have

$$(3.15) \quad R_{\beta\gamma} + (R/8)g_{\beta\gamma} = 0.$$

Thus, even A-flat space-time is not an Einstein space.

Proposition 3.2. *Curvature tensors having (skew) symmetric properties are the ones that satisfy the cyclic properties.*

Proposition 3.3. *Curvature tensors satisfying both symmetric and skew-symmetric properties satisfy both cyclic properties.*

Proposition 3.4. *Curvature tensors that do not satisfy skew-symmetric or symmetric properties, do not satisfy Bianchi differential identity even with Ricci tensor being Codazzi type.*

Studying the structures of $W_1, W_2, \dots, W_9$, it was found that⁹:

- (i) The vanishing of the divergence of all those tensors in which the vector field X is present in all the metric potentials, reduces the electromagnetic field to a purely electric field.
- (ii) All the tensors, in which the vector field X is distributed in such a fashion that it is either in both the metric potentials or in both the matter tensors alone, satisfy the cyclic properties with fixed X .
- (iii) The skew-symmetric parts defined by those skew-symmetric tensors in which the combinations of vector fields X, T and Y, Z are distributed in a term among the metric potentials and matter tensors with a negative sign, enable to extend the Pirani formulation of gravitational waves to the Einstein space.
- (iv) Rainich conditions for the existence of the non-null electrovariance can be obtained by the contracted part of the defined tensors, by replacing matter tensor by it.

Flat Space-times: Checking the various flat space-times it was noticed that:

- i) W_1 -flat and W_4 -flat space times signify an empty field.
- ii) W_2 -flat and P -flat space-times are Einstein spaces.
- iii) W_3 -flat space-time is not an Einstein space (but an almost Einstein space).
- iv) P_1 -flat (or A -flat) space-time is not an Einstein space.
- v) For W_2 -flat, W_3 -flat, P -flat and P_1 -flat: space-times satisfying Einstein field equations with cosmological term, there exists a Killing vector field ξ if and only if the Lie derivative of the energy-momentum tensor vanishes with respect to ξ . Also these space-times admit conformal Killing vector field if and only if the energy-momentum tensor has the symmetry inheritance property.

Scalar Invariants: The values of the various scalars were found to be:

- i) W_1, W_3 and P_1 are twice the scalar curvature R .
- ii) W_2 and P are identically equal to zero.
- iii) W_4 is equal to scalar curvature R .

Perfect Fluid Space Time satisfying Einstein Field Equations with Cosmological constant:

- i) If W_2 -flat, the matter content of the space-time obeys the vacuum-like equation of state.
- ii) If W_3 -flat, the energy-momentum tensor is proportional to the metric tensor. The isotropic pressure is a function of cosmological constant and non-zero gravitational constant.

Divergence of Curvature tensor and perfect fluid space-time:

- i) For space-time possessing harmonic curvature with divergence-free W_2 -tensor (W_3 - tensor), the Ricci tensor is covariantly constant.
- ii) For a V_4 the divergence of W_2 -tensor (W_3 -tensor) vanishes and Ricci tensor is covariantly constant, then the space-time is of constant curvature.

Discussion: It is seen that the used curvature tensors have some similar as well as dissimilar physical and geometrical properties. It would be interesting to investigate the possible reasons for these properties, some of which are suggested, but more to be found.

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