

Integrability conditions of an Almost Contact 3- Structure Manifold

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(Received August 9, 2011)

Abstract: In this paper, we have obtained the integrability conditions for an almost contact 3- structure manifold, some related results have also been discussed.

1. Introduction

Let $\{F_x, U_x, u^x\}$, where $x = 1, 2, 3$ be 3- structure on an odd dimensional manifold V_n , such that

$$(1.1a) \quad \varepsilon_{xyz} F_z = F_x F_y - u^x \otimes U_y + \delta_{xy} I_n,$$

$$(1.1b) \quad F_x U_y = \varepsilon_{xyz} U_z,$$

$$(1.1c) \quad u^x \circ F_y = \varepsilon_{xyz} u^z,$$

$$(1.1d) \quad u^x \left(U_y \right) = \delta_{xy}^x.$$

where $\varepsilon_{xyz} = 1$ or -1 according as xyz is an even or odd permutation of 123 and 0 otherwise. Then $\{F_x, U_x, u^x\}$, where $x = 1, 2, 3$ are said to define an almost contact 3- structure on V_n and the manifold is called an almost contact 3- structure manifold¹.

The Nijenhuis tensor with respect to the structure F_x on an almost contact 3-structure manifold is given by

$$(1.2) \quad N_x(X, Y) = \left[F_x X, F_x Y \right] + F_x^2 [X, Y] - F_x \left[F_x X, Y \right] - F_x \left[X, F_x Y \right], \\ = \left[F_x X, F_x Y \right] - [X, Y] + u([X, Y])U_x - F_x \left[F_x X, Y \right] - F_x \left[X, F_x Y \right].$$

Agreement 1.1. In what follows and the above the equations containing $X, Y, Z, \dots$, etc. hold for arbitrary vectors $X, Y, Z, \dots$, etc. in V_n , unless stated otherwise.

2. Integrability Conditions

Theorem 2.1. A manifold V_n with an almost 3-structure $\{F_x, U_x, u_x\}$, $x = 1, 2, 3$ is odd dimensional. If $n = 2m+1$, there is an eigen value 0, corresponding to the eigen vectors U_x , m - eigen values i and m - eigen values $-i$ of F_x , the corresponding pencils of eigen vectors being linear combination of linearly independent vectors T_{α_x} ($\alpha = 1, \dots, m$) and their complex conjugate S_{α_x} . Such that $u_x(T_{\alpha_x}) = u_x(S_{\alpha_x}) = 0$.

Proof . Let λ_x be an eigen value of F_x , the corresponding eigen vectors being L_x , then

$$(2.1) \quad F_x L_x = \lambda_x L_x.$$

Applying F_x in equation (2.1) and using equation (1.1)a, we get

$$(2.2) \quad (\lambda_x^2 + 1) L_x = u_x(L_x) U_x.$$

Now there are two possibilities:

(i) $L_x = a^x U_x$, $a^x \in \mathbb{R}$, then using (1.1)d in equation (2.2), we get $\lambda_x = 0$,

which gives the eigen value corresponding to the eigen vector U_x .

(ii) $\{L_x, U_x\}$ are linearly independent. Then from equation (2.2) $u_x(L_x) = 0$ and $\lambda_x = \pm i$.

Since the complex conjugate eigen values occur in pairs the dimension of V_n is odd, that is $n = 2m + 1$. Since the rank $((F))$ is $2m$, there are m - eigen values i and m - eigen values $-i$. The remaining part of the proof is obvious.

Theorem 2.2. *The necessary and sufficient condition that V_n (n odd) be an almost contact 3- structure manifold is that it contains a tangent bundle π_m of dimension m , a tangent bundle $\tilde{\pi}_m$ complex conjugate to π_m and a real line π_1 , such that $\pi_m \cap \pi_1 = \phi$, $\pi_m \cap \tilde{\pi}_m = \phi$, $\tilde{\pi}_m \cap \pi_1 = \phi$ and $\pi_1 \cup \pi_m \cup \tilde{\pi}_m =$ a tangent bundle of dimension $n=2m+1$. Projections l, m, n on $\pi_m, \tilde{\pi}_m, \pi_1$ being given by*

$$(2.3a) \quad \underline{l} \stackrel{\text{def}}{=} F^2 - i F,$$

$$(2.3b) \quad \underline{m} \stackrel{\text{def}}{=} F^2 + i F,$$

$$(2.3c) \quad \underline{n} \stackrel{\text{def}}{=} F^2 + I_n = u \otimes U.$$

Proof . Let V_n be an almost contact 3- structure manifold with the almost contact 3- structure $\{F, U, u\}, x = 1, 2, 3$ corresponding to the eigen value i , let $T_{\alpha_x} (\alpha = 1, \dots, m)$ be a m - linearly independent eigen vectors. Let S_{α_x} be complex conjugate to T_{α_x} . Then

$$a^{\alpha_x} T_{\alpha_x} = 0 \Rightarrow a^{\alpha_x} = 0 \forall \alpha_x;$$

$$b^{\alpha_x} S_{\alpha_x} = 0 \Rightarrow b^{\alpha_x} = 0 \forall \alpha_x;$$

Now

$$C^{\alpha_x} T_{\alpha_x} + d^{\alpha_x} S_{\alpha_x} + e^x U = 0 \Rightarrow c^{\alpha_x} F(T_{\alpha_x}) + d^{\alpha_x} F(S_{\alpha_x}) \Rightarrow c^{\alpha_x} T_{\alpha_x} - d^{\alpha_x} S_{\alpha_x} = 0.$$

All these equations imply

$$e^x U = c^{\alpha_x} T_{\alpha_x} = d^{\alpha_x} S_{\alpha_x} = 0.$$

or

$$e^x = c^{\alpha_x} = d^{\alpha_x} = 0, \forall \alpha_x \text{ and } x \text{ (} x \text{ is not summed).}$$

Thus $\{T_{\alpha_x}, S_{\alpha_x}, U\}$ is a linearly independent set. From equations (2.3)a, (2.3)b and (2.3)c, it can be easily seen that

$$(2.4a) \quad l_x T_{\alpha_x} = T_{\alpha_x}, \quad (2.4b) \quad l_x S_{\alpha_x} = 0, \quad (2.4c) \quad l_x U_x = 0.$$

$$(2.5a) \quad m_x T_{\alpha_x} = 0, \quad (2.5b) \quad m_x S_{\alpha_x} = S_{\alpha_x}, \quad (2.5c) \quad m_x U_x = 0.$$

$$(2.6a) \quad n_x T_{\alpha_x} = 0, \quad (2.6b) \quad n_x S_{\alpha_x} = 0, \quad (2.6c) \quad n_x U_x = U_x.$$

Thus, we prove that on an almost contact 3-structure manifold V_{2m+1} there is a tangent bundle π_m of dimension m , a tangent bundle $\tilde{\pi}_m$ complex conjugate to π_m and a real line π_1 , such that $\pi_m \cap \pi_1 = \phi$, $\pi_m \cap \tilde{\pi}_m = \phi$, $\tilde{\pi}_m \cap \pi_1 = \phi$ and $\pi_1 \cup \pi_m \cup \tilde{\pi}_m$ is a tangent bundle of dimension $n=2m+1$. Projections on π_m , $\tilde{\pi}_m$ and π_1 being l , m and n respectively.

Conversely, suppose that there is a tangent bundle π_m of dimension m , a tangent bundle $\tilde{\pi}_m$ complex conjugate to π_m and a real line π_1 , such that π_m , $\tilde{\pi}_m$ and π_1 have no common direction and span a linear manifold of dimension $2m+1$. Let T_{α_x} be a m -linearly independent vectors in π_m and S_{α_x} be complex conjugate to T_{α_x} be a m -linearly independent vectors in $\tilde{\pi}_m$ and U_x be vectors in π_1 .

Let $\{T_{\alpha_x}, S_{\alpha_x}, U_x\}$ span a tangent bundle of dimension $2m+1$. Then $\{T_{\alpha_x}, S_{\alpha_x}, U_x\}$ is a linearly independent set. Let us define the inverse set $\{T^{\alpha_x}, S^{\alpha_x}, u^x\}$, such that

$$(2.7) \quad X = T^{\alpha_x}(X) T_{\alpha_x} + S^{\alpha_x}(X) S_{\alpha_x} + u^x(X) U_x.$$

Let us put

$$(2.8) \quad F_x X \stackrel{\text{def}}{=} \{T^{\alpha_x}(X) T_{\alpha_x} - S^{\alpha_x}(X) S_{\alpha_x}\}.$$

and

$$(2.9) \quad F_x(F_x X) = i \{T^{\alpha_x}(F_x X) T_{\alpha_x} - S^{\alpha_x}(F_x X) S_{\alpha_x}\}.$$

Substituting from equations (2.7) and (2.8) in equation (2.9), we get

$$F^2 X = - \{ T^{\alpha_1}(X) T_{\alpha_1} + S^{\alpha_1}(X) S_{\alpha_1} \} = -X + u(X) U$$

Thus, we see that V_{2m+1} admits an almost contact 3- structure $\{F, U, u\}$.

Hence the conditions is sufficient.

Corollary 2.1. *We have*

$$(2.10a) \quad l = T^{\alpha_1} \otimes T_{\alpha_1}$$

$$(2.10b) \quad m = S^{\alpha_1} \otimes S_{\alpha_1}$$

$$(2.10c) \quad n = u \otimes U$$

Proof. From equations (2.3)a and (2.3)b, we have

$$(2.11) \quad F = i(l - m)$$

Comparing equations (2.8) and (2.11), we get the equations (2.10)a and (2.10)b. Equations (2.3) yields

$$(2.12) \quad l + m + n = I_n$$

Equation (2.10)c is obtained from equations (2.7),(2.12) and (2.10)a,b.

Corollary 2.2. *We have*

$$(2.13) \quad l m = m l = l n = n l = m n = n m = 0.$$

$$(2.14) \quad l^2 = l, \quad m^2 = m, \quad n^2 = n.$$

Proof. From equations (2.10)a, (2.10)b and (2.4)b, we have

$$l m = S^{\alpha_1} \otimes l S_{\alpha_1} = 0.$$

Other equations of (2.13) follow similarly. Equations (2.10)a and (2.4)a yield

$$l^2 = T^{\alpha_1} \otimes l T_{\alpha_1} = T^{\alpha_1} \otimes T_{\alpha_1} = l.$$

Other equations of (2.14) follow similarly.

Lemma 2.1. *We have*

$$(2.15a) \quad (d_x^l)(n_x X, n_x Y) = 0,$$

$$(2.15b) \quad (dT_x^a)(n_x X, n_x Y) = 0.$$

Proof. With the help of the equation (2.10)c, we have

$$(d_x^l)(n_x X, n_x Y) = (d_x^l)(u_x(X)U_x, u_x(Y)U_x) = 0.$$

Since (d_x^l) is skew-symmetric in both the slots. Similarly (2.15)b is proved

Lemma 2.2. We have

$$(2.16) \quad 2(dm_x)(l_x X, l_x Y) = F_x^2[l_x X, l_x Y] - iF_x[l_x X, l_x Y].$$

$$(2.17) \quad 8(dm_x)(l_x X, l_x Y) = F_x^2 N_x(F_x X, F_x Y) - iF_x N_x(F_x X, F_x Y).$$

Proof. In consequence of the equations (2.3) and $l_x m_x = 0 = m_x l_x$, we have

$$2(dm_x)(l_x X, l_x Y) = -2m_x[l_x X, l_x Y] = F_x^2[l_x X, l_x Y] - iF_x[l_x X, l_x Y].$$

Also

$$(2.18) \quad \begin{aligned} 8(dm_x)(l_x X, l_x Y) &= F_x^2[F_x^2 X + iF_x X, F_x^2 Y + iF_x Y] - \\ &iF_x[F_x^2 X + iF_x X, F_x^2 Y + iF_x Y] = F_x^2[F_x^2 X, F_x^2 Y] - F_x^2[F_x X, F_x Y] + iF_x^2 \\ &[F_x^2 X, F_x Y] + iF_x^2[F_x X + F_x^2 Y] - iF_x[F_x^2 X, F_x^2 Y] + iF_x[F_x X, F_x Y] \\ &+ F_x[F_x^2 X, F_x Y] + F_x[F_x X + F_x^2 Y]. \end{aligned}$$

Now, using equations(1.1) and (1.2) in equation (2.18), we get the equation (2.17).

Lemma 2.3. We have

$$(2.19) \quad (dn_x)(l_x X, l_x Y) = -u_x\left([l_x X, l_x Y]\right)U_x,$$

$$(2.20) \quad 4(dn_x)(l_x X, l_x Y) = F_x^2 N_x(X, Y) + N_x(X, Y) - F_x^2 N_x(F_x X, F_x Y)$$

$$-N(FX, FY) + i \{ F^2 N(FX, Y) + N(FX, Y) + F^2 N(X, FY) + N(X, FY) \}.$$

Proof. As

$$(dn)(lX, lY) = -n[lX, lY].$$

Using equation (2.3)c in this equation, we get the equation (2.19). Now,

$$(2.21) \quad 4(dn)(lX, lY) = [F^2 X + iF X, F^2 Y + iF Y].$$

Applying n in equation (2.21) and using equations (2.3)c and (1.1)c, we get the equation (2.20).

Theorem 2.3. *The distribution π_1 is integrable.*

Proof. From equation (2.12) π_1 is given by

$$(2.22a) \quad l = 0, \quad (2.22b) \quad m = 0, \quad (2.22c) \quad n = I_n.$$

In order that π_1 is integrable, it is necessary and sufficient that $l = 0$ and $m = 0$ be integrable, that is $(dl)(X, Y) = 0$, be satisfied for any vector satisfying (2.22)c. Thus, we have

$$(dl)(nX, nY) = 0.$$

But from lemma (2.1), we see that these equations are identically satisfied. Hence, we have the statement.

Theorem 2.4. *In order that π_m and $\tilde{\pi}_m$ are integrable, it is necessary and sufficient that*

$$(2.23a) \quad F^2[lX, lY] = iF[lX, lY], \quad u([lX, lY]) = 0,$$

or

$$(2.23b) \quad N(FX, FY) = N(X, Y) + F^2 N(X, Y).$$

Proof. From equation (2.12) π_m is given by

$$(2.24a) \quad m = 0, \quad (2.24b) \quad n = 0, \quad (2.24c) \quad l = I_n.$$

In order that π_m is completely integrable, it is necessary and sufficient that $m_x = 0$ and $n_x = 0$ be completely integrable, that is

$$(2.25a) \quad (dm)_x(X, Y) = 0, \quad (2.25b) \quad (dn)_x(X, Y) = 0.$$

be satisfied by any vector satisfying (2.24)c, substituting from (2.24)c in equations (2.25), we get

$$(2.26a) \quad (dm)_x(l_x X, l_x Y) = 0, \quad (2.26b) \quad (dn)_x(l_x X, l_x Y) = 0.$$

In consequence of equations (2.16) and (2.19), the equations (2.26) assume the form of the equation (2.23)a. From equations (2.17), (2.20) and (2.25), we get the equation (2.23)b.

Note 1.1. The necessary and sufficient that the almost contact 3 - structure manifold be integrable are the equations (2.23).

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