

Quasi-*Einstein Metric on Kenmotsu Manifolds

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Abstract: In this paper, we study Kenmotsu manifolds with quasi-*Einstein metric and obtain that quasi-*Einstein soliton is expanding or steady.

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1. Introduction

The methods of contact geometry play a crucial role in modern mathematics and has applications in physics, for example control theory, geometric quantization, geometric optics, integrable systems and to the classical mechanics. The origin of contact geometry lies in differential equations as the concept of contact transformation, as a geometric tool to study system of differential equations was introduced by Sophus Lie in 1872. The nature of a manifold depends on the nature of the metric defined on it. In 1958, Boothby and Wang¹ considered an odd dimensional differentiable manifold endowed with the almost contact and contact structures. Since then, various classes of the contact manifolds have been characterized and studied by many geometers. Later, Kenmotsu² investigated a class of almost contact metric(a.c.m) manifold satisfying certain tensorial relations and called it a Kenmotsu manifold.

Further, a Riemannian manifold (M, g) with $S^* = \mu g$, where S^* denote *-Ricci tensor and μ is a constant is called *Einstein. An important generalization of *Einstein metric is called *-Ricci soliton if ³

$$(1.1) \quad \frac{1}{2} \mathcal{L}_Z g + S^* = \mu g ,$$

where μ is a constant and Z is the potential field. If for a function $F \in C^\infty(M)$, $Z = \nabla F$, then (1.1) is called gradient $*$ -Ricci soliton.

On an a.c.m manifold M , the $*$ -Ricci tensor is defined as follows⁴:

$$(1.2) \quad S^*(W, U) = \frac{1}{2} \text{trace}(X \mapsto R(W, \phi U) \phi X), \forall W, U, X \in TM ,$$

where ϕ is a $(1,1)$ -tensor field and R is a Riemann curvature tensor.

Recently, much attention has been given to the study of the Einstein condition and its generalizations and have been examined extensively in contact geometry^{5,6}. A generalization of Einstein metric emerged from the m -Bakry-Emery Ricci tensor S_m^F defined as:

$$(1.3) \quad S_m^F = S + \nabla^2 F - \frac{1}{m} dF \otimes dF ,$$

has been studied in^{7,8}, where $0 < m \leq \infty$, S is the Ricci tensor and $\nabla^2 F$ denotes the Hessian form of F .

A Riemannian manifold (M, g, F, m) is called m -quasi-Einstein⁹ if

$$(1.4) \quad S + \text{Hess}F - \frac{1}{m} dF \otimes dF = \mu g ,$$

for some $m \in \mathbb{Z}^+$. Similarly, we call (M, g, F, m) , m -quasi- $*$ Einstein if

$$(1.5) \quad S^* + \text{Hess}F - \frac{1}{m} dF \otimes dF = \mu g .$$

If $m = \infty$, then (1.5) gives the gradient $*$ -Ricci soliton. We call a quasi $*$ Einstein metric steady, expanding, or shrinking, respectively, when $\mu = 0$, < 0 or > 0 and $*$ Einstein in case $F = \text{constant}$.

Quasi-Einstein metrics have been investigated in extent for a general manifold, and gap results and rigid properties were obtained in^{7,9}. Further, Ghosh⁸ examined quasi-Einstein contact metric on Sasakian manifolds and

on (κ, μ) -spaces. Recently, Venkatesha et al.¹⁰ investigated gradient almost $\ast$ -Ricci soliton on Kenmotsu manifolds. Chen¹¹ studied quasi-Einstein metric on almost cosymplectic manifolds.

However, only very few literature are available on the study of $\ast$ -Einstein and its generalization. This motivate us to study quasi- $\ast$ -Einstein structure associated with Kenmotsu manifolds.

2. Preliminaries

A Riemannian manifold M^{2n+1} is called an a.c.m manifold if it admits a tensor field ϕ , a vector field ξ , a 1-form η and a Riemannian metric g Satisfying¹²

$$(2.1) \quad \phi^2 W = -W + \eta(W)\xi, \quad \eta(\xi) = 1,$$

$$(2.2) \quad \eta o \phi = 0, \quad \phi \xi = 0,$$

$$(2.3) \quad \eta(W) = g(W, \xi), \quad g(\phi W, \phi U) + \eta(W)\eta(U) = g(W, U),$$

where $W, U \in TM$.

An a.c.m manifold which satisfies $d\Phi = 2\eta \wedge \Phi$ and $d\eta = 0$, where Φ is fundamental 2-form is said to be an almost Kenmotsu manifold. Further, a normal almost Kenmotsu manifold called a Kenmotsu manifold if²

$$(2.4) \quad (\nabla_W \phi)U = -\eta(U)\phi W + g(\phi W, U)\xi.$$

On a Kenmotsu manifold, we have^{2,10}

$$(2.5) \quad R(W, U)\xi = -\eta(U)W + \eta(W)U,$$

$$(2.6) \quad \nabla_W \xi = W - \eta(W)\xi,$$

$$(2.7) \quad R(W, \xi)\xi = \eta(W)\xi - W,$$

$$(2.8) \quad Q\xi = -2n\xi,$$

$$(2.9) \quad Q^*W = QW + (2n-1)W + \eta(W)\xi,$$

$$(2.10) \quad r^* = r + 4n^2,$$

where Q^* and r^* are $*$ -Ricci operator and $*$ -scalar curvature respectively.

3. Quasi- $*$ Einstein Metric on a Kenmotsu Manifold

In this section, we investigate quasi- $*$ Einstein metric on a Kenmotsu manifold.

Lemma 3.1: *The curvature tensor R of a Riemannian manifold M^{2n+1} with quasi- $*$ Einstein metric satisfies*

$$(3.1) \quad R(W, U)\nabla F = (\nabla_U Q^*)W - (\nabla_W Q^*)U - \frac{\mu}{m}\{(WF)U - (UF)W\} \\ + \frac{1}{m}\{(WF)Q^*U - (UF)Q^*W\},$$

$$(3.2) \quad S(U, \nabla F) = \frac{1}{2}Ur^* + \frac{2n\mu}{m}(UF) + \frac{1}{m}\{(Q^*U)F - r^*(UF)\},$$

for any $W, U \in TM$, where ∇F is gradient of F .

Proof: From (1.5), we have

$$(3.3) \quad Q^*U + \nabla_U \nabla F - \frac{1}{m}(UF)\nabla F = \mu U.$$

We know that

$$(3.4) \quad R(W, U)X = \nabla_W \nabla_U X - \nabla_U \nabla_W X - \nabla_{[W, U]}X.$$

Using (3.3) in (3.4), we obtain (3.1). Further, contracting (3.1) over W , we get (3.2).

Lemma 3.2: *On a Kenmotsu manifold M^{2n+1}*

$$(3.5) \quad Q^*\phi = \phi Q^*.$$

Proof: Putting $W = \xi$ in (2.9) and using (2.8), we get

$$(3.6) \quad Q^*\xi = 0.$$

Differentiating (3.6) along $W \in TM$, we obtain

$$(3.7) \quad (\nabla_W Q^*)\xi = -Q^*W.$$

If $U, W \in \{\xi^\perp\}$, then with respect to ϕ basis $\{W_i, W_{n+i} = \phi W_i, \xi\}$, we have

$$\begin{aligned} g(Q^* \phi W, \phi U) &= \frac{1}{2} \sum_{i=1}^{2n} g(R(\phi W, \phi^2 U) \phi W_i, W_i) + \frac{1}{2} g(R(\phi W, \phi^2 U) \phi \xi, \xi) \\ &= \frac{1}{2} \sum_{i=1}^{2n} g(R(U, \phi W) \phi W_i, W_i), \end{aligned}$$

which gives

$$(3.8) \quad g(Q^* \phi W, \phi U) = g(Q^* U, W).$$

From (2.9) and using the fact that Q is symmetric, we get

$$(3.9) \quad g(Q^* U, W) = g(Q^* W, U).$$

From (3.8) and (3.9), we get

$$(3.10) \quad -\phi Q^* \phi W = Q^* W.$$

Multiplying with ' ϕ ' on both sides of (3.10) and using (2.1), we find $Q^* \phi W = \phi Q^* W$. Thus proof is complete.

Theorem 3.1: Let M^{2n+1} be a Kenmotsu manifold, then $\mathcal{L}_\xi S^* = 0$ and $\nabla_\xi Q^* = -2Q^*$.

Proof:

$$(3.11) \quad \mathcal{L}_\xi (R_{W, \phi U} \phi X) = \mathcal{L}_\xi (\nabla_W \nabla_{\phi U} \phi X - \nabla_{\phi U} \nabla_W \phi X - \nabla_{[W, \phi U]} \phi X),$$

$$(3.12) \quad R_{\mathcal{L}_\xi W, \phi U} \phi X = \nabla_{\mathcal{L}_\xi W} \nabla_{\phi U} \phi X - \nabla_{\phi U} \nabla_{\mathcal{L}_\xi W} \phi X - \nabla_{[\mathcal{L}_\xi W, \phi U]} \phi X,$$

$$(3.13) \quad R_{W, \mathcal{L}_\xi \phi U} \phi X = \nabla_W \nabla_{\mathcal{L}_\xi \phi U} \phi X - \nabla_{\mathcal{L}_\xi \phi U} \nabla_W \phi X - \nabla_{[W, \mathcal{L}_\xi \phi U]} \phi X,$$

$$(3.14) \quad R_{W, \phi U} \mathcal{L}_\xi \phi X = \nabla_W \nabla_{\phi U} (\mathcal{L}_\xi \phi X) - \nabla_{\phi U} \nabla_W (\mathcal{L}_\xi \phi X) - \nabla_{[W, \phi U]} \mathcal{L}_\xi \phi X,$$

$$\forall W, U, X \in TM.$$

From (3.11)~(3.14), we obtain

$$\mathcal{L}_\xi(R_{W,\phi U}\phi X) = R_{\mathcal{L}_\xi W,\phi U}\phi X + R_{W,\mathcal{L}_\xi\phi U}\phi X + R_{W,\phi U}\mathcal{L}_\xi\phi X,$$

which gives $(\mathcal{L}_\xi R)_{W,\phi U}\phi X = 0$ and contracting it over X , we get $\mathcal{L}_\xi S^* = 0$.

This implies

$$(3.15) \quad \mathcal{L}_\xi(g(Q^*W, U)) - S^*(\mathcal{L}_\xi W, U) - S^*(W, \mathcal{L}_\xi U) = 0.$$

Simplifying (3.15), we find

$$(3.16) \quad g(Q^*W, \nabla_U \xi) + g((\nabla_\xi Q^*)W, U) + g(Q^* \nabla_W \xi, U) = 0.$$

Using (2.6) in (3.16), we get

$$(3.17) \quad \nabla_\xi Q^* = -2Q^*,$$

which completes the proof.

Theorem 3.2: Let M^{2n+1} be a Kenmotsu manifold satisfying (1.5), then either $\xi r^* = -2r^*$ and quasi- $*$ Einstein soliton is expanding or F is constant and quasi- $*$ Einstein soliton is steady.

Proof: Taking inner product of (3.1) with ξ and using (3.6) and (3.7), we get

$$(3.18) \quad g(R(W, U)\nabla F, \xi) = \frac{\mu}{m} \{ (UF)\eta(W) - (WF)\eta(U) \}.$$

Putting $U = \xi$ in (3.18) and using (2.5), we get

$$(3.19) \quad \left(1 + \frac{\mu}{m}\right) ((WF) - (\xi F)\eta(W)) = 0,$$

which implies either $\mu + m = 0$ or $\mu + m \neq 0$.

Case A: If $\mu + m = 0$, then putting $W = \xi$ in (3.1), we find

$$(3.20) \quad R(\xi, U)\nabla F = (\nabla_U Q^*)\xi - (\nabla_\xi Q^*)U + \{(\xi F)U - (UF)\xi\} \\ + \frac{1}{m} \{(\xi F)Q^*U - (UF)Q^*\xi\}.$$

From (2.5), we obtain

$$(3.21) \quad R(\xi, U)\nabla F = (\xi F)U - (UF)\xi.$$

Using (3.21) in (3.20), we get

$$(3.22) \quad (\nabla_U Q^*)\xi - (\nabla_\xi Q^*)U + \frac{1}{m}\{(\xi F)Q^*U - (UF)Q^*\xi\} = 0.$$

Using Theorem 3.1, (3.6) and (3.7) in (3.22), we obtain

$$(3.23) \quad \left(1 + \frac{\xi F}{m}\right)Q^*U = 0,$$

which implies either $\xi F = -m$ or $Q^* = 0$. If $Q^* = 0$, then $r^* = 0$. Using this in (3.2), we get

$$(3.24) \quad S(U, \nabla F) = -2n(UF).$$

On the other hand from (2.9), we get

$$(3.25) \quad QU = -(2n-1)U - \eta(U)\xi.$$

From (3.24) and (3.25), we get, $(UF) - \eta(U)(\xi F) = 0$, which implies F is constant¹³ so from (1.5), we obtain $\mu = 0$, that gives $m = 0$, a contradiction. Hence $Q^* \neq 0$. If $\xi F = -m$, putting $U = \xi$ in (3.2), we get

$$(3.26) \quad g(Q\xi, \nabla F) = \frac{1}{2}\xi r^* - 2n(\xi F) + \frac{1}{m}\{(Q^*\xi)F - r^*(\xi F)\}.$$

Using (2.8), (3.6) and $\xi F = -m$ in (3.26), we get

$$(3.27) \quad \xi r^* = -2r^*.$$

Further, since $\mu = -m$ and $m > 0$, so $\mu \neq 0$ and quasi-* Einstein soliton is expanding.

Case B: If $\mu + m \neq 0$, then from (3.19), we have, $(WF) = (\xi F)\eta(W)$, which implies F is constant¹³. Using this and (3.6) in (1.5), we get $\mu = 0$. Hence quasi-* Einstein soliton is steady.

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