

# Boundary Representations and Hyperrigidity for Spaces of Operations

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**Abstract:** We introduce the notion of local hyperrigidity for linear spaces in locally  $C^*$ -algebras and study its connections with local boundary representations. Also, an analogue of the Korovkin theorem in the setting of locally  $C^*$ -algebra is proved.

**Keywords:** Locally  $C^*$ -algebras, Local hyperrigidity, Local boundary representations.

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## 1. Introduction

The extremal theory regarding the classical notion of Choquet boundary of subalgebras of function algebras has seen immense progress in the recent past. Let  $X$  be a compact Hausdorff topological space and  $C(X)$  be the algebra of complex-valued continuous functions on  $X$ . For a uniform algebra  $A \subset C(X)$  and a point  $x \in X$ , if the evaluation functional on  $A$  corresponding to  $x$  admits a unique unital positive extension to  $C(X)$  then we say the point  $x$  is in the Choquet boundary<sup>1</sup> of  $A$ . The non-commutative analogue of this notion was proposed by Arveson<sup>2</sup> and called it boundary representations. The inspiration for Arveson to define boundary representations was double track: to furnish an intrinsic invariant for operator systems and to furnish a context for exhibiting that the noncommutative Silov boundary exists in general. Later, many other authors proceed with the program initiated by Arveson and the articles<sup>3-7</sup> are worth mentioning in this context.

Noncommutative approximation and extremal theories started by Arveson<sup>8</sup> in the context of operator systems in  $C^*$ -algebras is a closely related topic of interest. A powerful theorem in classical approximation theory is the famous Korovkin theorem<sup>9</sup> which concerns the convergence of positive linear maps on function algebras. The Korovkin theorem is as follows: for each  $n \in \mathbb{N}$ , let  $\Phi_n : C[0,1] \rightarrow C[0,1]$  be a positive linear map. If  $\lim_{n \rightarrow \infty} \|\Phi_n(f) - f\| = 0$  for every  $f \in \{1, x, x^2\}$ , then  $\lim_{n \rightarrow \infty} \|\Phi_n(f) - f\| = 0$  for every  $f \in C[0,1]$ . Motivated by this we have the definition of Korovkin sets. A set  $G \subseteq C[0,1]$  is said to be a Korovkin set if the sequence  $\{\phi_n(f)\}$  converges to  $f$  uniformly for every  $f \in G$  should imply the sequence  $\{\phi_n(f)\}$  converges to  $f$  uniformly for every  $f \in C[0,1]$ . Note that by Korovkin theorem the set  $G = \{1, x, x^2\}$  is a Korovkin set for  $[0,1]$ .

In 1966, Saskin<sup>10</sup> identified a strong relation between Korovkin sets in  $[0,1]$  and their Choquet boundaries. Saskin's theorem states that  $G$  is a Korovkin set in  $[0,1]$  if and only if the Choquet boundary of  $G$  is  $[0,1]$ .

The non-commutative analogue of Korovkin sets called hyperrigid set in the setting of  $C^*$ -algebras were introduced by Arveson<sup>8</sup> and proved that if an operator system  $S$  is hyperrigid in the generated  $C^*$ -algebra  $A = C^*(S)$ , then every irreducible representation of  $A$  is a boundary representation for  $S$ . The converse of this statement is called Arveson's hyperrigidity conjecture. That is, for an operator system  $S$  and the generated  $C^*$ -algebra  $A = C^*(S)$ , if every irreducible representation of  $A$  is a boundary representation for  $S$ , then, is  $S$  hyperrigid? There are several articles<sup>11-16</sup> studies hyperrigidity conjecture in different contexts. Arveson showed<sup>8</sup> that for a  $C^*$ -algebra with a countable spectrum the conjecture is valid. Davidson and Kennedy<sup>17</sup> verified in the case of commutative  $C^*$ -algebras. Some partial results in different contexts have also appeared in<sup>18,19</sup>. But, in general, conjecture is still open and is an active research area.

An unbounded version of the notion of boundary representations is introduced and studied in<sup>2</sup> using the theory of locally  $C^*$ -algebras (introduced in<sup>20</sup> and local completely positive maps. Where the concrete locally  $C^*$ -algebra is considered to be the locally  $C^*$ -algebra  $C_E^*(D)$  of unbounded operators on a quantized domain  $E$  with its union space  $D$

(introduced by A. Dosiev<sup>21</sup>. In this paper, we introduce a non-self-adjoint version of the notion of local hyperrigidity for linear spaces of unbounded operators and local completely contractive maps and study its connections with local boundary representations. Also, we prove a version of the non-commutative Korovkin theorem for this new setting of locally  $C^*$ -algebra.

## 2. Preliminaries

This section recalls the basic definitions and some standard theorems required for the main section.

**2.1 Locally  $C^*$ -algebras:** The notion of locally  $C^*$ -algebras was introduced by A. Inoue<sup>20</sup> to study the subalgebras of unbounded linear operators on Hilbert spaces. A seminorm  $p$  on an unital  $*$ -algebra  $A$  with unit  $1_A$  is said to be sub multiplicative, if  $p(1_A)=1$  and  $p(ab) \leq p(a)p(b)$  for every  $a, b \in A$ . If a sub multiplicative seminorm  $p$  satisfies the condition  $p(a^*a) = p(a)^2$ , then we say that  $p$  is a  $C^*$ -seminorm. A family of seminorms  $P = \{p_\lambda : \lambda \in \Omega\}$  is called an upward filtered family if  $\lambda \leq \gamma$  in  $\Omega$ , then  $p_\lambda(a) \leq p_\gamma(a)$  for every  $a \in A$ , where  $(\Omega, \leq)$  is a directed set. A locally  $C^*$ -algebra  $A$  is a locally convex topological  $*$ -algebra  $A$  whose topology is induced by an upward filtered family  $P = \{p_\lambda : \lambda \in \Omega\}$  of  $C^*$ -seminorms on  $A$  and it is complete and Hausdorff concerning the induced topology. Interestingly, each locally  $C^*$ -algebra is an inverse limit of a projective system of  $C^*$ -algebras<sup>22</sup>.

Let  $A$  be a locally  $C^*$ -algebra with a family of seminorms  $P = \{p_\lambda : \lambda \in \Omega\}$ . An element  $x \in A$  is said to be local positive if there exists  $y, z \in A$  and  $\lambda \in \Omega$  such that  $x = y^*y + z$  and  $p_\lambda(z) = 0$ . In that case, we say  $x$  is  $\lambda$ -positive and is denoted as  $x \geq_\lambda 0$ . Suppose  $M_n(A)$  denotes the set of all  $n \times n$  matrices over  $A, n \in \mathbb{N}$ .

Then  $M_n(A)$  is canonically a locally  $C^*$ -algebra under the family of seminorms  $\{p_\lambda^n : \lambda \in \Omega\}$  given by  $p_\lambda^n(\llbracket x_{ij} \rrbracket) = \|\pi_\lambda^{(n)}(\llbracket x_{ij} \rrbracket)\|_\lambda$  for  $\llbracket x_{ij} \rrbracket$  in  $M_n(A)$ , where  $\pi_\lambda$  is the canonical map from  $A$  to the  $C^*$ -algebra  $A_\lambda = A/I_\lambda$  and  $I_\lambda = \{x \in A : p_\lambda(x) = 0\}$ .

Let  $A_1$  and  $A_2$  be a locally  $C^*$ -algebras with the associated family of seminorms  $P = \{p_\lambda : \lambda \in \Omega_1\}$  and  $\{q_\beta : \beta \in \Omega_2\}$  respectively. Let  $\phi : A_1 \rightarrow A_2$

be a linear map. The  $n$ -amplification of the map  $\phi$  is denoted as  $\phi^{(n)}$  and is defined as  $\phi^{(n)} : M_n(A_1) \rightarrow M_n(A_2)$  by

$$\phi^{(n)}\left(\begin{bmatrix} x_{ij} \end{bmatrix}\right) = \begin{bmatrix} \phi(x_{ij}) \end{bmatrix} \text{ for } \begin{bmatrix} x_{ij} \end{bmatrix} \in M_n(A_1)$$

A linear map  $\phi : A_1 \rightarrow A_2$  is

1. Local positive: if for each  $\beta \in \Omega_2$  there corresponds  $\lambda \in \Omega_1$  such that  $\phi(x) \geq_\beta 0$  in  $A_2$  whenever  $x \geq_\lambda 0$  in  $A_1$ .
2. Local completely positive (local CP-map): if for each  $\beta \in \Omega_2$ , there exists  $\lambda \in \Omega_1$  such that

$$\phi^{(n)}\left(\begin{bmatrix} x_{ij} \end{bmatrix}\right) \geq_\beta 0 \in M_n(A_2) \text{ whenever } \begin{bmatrix} x_{ij} \end{bmatrix} \geq_\lambda 0 \in M_n(A_1).$$

3. Local contractive: if for each  $\beta \in \Omega_1$  there exists  $\lambda \in \Omega_2$  such that

$$q_\beta(\phi(x)) \leq p_\lambda(x) \text{ for all } x \in A_1.$$

4. Local completely contractive (local CC-map): if for each  $\beta \in \Omega_1$ , there exists  $\lambda \in \Omega_2$  such that

$$q_\beta^n\left(\begin{bmatrix} \phi(x_{ij}) \end{bmatrix}\right) \leq p_\beta^n\left(\begin{bmatrix} x_{ij} \end{bmatrix}\right), \text{ for every } n \in \mathbb{N}.$$

Similarly, one can define local contractivity and local complete contractivity for maps from linear subspaces of  $A_1$  into linear subspaces of  $A_2$ . An analogue of a non-self-adjoint version of Arveson extension theorem for local CC-map on linear subspaces of locally  $C^*$ -algebras is proved in<sup>2</sup>. More precisely, if  $S$  is a linear subspace of a locally  $C^*$ -algebra  $A$  and  $\phi$  is a local CC-map from  $S$  into a concrete locally  $C^*$ -algebra  $C_E^*(D)$ , then  $\phi$  can be extended to a local CP-map from  $A$  into  $C_E^*(D)$ . Where the concrete locally  $C^*$ -algebra  $C_E^*(D)$  is defined below.

A quantized domain in a complex Hilbert space  $H$  is a triple  $\{H, E, D\}$ , where  $E = \{H_\lambda : \lambda \in \Omega\}$  is an upward filtered family of closed subspaces of  $H$  such that the union space  $D = \bigcup_{\lambda \in \Omega} H_\lambda$  is dense in  $H$ . In a word, we say  $E$  is a quantized domain in  $H$  with its union space  $D$ . Denote the set of all linear transformations on  $D$  by  $L(D)$ , and the orthogonal projection of  $H$  onto the closed subspace  $H_\lambda$  by  $Q_\lambda$ . Then the set of all noncommutative continuous functions on a quantized domain  $E$  is defined as

$$C_D(E) = \{T \in L(D) : TQ_\lambda = Q_\lambda TQ_\lambda \in B(H), \text{ for all } \lambda \in \Omega\}.$$

It follows that  $C_D(E)$  is an algebra and if  $T \in L(D)$ , then  $T \in C_D(E)$  if and only if  $T(H_\lambda) \subseteq H_\lambda$  and  $T|_{H_\lambda} \in B(H_\lambda)$  for all  $\lambda \in \Omega$ . The  $*$ -algebra of all noncommutative continuous functions on a quantized domain  $E$  is defined as

$$C_E^*(D) = \{T \in C_D(E) : Q_\lambda T \subseteq TQ_\lambda, \text{ for all } \lambda \in \Omega\}.$$

It is easy to see that  $C_E^*(D)$  is a unital subalgebra of  $C_D(E)$ . Refer<sup>8</sup> for more details on the space  $C_E^*(D)$ . For  $T \in L(D)$ , it is easy to see that  $T \in C_E^*(D)$  if and only if for all  $\lambda \in \Omega$

$$T(H_\lambda) \subseteq H_\lambda, T|_{H_\lambda} \in B(H_\lambda) \wedge T(H_\lambda^\perp \cap D) \subseteq H_\lambda^\perp \cap D.$$

The space  $C_E^*(D)$  is a locally  $C^*$ -algebra with respect to the family of  $C^*$ -seminorms defined as  $q_\lambda : C_E^*(D) \rightarrow R$  by  $q_\lambda(T) = \|T|_{H_\lambda}\|$  for all  $T \in C_E^*(D)$ . If the indexing set  $\Omega$  of the quantized domain  $\{H, E, D\}$  is countable, then we say that  $E$  is quantized Frechet domain.

**2.2 Representations:** A representation of a locally  $C^*$ -algebra  $A$  is a local completely contractive  $*$ -homomorphism from  $A$  to  $C_E^*(D)$  for some quantized domain  $E$ . Let  $\pi : A \rightarrow C_E^*(D)$  be a representation. The representation  $\pi$  is called nondegenerate if  $[\pi(A)H_\lambda] = H_\lambda$  for all  $\lambda \in \Omega$ . An injective representation is called faithful representation. The

representation  $\pi$  is called irreducible<sup>23</sup> if  $\pi$ , where  $\pi$  is the commutant of  $\pi(A)$ . Anar Dosiev<sup>8</sup> proved the Stinespring representation theorem for unbounded operator-valued local completely positive maps. That is, a local CP-map on  $A$  with codomain  $C_E^*(D)$  can be dilated to a representation of  $A$ .

**2.3 Local boundary representations:** The notion of boundary representations for operator systems in  $C^*$ -algebras was introduced by Arveson<sup>2</sup>. Its unbounded counterpart is introduced and studied in<sup>23</sup>. Let us first recall the definition of local unique extension property.

**Definition<sup>2</sup> 2.1:** Let  $A$  be a locally  $C^*$ -algebra and  $S$  be a generating linear subspace for  $A$ . A representation  $\pi: A \rightarrow C_E^*(D)$  is said to have local unique extension property for  $S$  if the only local completely positive extension of  $\pi|_S$  to  $A$  is  $\pi$  itself.

**Definition<sup>2</sup> 2.2:** Let  $A$  be a locally  $C^*$ -algebra and  $S$  be a generating linear subspace for  $A$ . A representation  $\pi: A \rightarrow C_E^*(D)$  is called a local boundary representation for  $S$  if

- (i)  $\pi$  is irreducible
- (ii)  $\pi$  has local unique extension property for  $S$ .

### 3. Main Results

We introduce a non self-adjoint version of hyperrigidity for spaces of unbounded operators.

**Definition 3.1:** A set  $S$  of generators of a locally  $C^*$ -algebra  $A$  is said to be locally hyperrigid if for each  $\lambda \in \Omega$ , and for every faithful representation  $\pi$  fro  $A$ . to  $C_E^*(D)$  and every sequence of unital local completely contractive linear maps  $\phi_m: A \rightarrow C_E^*(D)$ ,

$$\lim_{m \rightarrow \infty} \|\phi_m(s) - \pi(s)\|_\lambda = 0, \forall s \in S \Rightarrow \lim_{m \rightarrow \infty} \|\phi_m(x) - \pi(x)\|_\lambda = 0, \forall x \in A$$

**Remark 3.2:** It is easy to see that a subset  $S$  of  $A$  is locally hyperrigid if and only if the linear subspace spanned by  $S$  is locally hyperrigid. Thus, local hyperrigidity is actually a notion of linear subspaces. Also, if the quantized domain  $E$  is a singleton  $\{H\}$ , then the local hyperrigidity coincides with Arveson's notion of hyperrigidity. In this sense Definition,

3.1 is more general and it is valid for both space of bounded operators and unbounded operators.

Now, we prove a characterization of local hyperrigidity of linear spaces, which leads to study the unbounded version of Saskin's theorem<sup>7,10</sup> relating local hyperrigidity and local boundary representations for spaces of unbounded operators, and a Korovkin type theorem for spaces of unbounded operators. The spirit of the characterization theorem is similar to that by Arveson, Theorem 2.1 in the context of operator systems in  $C^*$ -algebras. As Arveson's extension theorem for local CC-map<sup>2</sup> is available only for Frechet locally  $C^*$ -algebra, we always assume the locally  $C^*$ -algebra to be Frechet.

**Theorem 3.3:** *Let  $A$  be a Frechet locally  $C^*$ -algebra and  $S$  be a separable linear subspace of  $A$  such that  $A$  is generated by  $S$ . Then the following statements are equivalent:*

(i)  *$S$  is locally hyperrigid in  $A$ .*

(ii) *For every non-degenerate representation  $\pi: A \rightarrow C_E^*(D)$  of  $A$  on a quantized Frechet domain  $E = \{H_n : n \in \mathbb{N}\}$  of a separable Hilbert space  $H$  with union space  $D = \cup H_n$  and for any sequence of unital local CC-map  $\phi_m: A \rightarrow C_E^*(D)$  such that for each  $s \in S, \phi_m(a)$  converges to  $\pi(a)$  on  $H_n, \forall n \in \mathbb{N}$  then  $\phi_m(a)$  converges to  $\pi(a)$  on  $H_n, \forall a \in A, \forall n \in \mathbb{N}$ .*

(iii) *For every non-degenerate representation  $\pi: A \rightarrow C_E^*(D)$  of  $A$  on a quantized Frechet domain  $E$ , has the local unique extension property for  $S$*

(iv) *For any Frechet locally  $C^*$ -algebra  $A_1$ , and for any unital  $*$ -homomorphism  $\rho: A \rightarrow A_1$  and local CC-map  $\phi: A_1 \rightarrow A_1$  such that  $\phi(s) = s \forall s \in \rho(S)$  then  $\phi(a) = a \forall a \in \rho(A)$ .*

**Proof:** (i)  $\Rightarrow$  (ii): Let  $\sigma: A \rightarrow C_{E'}^*(D)$  be a faithful representation of  $A$  on a Frechet quantized domain  $E'$  of a Hilbert space  $H'$  with union space  $D' = \cup H_n'$ . Then the direct sum  $\sigma \oplus \pi: A \rightarrow C_{E \oplus E'}^*(D \oplus D')$  is also a faithful representation of  $A$ . Define  $\sigma_n: (\sigma \oplus \pi)(A) \rightarrow C_{E \oplus E'}^*(D \oplus D')$  by  $\sigma_n(\sigma \oplus \pi)(a) = \sigma(a) \oplus \phi_n(a)$ . Then the maps  $\sigma_n$  are all unital local CC. Applying Arveson-Dosiev extension theorem [2, Theorem 3.6] to  $\sigma_n$ , we have a local CC-map  $\sigma_n: C_{E \oplus E'}^*(D \oplus D') \rightarrow C_{E \oplus E'}^*(D \oplus D')$  that extends  $\sigma_n$ . Since  $\phi_m(s)$  converges to  $\pi(s)$  on  $H_n$  for each  $s \in S$ , we have that  $\sigma_n(s)$

converges  $(\sigma \oplus \pi)(s)$  on  $H_n \oplus H'_n$ . By assumption (i), it follows that  $\tilde{\sigma}_n(a)$  converges  $(\sigma \oplus \pi)(a)$  on  $H_n \oplus H'_n$  for all  $a \in A$ . Then  $\forall a \in A$ ,

$$\limsup_{n \rightarrow \infty} \|\phi_n(a) - \pi(a)\|_n \leq \limsup_{n \rightarrow \infty} \|\sigma(a) \oplus \phi_n(a) - \sigma(a) \oplus \pi(a)\|_n$$

$$\limsup_{n \rightarrow \infty} \|\sigma_n(\sigma(a) \oplus \phi_n(a)) - \sigma(a) \oplus \pi(a)\|_n = 0.$$

ii)  $\Rightarrow$  (iii): Follows easily.

(iii)  $\Rightarrow$  (iv): Let  $\rho: A \rightarrow A_1$  be an unital \*-homomorphism between the locally C\*-algebras  $A$  and  $A_1$ . Consider a local CC-map  $\phi: A_1 \rightarrow A_1$  such that  $\phi(s) = s \forall s \in \rho(S)$ . Let  $A_2$  be the separable locally C\*-algebra generated by  $\rho(A) \cup \phi(\rho(A)) \cup \phi^2(\rho(A)) \cup \dots$ . Then  $A_2$  is invariant for  $\phi$ . We may represent  $A_2$  faithfully on a quantized Frechet domain  $E$  with union space  $D$  and assume  $A_2 \subseteq C^*_E(D)$ . Applying Arveson-Dosiev extension theorem<sup>2</sup> to the local CC-map  $\phi$ , we have a local CC-map  $\tilde{\phi}: C^*_E(D) \rightarrow C^*_E(D)$  that extends  $\phi$  and such that  $\tilde{\phi}(\rho(a)) = \rho(a) \forall a \in A$ . Hence,  $\phi(x) = x \forall x \in \rho(A)$ .

(iv)  $\Rightarrow$  (i): Let  $\pi: A \rightarrow C^*_E(D)$  be a faithful representation of  $A$  on a Frechet quantized domain  $E$ . Then we may identify  $\pi(A)$  with  $A$ . Let  $\phi_m: C^*_E(D) \rightarrow C^*_E(D)$ ; be a sequence of unital local CC-maps such that

$$\lim_{m \rightarrow \infty} \|\phi_m(s) - s\|_n = 0 \forall s \in S, \forall n \in \mathbb{N}.$$

Take  $A_1$  to be the locally C\*-algebra  $C^*_E(D)$ . Consider the space  $l^\infty(A_1)$  of all bounded sequences in  $A_1$  together with the seminorms: for  $x = (x_1, x_2, \dots) \in l^\infty(A_1)$ ,  $\|x\|_n = \sup_i \|x_i\|_n$ . Then  $l^\infty(A_1)$  is a locally C\*-algebra.

Let  $C_0(A_1) = \{x = (x_1, x_2, \dots) \in l^\infty(A_1) : \|x_i\|_n \rightarrow 0 \text{ as } i \rightarrow \infty, \forall n \in \mathbb{N}\}$ . One can easily see that  $C_0(A_1)$  is a closed two-sided ideal in the algebra  $l^\infty(A_1)$ . Now, define a map  $\psi: l^\infty(A_1) \rightarrow l^\infty(A_1)$  by  $\psi(x_1, x_2, \dots) = (\phi_1(x_1), \phi_2(x_2), \dots)$ . The map  $\psi$  is an unital local CC map such that  $\psi$  maps  $C_0(A_1)$  into itself.

Consider the induced map  $\phi$  on the quotient locally  $C^*$ -algebra  $l^\infty(A_1)/C_0(A_1)$  given by

$$\phi(x + C_0(A_1)) = \psi(x) + C_0(A_1) \forall x \in l^\infty(A_1).$$

Note that the map  $\phi$  is local CC. Now, take  $\rho$  to be the canonical embedding of  $A$  into  $l^\infty(A_1)/C_0(A_1)$ . That is,  $\rho: A \rightarrow l^\infty(A_1)/C_0(A_1)$  defined by  $\rho(a) = (a, a, \dots) + C_0(A_1)$ . By assumption (iv), we have  $\phi(\rho(a)) = \rho(a) \forall a \in A$ . By hypothesis

$$\lim_{m \rightarrow \infty} \|\phi_m(s) - \pi(s)\|_n = 0 \forall s \in S \wedge \forall n \in \mathbb{N},$$

and thus

$$\phi(\rho(s)) = (\phi_1(s), \phi_2(s), \dots) + C_0(A_1) = (s, s, \dots) + C_0(A_1) = \rho(s)$$

Now, by applying the hypothesis (iv) to the local CC-map  $\phi$ , we can conclude that  $\phi(\rho(a)) = \rho(a) \forall a \in A$ . From the definitions of  $\rho$  and  $\phi \circ \rho$  we must have that the sequence  $\{\phi_m(a) - a\} \in C_0(A_1) \forall a \in A$ . Hence  $S$  is locally hyperrigid in  $A$ .

**Corollary 3.4:** If  $S$  is a locally hyperrigid separable linear subspace in a Frechet locally  $C^*$ -algebra  $A$ , then every irreducible representation of  $A$  is a local boundary representation for  $S$ .

**Proof:** Let  $S$  be a locally hyperrigid separable linear subspace in a Frechet locally  $C^*$ -algebra  $A$ . Then by statement (iii) of Theorem 3.3, every nondegenerate representation will have unique local extension property for  $S$ . Since an irreducible representation is a nondegenerate representation, every irreducible representation will also have unique local extension property for  $S$ . Thus, every irreducible representation of  $A$  is a local boundary representation for  $S$ .

**Conjecture 3.5:** Let  $S$  be a separable linear subspace in a Frechet locally  $C^*$ -algebra  $A$  such that  $A$  is generated by  $S$ . If every irreducible representation of  $A$  is a local boundary representation for  $S$ , then  $S$  is locally hyperrigid in  $A$ .

**Remark 3.6:** If Conjecture 3.5 is valid, then we have a non-commutative analogue of the classical Saskin's theorem in the setting of spaces of unbounded operators.

Now, we prove a non-commutative Korovkin type theorem for certain unbounded operators. Let  $E$  be a quantized Frechet domain with its union space  $D$  dense in a separable Hilbert space  $H$ .

**Theorem 3.7:** Let  $T \in C^*_E(D)$  be a self-adjoint operator and  $A$  be the locally  $C^*$ -algebra generated by  $I$  and  $T$ . For any sequence of unital local CC maps  $\phi_m: A \rightarrow C^*_E(D)$ , if

$$\forall n \in N, \lim_{m \rightarrow \infty} \|\phi_m(x) - x\|_n = 0, \forall x \in \{I, T, T^2\},$$

then

$$\lim_{m \rightarrow \infty} \|\phi_m(a) - a\|_n = 0, \forall a \in A, \forall n \in N.$$

**Proof:** Let  $S = \text{span}\{I, T, T^2\}$  and  $A$  be the locally  $C^*$ -algebra generated by  $S$  in  $C^*_E(D)$ . In view of Theorem 3.3, it is enough to show that every nondegenerate representation has unique local extension property for  $S$ . Let  $\rho: A \rightarrow C^*_E(D)$  be a nondegenerate representation and  $\phi: A \rightarrow C^*_E(D)$  be a local CC-map such that  $\phi(s) = \rho(s)$  for all  $s \in S$ . If  $(\pi, V, \{H'; E'; D'\})$  is a minimal Stinespring representation of  $\phi$ , then to prove  $\phi = \pi$  on  $A$  it is enough to show that  $V$  is unitary. Suppose  $\pi(A)V(D) \subseteq V(D)$  then by minimality  $H' = [\pi(A)V(D)] \subseteq [V(D)] \subseteq H'$  will imply  $V$  is unitary. To see  $\pi(A)V(D) \subseteq V(D)$ , let us first show that  $\pi(T)V(D) \subseteq V(D)$ . For that, we show that  $\pi(T)V(H_n) \subseteq V(H_n)$  for every  $n \in N$ . Let  $n \in N$  and  $h' \in H'_n$ . Direct computations show that

$$\begin{aligned} \|(I - VV^*)\pi(T)VV^*h'\|^2 &= \langle V\phi(T^2)V^*h' - V\phi(T)\phi(T)V^*h', h' \rangle \\ &\quad \langle V\rho(T^2)V^*h' - V\rho(T^2)V^*h', h' \rangle = 0. \end{aligned}$$

Thus  $(I - VV^*)\pi(T)VV^* = 0$  on  $H'_n$ . As  $T$  is self-adjoint,  $VV^*\pi(T)(I - VV^*) = 0$  on  $H'_n$ . Then using  $\pi$  is an isometry and  $\pi(T)V(H_n) \subseteq H'_n$  we have  $\pi(T)V(H_n) \subseteq V(H_n)$ . We conclude that

$\pi(T)V(D) \subseteq V(D)$ . To see  $\pi(A)V(D) \subseteq V(D)$ , let  $T_0 \in A$  and  $A_T = \text{span}\{I, T, T^2, T^3, \dots\}$ . Then  $T_0 = \lim T_\lambda$ , where  $T_\lambda \in A_T$ . For  $h \in H_n$ ,

$$\|\pi(T_\lambda)Vh - \pi(T_0)Vh\|_h = \|\pi(T_\lambda - T_0)Vh\|_h \leq \|T_\lambda - T_0\|_{h_0} \|h\|_h,$$

where  $n_0$  corresponds to  $n$  in the local contractivity of  $\pi$ . It follows that

$$\pi(T_0)Vh \in [\pi(T_\lambda)Vh].$$

Since  $\pi(T)$  leaves  $V(H_n)$  invariant, each element of  $A_T$  is also leaves  $V$  invariant. Then

$$\pi(T_0)Vh \in [\pi(T_\lambda)Vh] \subseteq [\pi(T_\lambda)V(H_n)] \subseteq [V(H_n)] = V(H_n)$$

Therefore  $\pi(A)V(H_n) \subseteq V(H_n)$  for every  $n$  and hence  $\pi(A)V(D) \subseteq V(D)$ .

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