

Numerical Solution of Two-Dimensional Fractional Order Diffusion Equation by using Elzaki Transform with Residual Power Series Method

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Abstract: The Elzaki transform with RPS method is an efficient and reliable approach for solution of non-integer order linear as well as non-linear differential equations. The major endeavour of current work is to find the solution of two dimensional non-integer order diffusion equation by Elzaki transform with RPS method. Firstly, Elzaki is applied on this equation. Secondly, inverse Elzaki is taken on this equation for finding the expression of solution. Thirdly, assumed approximate solution is substituted on this equation. Then unidentified coefficient functions are obtained by using residual function is equal to zero as well as combining its initial circumstances. At last, coefficient functions are substituted in power series form for finding finite approximate logical solutions. The judgment between exact solution and approximate analytic solutions with different number of terms of this equation are determined and compared for reliability. This method reduces the size of computational works of solution of non-integer order non-linear diffusion equation.

Keywords: Elzaki transforms, fractional order diffusion equation, residual power series method and inverse Elzaki transform.

1. Introduction

A number of transforms are available in research as a tool for solving differential equations that includes Laplace transform, Sumudu transform, Elzaki Transform, Binomial transform, Hankel transform, wavelet transform, fast wavelet transform, finite Legendre transform, Radon transform, Weierstrass transform, Mell transform, Laplace-Stieltjes transform, Linear canonical transform, Abel transform, Fourier transform, Hilbert transform, Jacobi transform, Legurre transform, Gabor transform, etc. One of the well-known transforms out of them is Elzaki Transform which has been used in

finding the numerical solution of the well-known differential equations existing as mathematical modelling of the phenomenon taking place in science and technology. Various transforms have been used by the researchers with different well-known approaches to solve the differential equations that include the solution of non-integer relaxation-oscillation differential equations¹, two-dimensional non-integer order diffusion equations, non-integer order BBM-Burger equations, non-integer order logistic differential equations etc.

Some non-integer order differential equations are solved by Elzaki transform with RPS technique. The RPS method is one of the efficient and reliable approaches for the solution of linear or non-linear differential problems of non-integer order. In the case of non-linear non-integer differential problems it is impossible to find the series solutions as well as it is very difficult to find the series coefficients. Then an RPS approach is brought to find the coefficients in sequential form as recurrence relation by using transformed functions. For this a reliable and efficient method is established and used by differentiating n^{th} partial sum of that power series in $(n - 1)$ times to find its n^{th} ordered coefficients that is the RPS method. In general for handling non-integer non-linear problems² ordinary derivatives are updated to fractional derivatives. Elzaki transform with RPS approach is established for solution of non-integer order differential models arising in different branches of mathematics, physics as well as engineering fields analytically.

The chief plan of current work is to boost the reliability and correctness of RPS method by using the Elzaki transform in the methodology of this equation in two dimensions³. The modified form of Laplace transform and Sumudu transform is the Elzaki transform and such transform is used to solve fractional order linear as well as non-linear differential equations. This reliable and efficient method Elzaki transform with RPS approach is applied to solve non-integer order diffusion equation in current work. The new approach applies Elzaki transform with RPS technique to solve non-integer diffusion equation.

The two-dimensional non-integer order diffusion equation is

$$(1) \quad D_t^\alpha u(x, y, t) = u_{xx}(x, y, t) + u_{yy}(x, y, t) \text{ with } 0 < \alpha \leq 1$$

with initial condition

$$(2) \quad u(x, y, 0) = \sin x \sin y$$

and exact solution is $u(x, y, t) = e^{-2t} \sin x \sin y$ for $\alpha = 1$

For $0 < \alpha \leq 1$ exact solution is

$$(3) \quad u(x, y, t) = E_\alpha(z) \sin x \sin y$$

where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}$, $z = -4it^\alpha$

The equation (1) is a non-integer (fractional) differential equation and it is solved by using Elzaki transform with RPS method. There are so many reliable, popular as well as efficient numerical and analytic methods having more reliability for finding the solutions of fractional order differential problems. Out of them residual power series method⁴, homotopy perturbation and analysis methods⁵, differential transform method⁶, iterative method⁷, Adomian decomposition method⁸, different forms of non-integer power series version⁹, biological engineering image dispensation¹⁰, physical model¹¹, risk analysis¹², Taylor's method¹³ and so many other methods have been used for solving fractional order linear or non-linear¹⁴ differential equations. The residual power series approach¹⁵ is also applied to solve various non-integer order differential equations in one or two dimensions.

The organization has been done as follows. In the first section, introduction is given. In the second section, the research methodology of the Elzaki transform with RPS method for the solution of a two-dimensional non-integer order diffusion equation is explained clearly. In the third section, the numerical solution of this equation by using Elzaki transform with RPS approach is given. In the fourth section, the numerical simulation of this equation by Elzaki transform with different terms is given. At last, in the fifth section, the conclusion of the research topic is outlined.

2. Methodology

The methodology of solution of diffusion equation¹⁶ by using Elzaki transform with RPS method¹⁷ is performed by following steps:

Step 1 Applying Elzaki transform on equation (1) as,

$$(4) \quad E(D_t^\alpha u(x, y, t)) = E(u_{xx}(x, y, t) + u_{yy}(x, y, t))$$

Applying the differentiation property of Elzaki transform,

$$E[D_t^\alpha u(x, y, t)] = \frac{1}{v^\alpha} \{E(u(x, y, t)) - g(x, y, t)\}$$

on equation (4), we get

$$\frac{1}{v^\alpha} \{E(u(x, y, t)) - g(x, y, t)\} = E\{(u(x, y, t))_{xx} + (u(x, y, t))_{yy}\} \text{ i.e.}$$

$$(5) \quad E(u(x, y, t)) = g(x, y, t) + v^\alpha E\{(u(x, y, t))_{xx} + (u(x, y, t))_{yy}\}$$

Step 2 Taking inverse Elzaki transform in equation (5), we get

$$(6) \quad u(x, y, t) = G(x, y, t) + E^{-1}[v^\alpha E\{(u(x, y, t))_{xx} + (u(x, y, t))_{yy}\}]$$

here $G(x, y, t)$ is the primary circumstance of given problem.

Step 3 By this method algorithm of $u(x, y, t)$ is proposed as,

$$(7) \quad u(x, y, t) = \sum_{n=0}^{\infty} f_n(x, y) \frac{t^{n\alpha}}{(n\alpha)!}$$

To find the mathematical solutions of (7), $u_i(x, y, t)$ may be written in the form

$$(8) \quad s_i = \sum_{n=0}^i u_n(x, y, t) = \sum_{n=0}^i f_n(x, y) \frac{t^{n\alpha}}{(n\alpha)!}$$

Step 4 The Elzaki residual function from (6) can be written as,

$$(9) \quad Res_i(x, y, t) = u_i(x, y, t) - G(x, y, t) - E^{-1}[v^\alpha E\{u_{i-1}(x, y, t)_{xx} + (u_{i-1}(x, y, t))_{yy}\}]$$

and hence the values of $f_n(x, y)$ may be obtained by putting $n = 0, 1, 2, \dots$ in the relation,

$$(10) \quad t^{-n\alpha} Res_n(x, y, t) /_{t=0} = 0$$

Substituting these values of $f_n(x, y)$ obtained from (10) in equation (7) the approximate solution of non-integer order non-linear diffusion equation¹⁸ is obtained analytically by using Elzaki transform with RPS method.

The pseudo code of the methodology can be demonstrated as in Figure 1.

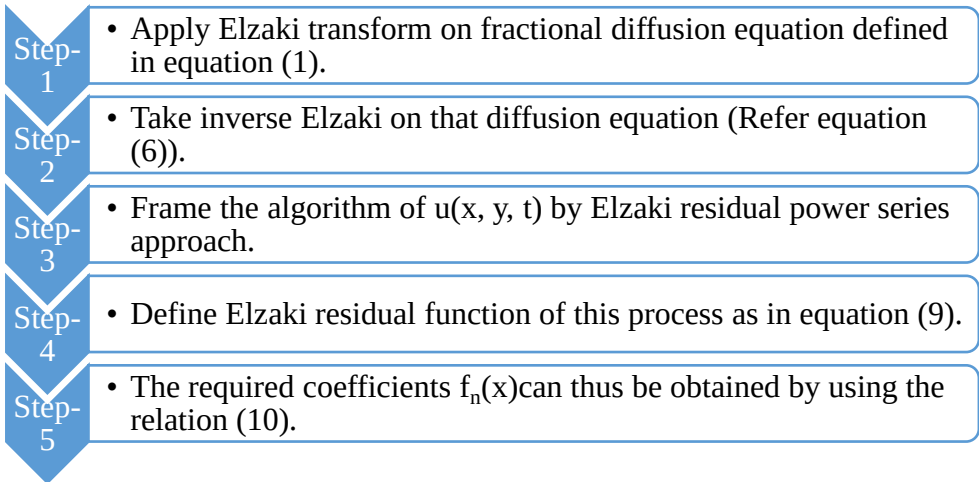


Figure 1 Pseudo code of the Methodology.

3. Numerical Solution

Using Elzaki transform on equation (1), then

$$(11) \quad E(D_t^\alpha u(x, y, t)) = E(u_{xx}(x, y, t) + u_{yy}(x, y, t))$$

Applying the differentiation property of Elzaki transform,

$$E[D_t^\alpha u(x, y, t)] = \frac{1}{v^\alpha} \{E(u(x, y, t)) - g(x, y, t)\} \text{ on equation (11), we get}$$

$$\frac{1}{v^\alpha} \{E(u(x, y, t)) - g(x, y, t)\} = E\{(u(x, y, t))_{xx} + (u(x, y, t))_{yy}\} \text{ i.e.}$$

$$(12) \quad E(u(x, y, t)) = g(x, y, t) + v^\alpha E\{(u(x, y, t))_{xx} + (u(x, y, t))_{yy}\}$$

Taking inverse Elzaki transform in equation (12), we get

$$(13) \quad u(x, y, t) = G(x, y, t) + E^{-1}[v^\alpha E\{(u(x, y, t))_{xx} + (u(x, y, t))_{yy}\}]$$

Now $u_i(x, y, t)$ may be written in the form,

$$(14) \quad s_i = \sum_{n=0}^i u_n(x, y, t) = \sum_{n=0}^i f_n(x, y) \frac{t^{n\alpha}}{(n\alpha)!}$$

Then the values of $f_n(x, y)$ can be obtained by using

$$(15) \quad Res_i(x, y, t) = u_i(x, y, t) - G(x, y, t) - E^{-1}[v^\alpha E\{(u_{i-1}(x, y, t))_{xx} + (u_{i-1}(x, y, t))_{yy}\}]$$

When $i = 0$ from equation (15),

$$Res_0(x, y, t) = u_0(x, y, t) - G(x, y, t)$$

and from equation (14),

$$(16) \quad \begin{aligned} u_0(x, y, t) &= G(x, y, t) \\ u_0(x, y, t) &= f_0(x, y) = \sin x \sin y \end{aligned}$$

When $i = 1$ from equation (15),

$$Res_1(x, y, t) = u_1(x, y, t) - G(x, y, t) - E^{-1}[v^\alpha E\{(u_0(x, y, t))_{xx} + (u_0(x, y, t))_{yy}\}]$$

with the conditions $u_1(x, y, t) = f_0(x, y) + f_1(x, y) \frac{t^\alpha}{\alpha!}$, we can obtain,

$$\begin{aligned} Res_1(x, y, t) &= f_0(x, y) + f_1(x, y) \frac{t^\alpha}{\alpha!} - G(x, y, t) - E^{-1}[v^\alpha E\{f_0(x, y)_{xx} + f_0(x, y)_{yy}\}] \\ &= \sin x \sin y + f_1(x, y) \frac{t^\alpha}{\alpha!} - \sin x \sin y - E^{-1}[v^\alpha E\{(\sin x \sin y)_{xx} + (\sin x \sin y)_{yy}\}] \\ &= f_1(x, y) \frac{t^\alpha}{\alpha!} - E^{-1}[v^\alpha E(-\sin x \sin y - \sin x \sin y)] \\ &= f_1(x, y) \frac{t^\alpha}{\alpha!} - E^{-1}[v^\alpha E(-2 \sin x \sin y)] \\ &= f_1(x, y) \frac{t^\alpha}{\alpha!} + 2 \sin x \sin y E^{-1}[v^\alpha E(1)] \\ &= f_1(x, y) \frac{t^\alpha}{\alpha!} + 2 \sin x \sin y E^{-1}[v^{\alpha+2}] \quad (\text{as } E(1) = v^2) \\ &= f_1(x, y) \frac{t^\alpha}{\alpha!} + 2 \sin x \sin y \frac{t^\alpha}{\alpha!} \quad (\text{as } E^{-1}(v^{\alpha+2}) = \frac{t^\alpha}{\alpha!}) \\ &= \{f_1(x, y) + 2 \sin x \sin y\} \frac{t^\alpha}{\alpha!} \end{aligned}$$

Then after solving $t^{-\alpha} Res_1(x, y, t)_{t=0} = 0$ gives that

$$(17) \quad f_1(x, y) = -2 \sin x \sin y.$$

When $i = 2$ from equation (15)

$$Res_2(x, y, t) = u_2(x, y, t) - G(x, y, t) - E^{-1}[v^\alpha E\{(u_1(x, y, t))_{xx} + (u_1(x, y, t))_{yy}\}],$$

with conditions

$$u_1(x, y, t) = f_0(x, y) + f_1(x, y) \frac{t^\alpha}{\alpha!}$$

and

$$u_2(x, y, t) = f_0(x, y) + f_1(x, y) \frac{t^\alpha}{\alpha!} + f_2(x, y) \frac{t^{2\alpha}}{(2\alpha)!},$$

we get

$$\begin{aligned} Res_2(x, y, t) &= f_0(x, y) + f_1(x, y) \frac{t^\alpha}{\alpha!} + f_2(x, y) \frac{t^{2\alpha}}{(2\alpha)!} - f_0(x, y) - \\ &E^{-1} \left[v^\alpha E \left\{ (f_0(x, y) + f_1(x, y) \frac{t^\alpha}{\alpha!})_{xx} + (f_0(x, y) + f_1(x, y) \frac{t^\alpha}{\alpha!})_{yy} \right\} \right] \\ &= \{f_2(x, y) - 4 \sin x \sin y\} \frac{t^{2\alpha}}{(2\alpha)!} \end{aligned}$$

Therefore, from $t^{-2\alpha} Res_2(x, y, t)/_{t=0} = 0$ we have,

$$(18) \quad f_2(x, y) = 4 \sin x \sin y.$$

Now, the second approximate solution is

$$u_2(x, y, t) = \sin x \sin y - 2 \sin x \sin y \frac{t^\alpha}{\alpha!} + 4 \sin x \sin y \frac{t^{2\alpha}}{(2\alpha)!}.$$

When $i = 3$ from equation (15)

$$Res_3(x, y, t) = u_3(x, y, t) - G(x, y, t) - E^{-1}[v^\alpha E\{(u_2(x, y, t))_{xx} + (u_2(x, y, t))_{yy}\}],$$

with conditions $u_2(x, y, t) = f_0(x, y) + f_1(x, y) \frac{t^\alpha}{\alpha!} + f_2(x, y) \frac{t^{2\alpha}}{(2\alpha)!}$ and

$$u_3(x, y, t) = f_0(x, y) + f_1(x, y) \frac{t^\alpha}{\alpha!} + f_2(x, y) \frac{t^{2\alpha}}{(2\alpha)!} + f_3(x, y) \frac{t^{3\alpha}}{(3\alpha)!}$$

we get,

$$\begin{aligned} Res_3(x, y, t) &= f_0(x, y) + f_1(x, y) \frac{t^\alpha}{\alpha!} + f_2(x, y) \frac{t^{2\alpha}}{(2\alpha)!} + f_3(x, y) \frac{t^{3\alpha}}{(3\alpha)!} - f_0(x, y) - \\ &E^{-1} \left[v^\alpha E \left\{ (f_0(x, y) + f_1(x, y) \frac{t^\alpha}{\alpha!} + f_2(x, y) \frac{t^{2\alpha}}{(2\alpha)!})_{xx} + \right. \right. \\ &\quad \left. \left. (f_0(x, y) + f_1(x, y) \frac{t^\alpha}{\alpha!} + f_2(x, y) \frac{t^{2\alpha}}{(2\alpha)!})_{yy} \right\} \right] \\ &= \{f_3(x, y) + 8 \sin x \sin y\} \frac{t^{3\alpha}}{(3\alpha)!} \end{aligned}$$

Therefore, from $t^{-3\alpha} Res_3(x, y, t)/_{t=0} = 0$ we have

$$(19) \quad f_3(x, y) = -8 \sin x \sin y$$

Now, the third approximate solution is

$$u_3(x, y, t) = \sin x \sin y - 2 \sin x \sin y \frac{t^\alpha}{\alpha!} + 4 \sin x \sin y \frac{t^{2\alpha}}{(2\alpha)!} - 8 \sin x \sin y \frac{t^{3\alpha}}{(3\alpha)!}$$

Similarly, the n^{th} coefficient of $u(x, y, t)$ is

$$f_n(x, y) = (-2)^n \sin x \sin y$$

At last the n^{th} ERPSM approximate solutions of $u(x, y, t)$ is

$$(20) \quad u_n(x, y, t) = \sin x \sin y \sum_{n=0}^i \frac{(-2t^\alpha)^n}{(n\alpha)!}$$

4. Numerical Simulation

The numerical solution of this equation has been obtained for the domain $[0,1]$ for both x and y and the results are presented at $t = 0.5$ for $\alpha = 0.5, 0.8$ and 1 in Table 1-3 and Figure 2-4. In order to present the efficiency of the obtained results the maximum errors are presented at different time levels at α equals to one in Table 4. The obtained results are satisfying the exact solution with a good level of accuracy.

Table 1 Solution when value of $\alpha = 0.5$.

y/x	0.2	0.4	0.6	0.8	1.0
0.1	0.006805	0.01334	0.019342	0.024573	0.028825
0.2	0.013543	0.026546	0.038491	0.048901	0.057361
0.3	0.020145	0.039487	0.057255	0.07274	0.085325
0.4	0.026546	0.052033	0.075447	0.095852	0.112436
0.5	0.032681	0.06406	0.092885	0.118006	0.138423
0.6	0.038491	0.075447	0.109395	0.138982	0.163028
0.7	0.043915	0.086079	0.124812	0.158569	0.186004
0.8	0.048901	0.095852	0.138982	0.176571	0.207121
0.9	0.053398	0.104667	0.151763	0.192809	0.226168
1.0	0.057361	0.112436	0.163028	0.207121	0.242956

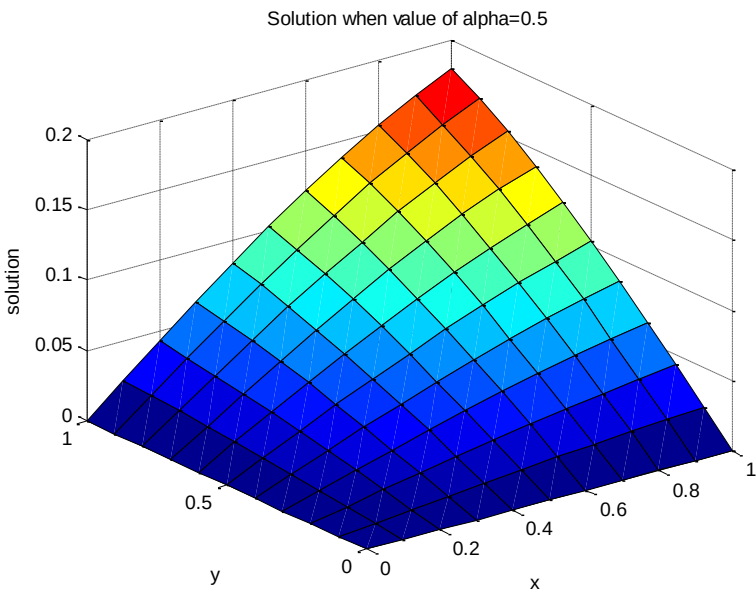


Figure 2 Solution when value of $\alpha=0.5$.

Table 2 Solution when value of alpha=0.8.

y/x	0.2	0.4	0.6	0.8	1.0
0.1	0.003764	0.007379	0.010699	0.013593	0.015944
0.2	0.007491	0.014684	0.021291	0.027049	0.031729
0.3	0.011143	0.021842	0.03167	0.040236	0.047197
0.4	0.014684	0.028782	0.041733	0.05302	0.062193
0.5	0.018078	0.035434	0.051379	0.065275	0.076568
0.6	0.021291	0.041733	0.060511	0.076877	0.090178
0.7	0.024291	0.047614	0.069039	0.087711	0.102887
0.8	0.027049	0.05302	0.076877	0.097669	0.114568
0.9	0.029537	0.057896	0.083947	0.106651	0.125104
1.0	0.031729	0.062193	0.090178	0.114568	0.13439

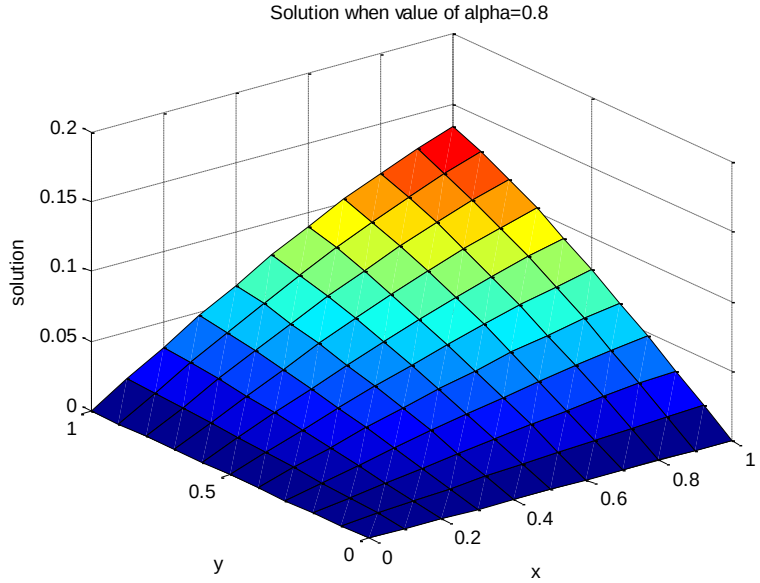


Figure 3 Solution when value of alpha =0.8

Table 3 Solution when value of alpha=1.0

y/x	0.2	0.4	0.6	0.8	1.0
0.1	0.002684	0.005261	0.007629	0.009692	0.011369
0.2	0.005342	0.01047	0.015182	0.019288	0.022625
0.3	0.007946	0.015575	0.022582	0.02869	0.033654
0.4	0.01047	0.020523	0.029758	0.037806	0.044347
0.5	0.01289	0.025267	0.036636	0.046544	0.054597
0.6	0.015182	0.029758	0.043148	0.054818	0.064302

0.7	0.017321	0.033952	0.049229	0.062543	0.073364
0.8	0.019288	0.037806	0.054818	0.069644	0.081693
0.9	0.021061	0.041283	0.059859	0.076048	0.089206
1.0	0.022625	0.044347	0.064302	0.081693	0.095827

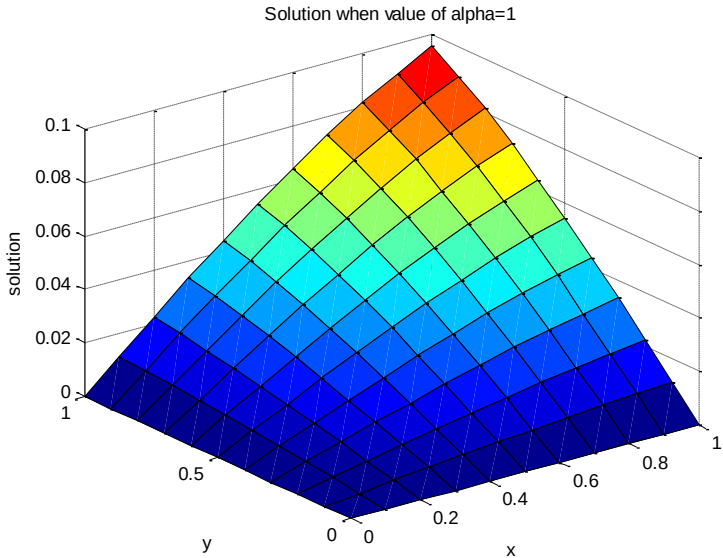


Figure 4 Solution when value of alpha=1.0.

Table 4 Errors in the solution for alpha=1.0 at different time levels.

T	l_{∞}
0.1	4.4408e-16
0.2	7.1981e-13
0.3	6.1280e-11
0.4	1.4280e-09
0.5	1.6366e-08
0.6	1.1974e-07
0.7	6.4273e-07
0.8	2.7504e-06
0.9	9.8999e-06
1.0	3.1088e-05

5. Conclusion

In current paper, a new and reliable approach is constructed to find the approximate solutions of two dimensional non-integer order diffusion equation by using Elzaki transform with RPS method. This approach combines the Elzaki with RPS method that is the enhancement of classical

RPS method. The advantage of this approach is to decrease the computational work with less error for finding the solution in RPS form. The coefficients of this power series solution are determined in the above successive steps. The two dimensional non-integer order diffusion equation is solved by Elzaki transform with RPS approach that verified its ability to solve non-linear non-integer order diffusion equation with sufficient correctness and reliable computation steps. This new method Elzaki transform with RPS approach is beneficial for solution of non-integer order diffusion equation. This method also provides simple as well as accurate algorithms for finding approximate solutions of non-integer order diffusion equation.

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