

Study of $N(k)$ -Quasi Einstein Manifolds with Certain Curvature Restrictions

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Abstract: This paper deals with the study of $N(k)$ -quasi-Einstein manifolds under certain curvature restrictions. We construct the non-trivial physical and geometrical examples of $N(k)$ -quasi-Einstein manifolds, which validate the existence of such manifolds. The necessary and sufficient conditions for which the conformally and quasi-conformally flat $N(k)$ -quasi-Einstein manifolds satisfy certain curvature restrictions are derived. We also prove that the generator of the manifold is a Killing vector field and the integral curves generated by the characteristic vector field are geodesics.

Keywords: Quasi Einstein manifolds, $N(k)$ -quasi Einstein manifolds, C -Fubinian manifolds, nullity distribution, $f(r, T)$ -gravity, different curvature tensors.

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1. Introduction

Einstein manifolds are geometric objects, and play a central role in differential geometry and mathematical physics. In mathematical physics, they are used to describe the spacetimes in Einstein's theory of general

relativity whereas in mathematics, they are basic building blocks of more complicated geometries. A quasi-Einstein manifold is a natural generalization or modification of the Einstein manifold. During the study of exact solutions of Einstein's field equations and quasi-umbilical hypersurfaces, we perceived the presence of quasi-Einstein manifolds. For illustration, the Robertson-Walker spacetimes in general relativity; the toy universe in cosmology; models of gravitational collapse in astrophysics are quasi-Einstein manifolds¹⁻¹⁰. Let M_n be an n -dimensional Riemannian manifold endowed with a Riemannian metric g , then (M_n, g) , ($n > 2$), is said to be a quasi-Einstein manifold if its non-vanishing Ricci tensor S satisfies

$$(1) \quad S(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y), \quad \forall X, Y \in TM$$

for smooth functions α and $\beta (\neq 0)$, where η is a non-zero 1-form associated with the Riemannian metric g as

$$(2) \quad g(X, \xi) = \eta(X), \quad g(\xi, \xi) = \eta(\xi) = 1$$

for all vector field X and the associated unit vector field ξ on M_n ¹¹. The unit vector field ξ and 1-form η are called the generator and the associated 1-form of M_n , respectively. Chaubey *et al.*¹² proved that a collection of non-interacting pressure less perfect fluid of general relativity is a four-dimensional semi-Riemannian quasi-Einstein manifold whose associated scalars are $\frac{r}{2}$ and $\kappa\rho$, where κ is the gravitational constant, ρ and r are the energy density and scalar curvature, the generator of the manifold being the timelike velocity vector field of the perfect fluid. Debnath *et al.*¹³ showed that the quasi-Einstein manifold is locally a product manifold of one-dimensional distribution U and $(n-1)$ dimensional distribution $U^\perp$, provided the generator of a quasi-Einstein manifold is a parallel vector field, where $U^\perp$ is involutive and integrable. Chaki *et al.*¹¹ have proved that the Ricci tensor of an n -dimensional quasi-Einstein manifold has precisely two distinct eigenvalues α and $\alpha + \beta$, where the multiplicity of α is $n-1$ and $\alpha + \beta$ is simple. It is noticed that a proper η -Einstein contact metric manifold is a natural example of a quasi-Einstein manifold^{14,15}. The physical and geometrical properties of the quasi-Einstein manifolds have been studied¹⁷⁻²⁴ and also by others.

Let k be a smooth function on M_n , then the k -nullity distribution $N(k)$ of a Riemannian manifold M_n is defined by

$$(3) \quad N(k): p \rightarrow N_p(k) = \{Z \in T_p M : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\}$$

for arbitrary vector fields X, Y and Z on M_n , where R represents the Riemannian curvature tensor M_n ²⁵. A quasi-Einstein manifold is called an $N(k)$ -quasi-Einstein manifold if the generator ξ of M_n belongs to the k -

nullity distribution²⁶. Many geometers^{17-21, 26-30} have studied the physical and geometrical properties of an $N(k)$ -quasi-Einstein manifold.

In Cosmology, the reason for studying various types of spacetime models is mainly to represent the different phases in the evolution of the universe. The evolution of the universe to its present state can be divided into three phases: The initial phase just after the big bang when the effects of both viscosity and heat flux were quite pronounced. The intermediate phase, when the effect of viscosity was no longer significant but the heat flux was still not negligible. The final phase, which extends to the present state of the universe when both the effects of viscosity and the heat flux have become negligible and the matter content of the universe may be assumed to be a perfect fluid. The importance of the study of a quasi-Einstein manifold lies in the fact that these spacetimes represent the second and the third phase, respectively, in the evolution of the universe²³. These factors motivate us to study the geometrical and physical properties of $N(k)$ -quasi-Einstein manifolds. We organize our work as:

Section 2 deals with the basic results of an $N(k)$ -quasi-Einstein manifold and different curvature tensors. We prove the existence of $N(k)$ -quasi-Einstein manifolds by constructing non-trivial physical and geometrical examples in Section 3. In Section 4, we explore $N(k)$ -quasi Einstein manifolds satisfying $H(\xi, X) \cdot C = 0$ and proved that the generator of such manifolds is Killing, and the integral curves generated by the characteristic vector field are geodesics. In Section 5 and Section 6, we show that an $N(k)$ -quasi-Einstein manifold is conformally and quasi-conformally flat under certain curvature restrictions. The conditions for the non-existence of $N(k)$ -quasi-Einstein manifolds are derived in Section 7 and Section 8, respectively.

2. Preliminaries

From the equations (1)-(3), we get

$$(4) \quad S(X, \xi) = (\alpha + \beta)\eta(X), \quad r = n\alpha + \beta$$

and

$$(5) \quad k = \frac{\alpha + \beta}{n-1},$$

where r is the scalar curvature of (M_n, g) . An $N(k)$ -quasi-Einstein manifold (M_n, g) satisfies the following relations:

$$(6) \quad R(X, \xi)Y = k[\eta(Y)X - g(X, Y)\xi] = -R(\xi, X)Y,$$

$$(7) \quad R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y],$$

$$(8) \quad \eta(R(X, Y)Z) = k[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]$$

for arbitrary vector fields X, Y and Z on M_n^{21} .

Projective curvature tensor P^{31} , concircular curvature tensor $\hat{C}$, conharmonic curvature H^{32-33} , Weyl conformal curvature tensor C^{31} and the quasi-conformal curvature tensor $\tilde{C}^{34}$ on (M_n, g) are defined by

$$(9) \quad P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y],$$

$$(10) \quad \hat{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y],$$

$$(11) \quad H(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY\},$$

$$(12) \quad C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y]$$

and

$$(13) \quad \tilde{C}(X, Y)Z = \lambda R(X, Y)Z + \mu[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] - \frac{an+b}{n}\left[\frac{\lambda}{n-1} + 2\mu\right]\{g(Y, Z)X - g(X, Z)Y\},$$

respectively, for arbitrary vector fields X, Y, Z on M_n and $\lambda, \mu \in \mathbb{R}$; where $S(X, Y) = g(QX, Y)$. Here Q denotes the Ricci operator of M_n . If $\mathcal{T}$ is a tensor field of type $(1,3)$ on M_n , then

$$(14) \quad (\hat{C}(W, X) \cdot \mathcal{T})(Y, Z)U = \hat{C}(W, X)\mathcal{T}(Y, Z)U - \mathcal{T}(\hat{C}(W, X)Y, Z)U - \mathcal{T}(Y, \hat{C}(W, X)Z)U - \mathcal{T}(Y, Z)\hat{C}(W, X)U,$$

for all vector fields W, X, Y, Z and U on M_n , where $\hat{C}(W, X)$ acts on $\mathcal{T}$ as a deviation.

3. Existence of some classes of $N(k)$ -quasi Einstein manifolds

In relativistic cosmology, the theory of uniform model universes is based on Einstein's equations, derived from a variational principle³⁵. In 1970, H. A. Buchdahl was first proposed the idea of $f(r)$ gravity. It is actually a family of theories in which each one is defined by a different function of the Ricci scalar. Thus we can say that the $f(r)$ gravity is a type of the modified gravity theory which generalizes the Einstein's theory of relativity. Harko *et al.*³⁶ developed the theory of $f(r, T)$ -gravity in 2011, which is the generalization or modification of the general theory of relativity and $f(r)$ -gravity. In the theory of $f(r, T)$ -gravity, the gravitational Lagrangian is assumed as an arbitrary function of the scalar curvature r and the trace of stress energy tensor

T . Let L_m denote the matter Lagrangian density of matter, then the field equations for the $f(r, T)$ -gravity are derived from the Hilbert-Einstein type variational by considering the action

$$(15) \quad A = \frac{1}{16\pi} \int [f(r, T) + L_m] \sqrt{-g} d^4x.$$

If possible, we consider that the matter Lagrangian density depends only on the metric g_{ij} , then the stress-energy tensor of the matter is given by

$$(16) \quad T_{ij} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g})L_m}{\delta g^{ij}}, \quad (i, j = 1, 2, 3, 4)$$

and the field equations of $f(r, T)$ -gravity is given by

$$(17) \quad (g_{ij} \nabla_k \nabla^k - \nabla_i \nabla_j) f_r(r, T) + f_r(r, T) S_{ij} - \frac{1}{2} f(r, T) g_{ij} \\ = 8\pi T_{ij} - f_T(r, T) T_{ij} - f_T(r, T) \Theta_{ij},$$

where f_r and f_T represent the partial derivative of f with respect to r and T , respectively, and

$$\Theta_{ij} = g_{ij} L_m - 2T_{ij} - 2g^{lk} \frac{\partial^2 L_m}{\partial g^{ij} \partial g^{lk}}.$$

Here ∇_i denotes the covariant derivative. As a particular, if we take $f(r, T) = f(r)$, then (15) and (16) give the field equations of the $f(r)$ -gravity³⁷.

Let us suppose that a matter be a 4-dimensional conformally flat perfect fluid. If T_{ij} is the stress energy tensor of the perfect fluid, then it is given by

$$(18) \quad T_{ij} = (\rho + p)u_i u_j - p g_{ij},$$

where ρ is the energy density, p is the pressure and u^i is the four-velocity of perfect fluid. It is well known that

$$(19) \quad u^i \nabla_j u_i = 0, \quad u_i \cdot u^i = 1.$$

As we know that there is no unique definition of the matter Lagrangian, therefore we choose $L_m = -p$ and thus the variation of stress energy of the perfect fluid together with equation (18) consider the form

$$(20) \quad \Theta_{ij} = -2T_{ij} - p g_{ij}.$$

In general, the field equations depend on the physical nature of the matter field and therefore for each choice of r and T , we get a theoretical model for $f(r, T)$ -gravity. As an example, we take

$$(21) \quad f(r, T) = r + f(T),$$

where $f(T)$ is the arbitrary function of the trace of stress energy tensor of the matter. In consequence of (21) and (17), we find

$$(22) \quad S_{ij} - \frac{1}{2}r g_{ij} = 8\pi T_{ij} - f'(T)T_{ij} - f'(T)\Theta_{ij} + \frac{1}{2}f(T)g_{ij}.$$

With the help of (18), (19) and (20), equation (22) assumes the form

$$(23) \quad S_{ij} = \frac{1}{2}\{r + f(T) - 16p\pi\}g_{ij} + (p + \rho)(8\pi + f'(T))u_i u_j.$$

Contracting (23) for i and j , we get

$$(24) \quad r = 24\pi p - (p + \rho)f'(T) - 2f(T) - 8\pi\rho.$$

In light of (24), equation (23) takes the form

$$(25) \quad S_{ij} = \frac{1}{2}\{8\pi p - 8\pi\rho - (p + \rho)f'(T) - f(T)\}g_{ij} \\ + (p + \rho)(8\pi + f'(T))u_i u_j.$$

Equation (25) shows that the perfect fluid under consideration is a certain class of the quasi-Einstein manifold with the associated scalars $\frac{1}{2}\{8\pi p - 8\pi\rho - (p + \rho)f'(T) - f(T)\}$ and $(p + \rho)(8\pi + f'(T))$. It is well known that a conformally flat quasi-Einstein manifold is an $N(k)$ -quasi-Einstein manifold²⁶. Hence from (5) and (25), we can find

$$k = \frac{1}{6}\{8\pi(3p + \rho) + (p + \rho)f'(T) - f(T)\}.$$

From the above discussions, we can state the following examples:

Example 3.1: A conformally flat perfect fluid of dimension four is an $N(\frac{1}{6}\{8\pi(3p + \rho) + (p + \rho)f'(T) - f(T)\})$ -quasi Einstein manifold.

In particular, if we suppose that $f(T) = \varrho = \text{constant}$, then we have $k = \frac{1}{6}\{8\pi(3p + \rho) - \varrho\}$. Hence we can state:

Example 3.2: A 4-dimensional conformally flat perfect fluid together with $f(T) = \varrho$ is an $N(\frac{1}{6}\{8\pi(3p + \rho) - \varrho\})$ -quasi Einstein manifold.

A dust universe ($p = 0, T = \rho$) is a simplest cosmological model of the perfect fluid. If possible, we assume that the matter space is a conformally flat dust universe along with $f(T) = \delta T$, where δ is a constant. Then from (25), it is obvious that the Ricci tensor of the matter takes the form

$$(26) \quad S_{ij} = -(4\pi + \delta)\rho g_{ij} + \rho(8\pi + \delta)u_i u_j.$$

The contraction of equation (26) for i and j gives

$$r = -3\delta\rho.$$

From (26), it is obvious that the dust universe is a certain class of the quasi-Einstein manifold with the associated scalars $-(4\pi + \delta)\rho$ and $\rho(8\pi + \delta)$. Therefore with the help of (5) and (26), we can easily conclude that

$$k = \frac{4\pi\rho}{3}.$$

Hence we can state:

Example 3.3: Every 4-dimensional conformally flat dust universe is an $N(\frac{4\pi\rho}{3})$ -quasi-Einstein manifold.

Remark 3.1: It is noticed that the geometrical properties of $f(R)$ -gravity and $f(R, T)$ -gravity have been studied from a geometry point of view in¹⁷.

Let $\bar{M}$ be an $n(= 2m)$ -dimensional Kähler manifold. A Kähler manifold $\bar{M}$ is said to be a locally Fubinian manifold of constant holomorphic sectional curvature if the holomorphic sectional curvature tensor at every point of $\bar{M}$ is independent of directions at the point, i.e.,

$$\bar{R}_{lkji} = \kappa\{g_{li}g_{kj} - g_{ki}g_{lj} + F_{li}F_{kj} - F_{ki}F_{lj} - 2F_{lk}F_{ji}\},$$

where g is the Hermitian metric, $\bar{R}$ is the curvature tensor with respect to the Riemannian connection of the metric g , F is a skew-symmetric tensor of type $(0, 2)$ and κ is a constant. A locally Fubinian manifold is an Einstein manifold³⁸⁻³⁹. In 1963, Y. Tashiro and Tachibana⁴⁰ proved that a C -umbilical hypersurface in a locally Fubinian manifold is a locally C -Fubinian manifold. In other words, a Sasakian manifold is called a C -Fubinian manifold or a manifold of constant holomorphic sectional curvature tensor, if the holomorphic sectional curvature at each point is independent of direction at the point, and its curvature tensor in index form notation is given by the relation

$$\begin{aligned} R_{kjih} = & (\kappa + 1)\{g_{kh}g_{ji} - g_{jh}g_{ki}\} + \kappa\{f_{kh}f_{ji} - f_{jh}f_{ki} - 2f_{kj}f_{ih}\} \\ & - \kappa\{g_{kh}f_{jfi} + g_{ji}f_{kfh} - g_{ih}f_{kfi} - g_{ki}f_{jfk}\}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} (27) \quad R(X, Y)Z = & (\kappa + 1)\{g(Y, Z)X - g(X, Z)Y\} + \kappa\{\Phi(Y, Z)\phi X \\ & - \Phi(X, Z)\phi Y - 2\Phi(X, Y)\phi Z\} - \kappa\{g(Y, Z)\eta(X)\xi \\ & - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}, \end{aligned}$$

where $\Phi(X, Y) = g(\phi X, Y)$ and ϕ is a tensor field of type $(1, 1)$. Contracting (27) along the vector X and then using $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ and the equation (2), we find that

$$(28) \quad S(Y, Z) = \{\kappa(n + 1) + (n - 1)\}g(Y, Z) - \kappa(n + 1)\eta(Y)\eta(Z),$$

which shows that the manifold is a certain class of quasi-Einstein manifold with associated scalars $\kappa(n+1) + (n-1)$ and $-\kappa(n+1)$. Replacing $Z = \xi$ in (27) and then using (2) and $\phi\xi = 0$, we obtain

$$(29) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

which reflects that the generator ξ of the manifold belongs to 1-nullity distribution. From (5) and (27), we conclude that

$$k = \frac{\alpha+\beta}{n-1} = 1.$$

Thus from above discussions, we can say

Example 3.4: Every C-Fubinian manifold is an $N(1)$ -quasi-Einstein manifold.

Putting $Y = \xi$ in (29) and then use of (2) gives

$$(30) \quad R(X, \xi)\xi = X - \eta(X)\xi \quad (\neq 0).$$

From (28)-(30) and Example 3, we conclude the following lemma:

Lemma: The following statements hold on a C-Fubinian manifold (M_n, g) .

- (i) (M_n, g) is an $N(1)$ -quasi Einstein manifold.
- (ii) (M_n, g) is regular, that is, $R(X, \xi)\xi \neq 0$.
- (iii) ξ is not a semi-torse forming vector field (ξ is called a semi-torse forming vector field if $R(X, \xi)\xi = 0$).
- (iv) $S(\xi, \xi) \neq 0$, i. e., ξ is isotropic with respect to S .
- (v) $Q\xi \neq 0$, i. e., $\xi \notin \text{Ker}(Q)$.

Example 3.5: Let $\mathbb{R}^n$ be an n -dimensional real space and M is a 4-dimensional differentiable manifold defined by $M = \{(x^1, x^2, x^3, x^4) : (x^1, x^2, x^3, x^4) \in \mathbb{R}^4 \text{ and } \pi q < x^1 < (q+1)\pi, q \in \mathbb{Z}\}$, ($\mathbb{Z}$ is the set of integer). If g is the Lorentzian metric of M defined by

$$ds^2 = g_{ij}dx^i dx^j = \{\cot(x^1) - \csc(x^1)\}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2,$$

where $i, j = 1, 2, 3, 4$, then we have

$$g_{ij} = \begin{pmatrix} \cot(x^1) - \csc(x^1) & 0 & 0 & 0 \\ 0 & \cot(x^1) - \csc(x^1) & 0 & 0 \\ 0 & 0 & \cot(x^1) - \csc(x^1) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and its associated components g^{ij} are given by

$$g^{ij} = \begin{pmatrix} \frac{\sin(x^1)}{\cot(x^1)-1} & 0 & 0 & 0 \\ 0 & \frac{\sin(x^1)}{\cot(x^1)-1} & 0 & 0 \\ 0 & 0 & \frac{\sin(x^1)}{\cot(x^1)-1} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

In consequence of the last two equations, it can be found that the non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor and the scalar curvature are given by

$$(31) \quad \Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{13}^3 = \frac{\csc(x^1)}{2}, \quad \Gamma_{22}^1 = \Gamma_{33}^1 = -\frac{\csc(x^1)}{2}, \quad S_{33} = \frac{\csc^2(x^1)}{4},$$

$$R_{1331} = \frac{\csc^2(x^1)(\csc(x^1) - \cot(x^1))}{4}, \quad r = \frac{\csc(x^1)}{4(\cos(x^1)-1)} (\neq 0).$$

With the help of equation (31), we can easily verify that the manifold $(\mathbb{R}^4, g)$ is a Lorentzian manifold. To achieve our goal, we will show that $(\mathbb{R}^4, g)$ is a certain class of the $N(k)$ -quasi-Einstein manifold. Define the associated smooth functions α and β as follows:

$$(32) \quad \alpha = 0, \quad \beta = \frac{\csc(x^1)}{4(1-\cos(x^1))} (\neq 0).$$

We define the 1-forms A_i as:

$$(33) \quad A_i = \begin{cases} \sqrt{\frac{1-\cos(x^1)}{\sin(x^1)}}, & i = 3 \\ 0, & \text{otherwise} \end{cases}$$

Now, we prove

$$S_{ij} = \alpha g_{ij} + \beta A_i A_j$$

for $i, j = 1, 2, 3, 4$. For instant, we have to show that

$$(34) \quad S_{33} = \alpha g_{33} + \beta A_3 A_3.$$

From equation (31), it is obvious that $S_{33} = \frac{\csc^2(x^1)}{4}$, which is the left hand side of (34). In view of (32) and (33), right hand side of (34) $= \alpha g_{33} + \beta A_3 A_3 = \frac{\csc^2(x^1)}{4}$. In the same fashion, we can verify for other components of S_{ij} . From (1)-(3) and (32), it is clear that

$$k = \frac{\alpha - \beta}{n-1} = \frac{\csc(x^1)}{12(\cos(x^1)-1)}$$

and $r = 4\alpha - \beta$. Hence $(\mathbb{R}^4, g)$ is an $N\left(\frac{\csc(x^1)}{12(\cos(x^1)-1)}\right)$ -quasi-Einstein manifold.

Example 3.6: Let us suppose that $\mathbb{R}^3 = \{(x^1, x^2, x^3) : x^i \in \mathbb{R}, i = 1, 2, 3\}$ be a three dimensional real number space. If M is a differentiable manifold of dimension 3, then we define $M = \{(x^1, x^2, x^3) : (x^1, x^2, x^3) \neq 0\}$. Let g_{ij} be a Lorentzian metric on M defined by

$$ds^2 = g_{ij}dx^i dx^j = -(dx^1)^2 + f(x^1)[(dx^2)^2 + (dx^3)^2],$$

where $i, j = 1, 2, 3$ and f is a non-zero function of x^1 such that $4ff'' \neq (f')^2$. Here f' and f'' are the first and second derivatives of f with respect to x^1 , respectively. The non-vanishing components of the Lorentzian metric g_{ij} are given by

$$(35) \quad g_{11} = -1, \quad g_{22} = f = g_{33}$$

and its associated components g^{ij} are

$$(36) \quad g^{11} = -1, \quad g^{22} = g^{33} = \frac{1}{f}.$$

The non-vanishing components of the Christoffel symbols, curvature and the Ricci tensors are obtained from the equations (35) and (36). The components are

$$(37) \quad \Gamma_{22}^1 = \Gamma_{33}^1 = \frac{f'}{2}, \quad \Gamma_{12}^2 = \Gamma_{13}^3 = \frac{f'}{2f}, \quad R_{1212} = R_{1313} = f \left\{ \left(\frac{f'}{2f} \right)' - \left(\frac{f'}{2f} \right)^2 \right\},$$

$$R_{2323} = \frac{(f')^2}{4}, \quad S_{11} = \frac{2ff'' - (f')^2}{2f^2}, \quad S_{22} = S_{33} = -\frac{f''}{2},$$

and other components can be obtained from symmetric properties. In consequence of equation (37), we can prove that the differentiable manifold M of dimension 3 equipped a Lorentzian metric g is a 3-dimensional Lorentzian manifold. From (35)-(37), we can find that

$$r = -\frac{2f''}{f} + \frac{(f')^2}{2f^2} \neq 0.$$

Now, we choose the scalars α and β as follows:

$$(38) \quad \alpha = -\frac{f''}{2f}, \quad \beta = \frac{ff'' - (f')^2}{2f^2} \neq 0.$$

We define the 1-form A as follows:

$$(39) \quad A_i = \begin{cases} 1 & \text{for } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Now, we have to prove

$$(40) \quad S_{ij} = \alpha g_{ij} + \beta A_i A_j.$$

In consequence of (35) and (37)-(40), it can be easily verified that $S_{11} = \alpha g_{11} + \beta A_1 A_1$ and all other components of S satisfy (40). Hence (M^3, g) is

a certain class of quasi-Einstein manifold. Now, $k = \frac{\alpha - \beta}{n-1} = \frac{(f')^2 - 2ff''}{4f^2}$ and $r = 3\alpha - \beta$ hold on $(\mathbb{R}^3, g)$, therefore $(\mathbb{R}^3, g)$ is an $N\left(\frac{(f')^2 - 2ff''}{4f^2}\right)$ -quasi Einstein manifold.

Example 3.7:¹⁷ A four-dimensional conformally flat perfect fluid (M_4, g) is an $N\left(\frac{1}{3}\left\{\frac{1}{2}r + f(T) + 8\pi\rho + 2(p + \rho)f'(T)\right\}\right)$ -quasi Einstein manifold.

Example 3.8:¹⁹ A conformally flat perfect fluid spacetime (M^4, g) satisfying Einstein's equation with cosmological constant λ is an $N\left(\frac{\lambda}{3} + \frac{\kappa(3\sigma + p)}{6}\right)$ -quasi Einstein manifold.

4. $N(k)$ -quasi Einstein manifolds satisfying $H(\xi, X) \cdot C = 0$

In view of equations (2), (3), (6) and (11), we find

$$(41) \quad H(\xi, Y)Z = -\frac{\alpha n + \beta}{(n-1)(n-2)}[g(Y, Z)\xi - \eta(Z)Y].$$

From (14), we have

$$(42) \quad (H(W, X) \cdot C)(Y, Z)U = H(W, X)C(Y, Z)U - C(H(W, X)Y, Z)U \\ - C(Y, H(W, X)Z)U - C(Y, Z)H(W, X)U.$$

Setting $W = \xi$ and $H(\xi, X) \cdot C = 0$ in (42), we get

$$H(\xi, X)C(Y, Z)U - C(H(\xi, X)Y, Z)U - C(Y, H(\xi, X)Z)U \\ - C(Y, Z)H(\xi, X)U = 0,$$

which becomes from (41)

$$\frac{\alpha n + \beta}{(n-1)(n-2)}[g(C(Y, Z)U, X)\xi - \eta(C(Y, Z)U)X \\ + \eta(Z)C(Y, X)U + \eta(Y)C(X, Z)U - g(X, Z)C(Y, \xi)U \\ - g(X, Y)C(\xi, Z)U - g(X, U)C(Y, Z)\xi + \eta(U)C(Y, Z)X] = 0.$$

In light of (41), the above equation becomes

$$(43) \quad \frac{\alpha n + \beta}{(n-1)(n-2)}\{g(C(Y, Z)U, X) - \eta(C(Y, Z)U)\eta(X) - g(X, Y)\eta(C(\xi, Z)U) \\ + \eta(Y)\eta(C(X, Z)U) - g(X, Z)\eta(C(Y, \xi)U) + \eta(Z)\eta(C(Y, X)U) \\ - g(X, U)\eta(C(Y, Z)\xi) + \eta(U)\eta(C(Y, Z)X)\} = 0.$$

In view of (1), (2), (4), (7), (8) and (12), we have

$$(44) \quad \eta(C(X, Y)Z) = 0.$$

With the help of (44), (43) becomes

$$\frac{\alpha n + \beta}{(n-1)(n-2)} {}'C(X, Y, Z, U) = 0,$$

which implies that either

$$(45) \quad \alpha n + \beta = 0 \quad \text{or} \quad {}'C(X, Y, Z, U) = 0,$$

where ${}'C(Y, Z, U, X) = g(C(Y, Z)U, X)$. Converse part is trivial from (41), (42) and (45). Thus we state the following theorem as:

Theorem 4.1: Let M_n ($n > 2$) be an n -dimensional $N(k)$ -quasi Einstein manifold. Then $H(\xi, X) \cdot C = 0$ holds on M_n if and only if either the scalar curvature of M_n vanishes or M_n is conformally flat.

Remark 4.1: Let us suppose that the function f defined in Example 3.6 satisfies the non-linear differential equation $4ff'' = (f')^2$, then (3.6) shows that the scalar curvature is zero, i.e. $r = 0$, and hence the Theorem 4.1.

Let us consider that $\alpha n + \beta = 0$ and therefore

$$(46) \quad n\alpha(X) + d\beta(X) = 0.$$

Taking covariant derivative of (1) and then using (2), we find

$$(47) \quad (\nabla_X S)(Y, Z) = d\alpha(X)g(Y, Z) + d\beta(X)\eta(Z)\eta(Y) \\ + \beta\{(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y)\}.$$

From (47), we have

$$(48) \quad (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = d\alpha(X)g(Y, Z) \\ + d\beta(X)\eta(Z)\eta(Y) + d\alpha(Y)g(Z, X) + d\beta(Y)\eta(Z)\eta(X) \\ + d\alpha(Z)g(X, Y) + d\beta(Z)\eta(X)\eta(Y) + \beta\{(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y) \\ + (\nabla_Y \eta)(Z)\eta(X) + (\nabla_Y \eta)(X)\eta(Z) + (\nabla_Z \eta)(X)\eta(Y) + (\nabla_Z \eta)(Y)\eta(X)\}.$$

Let us suppose that the Ricci tensor of (M_n, g) be cyclic parallel, i.e.,

$$(49) \quad (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.$$

From (47) and (49), we have

$$(50) \quad d\alpha(X)g(Y, Z) + d\beta(X)\eta(Z)\eta(Y) + d\alpha(Y)g(Z, X) \\ + d\beta(Y)\eta(Z)\eta(X) + d\alpha(Z)g(X, Y) + d\beta(Z)\eta(X)\eta(Y) \\ + \beta\{(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y) + (\nabla_Y \eta)(Z)\eta(X) \\ + (\nabla_Y \eta)(X)\eta(Z) + (\nabla_Z \eta)(X)\eta(Y) + (\nabla_Z \eta)(Y)\eta(X)\} = 0.$$

Replacing Y and Z with the vector field ξ in (50) and using (2), we find that

$$(51) \quad d\alpha(X) + d\beta(X) + 2\{d\alpha(\xi)\eta(X) + d\beta(\xi)\eta(X) + \beta(\nabla_\xi\eta)(X)\} = 0.$$

Let $\{e_i, i = 1, 2, \dots, n\}$ be an orthonormal basis of the tangent space at any point of (M_n, g) . Then putting $Y = Z = e_i$ in (50) and taking summation over i , $1 \leq i \leq n$, we get

$$(52) \quad (n+2)d\alpha(X) + d\beta(X) + 2d\beta(\xi)\eta(X) + 2\beta\{(\nabla_\xi\eta)(X) + \sum_{i=1}^n (\nabla_{e_i}\eta)(e_i)\eta(X)\} = 0.$$

In view of (46), (51) gives

$$(53) \quad 2\beta(\nabla_\xi\eta)(X) = \frac{n-1}{n}\{2d\beta(\xi)\eta(X) - d\beta(X)\}.$$

Again from (46) and (53), (52) becomes

$$(54) \quad 2\beta \sum_{i=1}^n (\nabla_{e_i}\eta)(e_i)\eta(X) = \left(\frac{1}{n} + 1\right)d\beta(X) - 2\left(2 - \frac{1}{n}\right)d\beta(\xi)\eta(X).$$

Setting $X = \xi$ in (54) and then using (2), we get

$$(55) \quad 2\beta \sum_{i=1}^n (\nabla_{e_i}\eta)(e_i) = -3\left(\frac{n-1}{n}\right)d\beta(\xi).$$

Replacing X by ξ in (53) and then using (2), we conclude that

$$(56) \quad d\beta(\xi) = 0.$$

From (55) and (56), we get

$$(57) \quad \sum_{i=1}^n (\nabla_{e_i}\eta)(e_i) = 0.$$

In consequence of (53) and (56), we have

$$2\beta(\nabla_\xi\eta)(X) = -\frac{n-1}{n}d\beta(X).$$

In view of (56) and (57), (54) turns into the form

$$(58) \quad d\beta(X) = 0 \Rightarrow \beta = \text{constant}$$

and therefore from (46) and (58), we have $\alpha = \text{constant}$. With the help of (58), (4) becomes

$$(59) \quad (\nabla_\xi\eta)(X) = 0.$$

From (58) and (59), (50) converts into the form

$$(60) \quad (\nabla_X\eta)(Y)\eta(Z) + (\nabla_X\eta)(Z)\eta(Y) + (\nabla_Y\eta)(Z)\eta(X) + (\nabla_Y\eta)(X)\eta(Z) + (\nabla_Z\eta)(X)\eta(Y) + (\nabla_Z\eta)(Y)\eta(X) = 0.$$

Setting $Z = \xi$ in (60) and then using (2) and (59), we have

$$(61) \quad (\nabla_X\eta)(Y) + (\nabla_Y\eta)(X) = 0,$$

which shows that the generator of the manifold is a Killing vector field. From (48), (58) and (61), we get (49). Thus we state the following theorem as:

Theorem 4.2: *If the scalar curvature of M_n vanishes identically, then the Ricci tensor of M_n is cyclic parallel if and only if the characteristic vector field of M_n is a Killing vector field.*

From Theorem 4.1 and Theorem 4.2, we state

Corollary 4.1: *Let (M_n, g) , $(n > 2)$, be a non-conformally flat $N(k)$ -quasi Einstein manifold and its Ricci tensor is cyclic parallel. If M_n satisfies $H(\xi, X) \cdot C = 0$, then the generator of M_n is a Killing vector field.*

Setting $Y = \xi$ in (61), we obtain

$$(\nabla_\xi \eta)(X) = 0 \Rightarrow \nabla_\xi \xi = 0,$$

which shows that the integral curves generated by the generator ξ are geodesics. Hence we have

Corollary 4.2: *If an n -dimensional non-conformally flat $N(k)$ -quasi Einstein manifold (M_n, g) , $(n > 2)$, equipped with a cyclic parallel Ricci tensor satisfies the condition $H(\xi, X) \cdot C = 0$, then the integral curves generated by the vector field ξ are geodesics.*

5. $N(k)$ -quasi Einstein manifolds satisfying $C(\xi, X) \cdot C = 0$

Let us consider that $\hat{C}(\xi, X) \cdot C = 0$, then from (14), we have

$$(62) \hat{C}(\xi, X)C(Y, Z)U - C(\hat{C}(\xi, X)Y, Z)U - C(Y, \hat{C}(\xi, X)Z)U - C(Y, Z)\hat{C}(\xi, X)U = 0.$$

From (2)-(6) and (10), it follows that

$$(63) \quad \hat{C}(\xi, Y)Z = \frac{\beta}{n} [g(Y, Z)\xi - \eta(Z)Y].$$

In view of (3)-(6), (10) and (63), (62) becomes

$$\begin{aligned} & \frac{\beta}{n} [g(C(Y, Z)U, X)\xi - \eta(C(Y, Z)U)X - g(X, Y)C(\xi, Z)U \\ & + \eta(Y)C(X, Z)U - g(X, Z)C(Y, \xi)U + \eta(Z)C(Y, X)U \\ & - g(X, U)C(Y, Z)\xi + \eta(U)C(Y, Z)X] = 0, \end{aligned}$$

which gives

$$\begin{aligned} (64) \quad & \frac{\beta}{n} ['C(Y, Z, U, X) - g(X, U)\eta(C(Y, Z)\xi) - g(X, Y)\eta(C(\xi, Z)U) \\ & + \eta(Y)\eta(C(X, Z)U) + \eta(Z)\eta(C(Y, X)U) + \eta(U)\eta(C(Y, Z)X) \\ & - \eta(C(Y, Z)U)\eta(X) - g(X, Z)\eta(C(Y, \xi)U)] = 0, \end{aligned}$$

where $'C(Y, Z, U, X) = g(C(Y, Z)U, X)$. Since $\beta \neq 0$, therefore (64) gives

$$(65) \quad \begin{aligned} {}'C(Y, Z, U, X) = & g(X, U)\eta(C(Y, Z)\xi) + g(X, Y)\eta(C(\xi, Z)U) \\ & + g(X, Z)\eta(C(Y, \xi)U) - \eta(Y)\eta(C(X, Z)U) - \eta(Z)\eta(C(Y, X)U) \\ & - \eta(U)\eta(C(Y, Z)X) + \eta(C(Y, Z)U)\eta(X). \end{aligned}$$

From (44) and (65), it follows that

$${}'C(Y, Z, U, X) = 0.$$

Hence we state the following theorem as:

Theorem 5.1: *An n -dimensional $N(k)$ -quasi Einstein manifold (M_n, g) , ($n \geq 3$), satisfies the condition $\hat{C}(\xi, X) \cdot C = 0$ if and only if it is conformally flat.*

From Theorem 4 and Theorem 5, we conclude the following corollary:

Corollary 5.1: *If the scalar curvature of an n -dimensional $N(k)$ -quasi Einstein manifold (M_n, g) , ($n > 2$), is non-zero, then the following conditions are equivalent:*

- (i) $H(\xi, X) \cdot C = 0$,
- (ii) $\hat{C}(\xi, X) \cdot C = 0$,
- (iii) M_n is conformally flat.

6. $N(k)$ -quasi Einstein manifolds satisfying $\hat{C}(\xi, X) \cdot \tilde{C} = 0$

Theorem 6.1: *An n -dimensional $N(k)$ -quasi Einstein manifold (M_n, g) satisfies $\hat{C}(\xi, X) \cdot \tilde{C} = 0$ if and only if it is quasi-conformally flat.*

Proof. We suppose that the $N(k)$ -quasi Einstein manifolds satisfying $\hat{C}(\xi, X) \cdot \tilde{C} = 0$, then from (14) we have

$$\hat{C}(\xi, X)\tilde{C}(Y, Z)U - \tilde{C}(\hat{C}(\xi, X)Y, Z)U - \tilde{C}(Y, \hat{C}(\xi, X)Z)U - \tilde{C}(Y, Z)\hat{C}(\xi, X)U = 0.$$

In consequence of (63), the above equation gives

$$\begin{aligned} \frac{\beta}{n} [& g(\tilde{C}(Y, Z)U, X)\xi - \eta(\tilde{C}(Y, Z)U)X - g(X, Y)\tilde{C}(\xi, Z)U \\ & + \eta(Y)\tilde{C}(X, Z)U - g(X, Z)\tilde{C}(Y, \xi)U + \eta(Z)\tilde{C}(Y, X)U \\ & - g(X, U)\tilde{C}(Y, Z)\xi + \eta(U)\tilde{C}(Y, Z)X] = 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \frac{\beta}{n} [& {}'\tilde{C}(Y, Z, U, X) - g(X, U)\eta(\tilde{C}(Y, Z)\xi) - g(X, Y)\eta(\tilde{C}(\xi, Z)U) \\ & - g(X, Z)\eta(\tilde{C}(Y, \xi)U) + \eta(Y)\eta(\tilde{C}(X, Z)U) + \eta(Z)\eta(\tilde{C}(Y, X)U) \\ & + \eta(U)\eta(\tilde{C}(Y, Z)X) - \eta(\tilde{C}(Y, Z)U)\eta(X)] = 0, \end{aligned}$$

where ${}'\tilde{C}(Y, Z, U, X) = g(\tilde{C}(Y, Z)U, X)$. Since $\beta \neq 0$, therefore the above equation assumes the form

$$(66) \quad {}'\tilde{C}(Y, Z, U, X) = g(X, U)\eta(\tilde{C}(Y, Z)\xi) + g(X, Y)\eta(\tilde{C}(\xi, Z)U) \\ + g(X, Z)\eta(\tilde{C}(Y, \xi)U) - \eta(Y)\eta(\tilde{C}(X, Z)U) - \eta(Z)\eta(\tilde{C}(Y, X)U) \\ - \eta(U)\eta(\tilde{C}(Y, Z)X) + \eta(\tilde{C}(Y, Z)U)\eta(X).$$

By virtue of (2)-(5), (8) and (13), we obtain

$$(67) \quad \eta(\tilde{C}(Y, Z)U) = \frac{\beta}{n} [\mu(n-2) + \lambda] \{ \eta(Y)g(U, Z) - \eta(Z)g(Y, U) \}.$$

From (66) and (67), it can be found that

$$(68) \quad {}'\tilde{C}(Y, Z, U, X) = \frac{\beta}{n} [\mu(n-2) + \lambda] \{ g(X, Y)g(Z, U) - g(X, Z)g(Y, U) \}.$$

In consequence of (13), (68) becomes

$$(69) \quad \lambda'R(Y, Z, U, X) + \mu[S(Z, U)g(X, Y) - S(Y, U)g(X, Z) \\ + g(Z, U)S(X, Y) - g(Y, U)S(X, Z)] \\ - \frac{\alpha n + \beta}{n} \left[\frac{\lambda}{n-1} + 2\mu \right] \{ g(X, Y)g(Z, U) - g(X, Z)g(Y, U) \} \\ = \frac{\beta}{n} [\mu(n-2) + \lambda] \{ g(X, Y)g(Z, U) - g(X, Z)g(Y, U) \}.$$

Let $\{e_i, i = 1, 2, \dots, n\}$ be an orthonormal basis of the tangent space. Then putting $X = Y = e_i$ in (69) and taking summation over i , $1 \leq i \leq n$, we get

$$[\lambda + \mu(n-2)][S(Z, U) - (\alpha + \beta)g(Z, U)] = 0.$$

Since (M_n, g) is not Einstein, i. e., $S \neq (\alpha + \beta)g$, therefore $\lambda = \mu(2 - n)$. Hence (67) gives

$$(70) \quad \eta(\tilde{C}(Y, Z)U) = 0,$$

for arbitrary vector fields Y, Z and U . From (66) and (70), it can be seen that

$${}'\tilde{C}(Y, Z, U, X) = 0.$$

The converse part is obvious. □

K. Amur and Y. B. Maralabhavi⁴¹ proved that a quasi conformally flat manifold is either conformally flat or Einstein. Therefore in view of Theorem 6, we state the following corollary as:

Corollary 6.1: If (M_n, g) is an n -dimensional quasi-conformally flat $N(k)$ -quasi Einstein manifold, then it satisfies $\hat{C}(\xi, X) \cdot C = 0$.

From Theorem 5, Theorem 6 and Corollary 6, we state

Corollary 6.2: If an n -dimensional $N(k)$ -quasi Einstein manifold (M_n, g) satisfies $\hat{C}(\xi, X) \cdot \tilde{C} = 0$, then $\hat{C}(\xi, X) \cdot C = 0$ holds on M_n .

Corollary 6.3: On an n -dimensional $N(k)$ -quasi Einstein manifold (M_n, g) , the following conditions are equivalent:

- (i) $\hat{C}(\xi, X) \cdot C = 0$ with $\lambda = \mu(2 - n)$,
- (ii) $\hat{C}(\xi, X) \cdot \tilde{C} = 0$,
- (iii) (M_n, g) is conformally flat with $\lambda = \mu(2 - n)$.

7 $N(k)$ -quasi Einstein manifolds satisfying $\hat{C}(\xi, X) \cdot P = 0$

Let us assume that $\hat{C}(\xi, X) \cdot P = 0$, then (9) gives

$$\hat{C}(\xi, X)P(Y, Z)U - P(\hat{C}(\xi, X)Y, Z)U - P(Y, \hat{C}(\xi, X)Z)U - P(Y, Z)\hat{C}(\xi, X)U = 0.$$

In consequence of (4)-(6) and (63), the above equation becomes

$$\begin{aligned} \frac{\beta}{n} [g(P(Y, Z)U, X)\xi - \eta(P(Y, Z)U)X - g(X, Y)P(\xi, Z)U \\ + \eta(Y)P(X, Z)U - g(X, Z)P(Y, \xi)U + \eta(Z)P(Y, X)U \\ - g(X, U)P(Y, Z)\xi + \eta(U)P(Y, Z)X] = 0, \end{aligned}$$

which gives

$$\begin{aligned} \frac{\beta}{n} [{}'P(Y, Z, U, X) - g(X, U)\eta(P(Y, Z)\xi) \\ - g(X, Y)\eta(P(\xi, Z)U) - g(X, Z)\eta(P(Y, \xi)U) + \eta(Y)\eta(P(X, Z)U) \\ + \eta(Z)\eta(P(Y, X)U) + \eta(U)\eta(P(Y, Z)X) - \eta(P(Y, Z)U)\eta(X)] = 0, \end{aligned}$$

where $'P(Y, Z, U, X) = g(P(Y, Z)U, X)$. Since $\beta \neq 0$, therefore the above equation gives

$$\begin{aligned} {}'P(Y, Z, U, X) = g(X, U)\eta(P(Y, Z)\xi) + g(X, Y)\eta(P(\xi, Z)U) \\ + g(X, Z)\eta(P(Y, \xi)U) - \eta(Y)\eta(P(X, Z)U) \\ - \eta(Z)\eta(P(Y, X)U) - \eta(U)\eta(P(Y, Z)X) + \eta(P(Y, Z)U)\eta(X). \end{aligned}$$

In view of (3), (8) and (9), we find

$$\eta(P(Y, Z)U) = \frac{\beta}{n-1} \{\eta(Y)g(U, Z) - \eta(Z)g(U, Y)\}.$$

From last two expressions, we have

$$'P(Y, Z, U, X) = \frac{\beta}{n-1} \{g(X, Y)g(Z, U) - g(X, Z)g(Y, U)\},$$

which is equivalent to

$$\begin{aligned} (71) \quad R(X, Y)Z - \frac{1}{n-1} [S(Y, Z)X - S(X, Z)Y] \\ = \frac{\beta}{n-1} \{g(X, Y)g(Z, U) - g(X, Z)g(Y, U)\}. \end{aligned}$$

Let $\{e_i, i = 1, 2, \dots, n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i$ in (71) and then taking summation over i , $1 \leq i \leq n$, we get

$$\beta g(Z, U) = 0,$$

which is a contradiction. Hence we can state the following theorem as:

Theorem 7.1: *There is no $N(k)$ -quasi Einstein manifold (M_n, g) satisfying the condition $\hat{C}(\xi, X) \cdot P = 0$.*

8. $N(k)$ -quasi Einstein manifolds satisfying $\hat{C}(\xi, X) \cdot H = 0$

From equations (2)-(5), (8) and (11), it follows that

$$(72) \quad \eta(H(Y, Z)U) = -\frac{\alpha n + \beta}{(n-1)(n-2)} \{g(Z, U)\eta(Y) - g(Y, U)\eta(Z)\}.$$

Let us suppose that an $N(k)$ -quasi Einstein manifold satisfy $\hat{C}(\xi, X) \cdot H = 0$, then (14) gives

$$\hat{C}(\xi, X)H(Y, Z)U - H(\hat{C}(\xi, X)Y, Z)U - H(Y, \hat{C}(\xi, X)Z)U - H(Y, Z)\hat{C}(\xi, X)U = 0.$$

With the help of (4)-(6) and (63), the above equation becomes

$$\begin{aligned} \frac{\beta}{n} [g(H(Y, Z)U, X)\xi - \eta(H(Y, Z)U)X - g(X, Y)H(\xi, Z)U \\ + \eta(Y)H(X, Z)U - g(X, Z)H(Y, \xi)U + \eta(Z)H(Y, X)U \\ - g(X, U)H(Y, Z)\xi + \eta(U)H(Y, Z)X] = 0, \end{aligned}$$

which gives

$$\begin{aligned} \frac{\beta}{n} ['H(Y, Z, U, X) - g(X, U)\eta(H(Y, Z)\xi) - g(X, Y)\eta(H(\xi, Z)U) \\ - g(X, Z)\eta(H(Y, \xi)U) + \eta(Y)\eta(H(X, Z)U) + \eta(Z)\eta(H(Y, X)U) \\ + \eta(U)\eta(H(Y, Z)X) - \eta(H(Y, Z)U)\eta(X)] = 0, \end{aligned}$$

where $'H(Y, Z, U, X) = g(H(Y, Z)U, X)$. Since $\beta \neq 0$, therefore the above equation gives

$$\begin{aligned} 'H(Y, Z, U, X) - g(X, U)\eta(H(Y, Z)\xi) - g(X, Y)\eta(H(\xi, Z)U) \\ - g(X, Z)\eta(H(Y, \xi)U) + \eta(Y)\eta(H(X, Z)U) + \eta(Z)\eta(H(Y, X)U) \\ + \eta(U)\eta(H(Y, Z)X) - \eta(H(Y, Z)U)\eta(X) = 0, \end{aligned}$$

In view of (3) and (72), the above equation converts into the form

$$'H(Y, Z, U, X) = -\frac{\alpha n + \beta}{(n-1)(n-2)} \{g(X, Y)g(Z, U) - g(X, Z)g(Y, U)\}.$$

With the help of (11), above equation converted into

$$\begin{aligned} 'R(Y, Z, U, X) = \left(\frac{\alpha(n-2) - \beta}{(n-1)(n-2)} \right) \{g(X, Y)g(Z, U) - g(X, Z)g(Y, U)\} \\ + \frac{\beta}{n-1} [\eta(Z)\eta(U)g(X, Y) - \eta(Y)\eta(U)g(X, Z) \\ + \eta(X)\eta(Y)g(Z, U) - \eta(X)\eta(Z)g(Y, U)], \end{aligned}$$

which gives

$$R(Y, Z)U = \left(\frac{\alpha(n-2)-\beta}{(n-1)(n-2)} \right) \{g(Z, U)Y - g(Y, U)Z\} \\ + \frac{\beta}{n-1} [\eta(Z)\eta(U)Y - \eta(Y)\eta(U)Z + \eta(Y)g(Z, U)\xi - \eta(Z)g(Y, U)\xi].$$

Setting $U = \xi$ in the last equation and then using (3), (5) and (7), we obtain $\beta = 0$, which is not possible. Thus we state:

Theorem 8.1: *There is no $N(k)$ -quasi Einstein manifold satisfying $\mathcal{C}(\xi, X) \cdot H = 0$.*

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