

Some Fixed Point Theorems for α -Admissible Mappings in Bipolar Metric Spaces

Manoj Kumar, Vinit Mor and Pankaj*

Department of Mathematics,

Baba Mastnath University, Asthal Bohar, Rohtak, Haryana, India

Email: manojantil18@gmail.com, vinitrpm@gmail.com, maypankajkumar@gmail.com*

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Abstract: In the present paper, some new contractions are introduced and fixed point theorems for such contractions are proved in bipolar metric spaces. Some examples are also presented to illustrate the main results.

Keywords: Bipolar metric spaces, fixed points, α -admissible mappings.

1. Introduction

Now a days, fixed point theory is the major area of research in non-linear analysis. In 1922, Banach¹ gave a contraction principle to obtain the fixed point of a self-map. Due to lots of applications of fixed point theory in other subjects, the researchers have generalized metric space as well as Banach contraction principle. In 2013, Farhana *et al.*² provided some fixed point theorems for α -admissible mappings. In 2016, Mutlu *et al.*³ introduced a new metric space, called bipolar metric space. Since then, many authors proved various fixed point theorems in bipolar metric space⁴⁻⁶. In this paper, we shall also introduce some α -admissible mappings.

Definition 1.1.³ Let $\mathcal{J}_h$ and $\mathcal{H}_j$ are two non-void sets and $\mathcal{H} : \mathcal{J}_h \times \mathcal{H}_j \rightarrow [0, \infty)$ be a function such that:

(BP1) $\mathcal{H}(\mathfrak{z}, \mathfrak{c}) = 0 \Leftrightarrow \mathfrak{z} = \mathfrak{c}$, where $(\mathfrak{z}, \mathfrak{c}) \in \mathcal{J}_h \times \mathcal{H}_j$;

(BP2) $\mathcal{H}(\mathfrak{z}, \mathfrak{c}) = \mathcal{H}(\mathfrak{c}, \mathfrak{z})$ for all $\mathfrak{z}, \mathfrak{c} \in \mathcal{J}_h \cap \mathcal{H}_j$;

(BP3) $\mathcal{H}(\mathfrak{z}_1, \mathfrak{c}_2) \leq \mathcal{H}(\mathfrak{z}_1, \mathfrak{c}_1) + \mathcal{H}(\mathfrak{z}_2, \mathfrak{c}_1) + \mathcal{H}(\mathfrak{z}_2, \mathfrak{c}_2)$ for all $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{J}_h$ and $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathcal{H}_j$.

Then $\mathcal{H}$ is bipolar metric and $(\mathcal{J}_h, \mathcal{H}_j, \mathcal{H})$ is bipolar metric space.

If $\mathcal{J}_h \cap \mathcal{H}_j = \emptyset$, then space is called disjoint otherwise joint. The set $\mathcal{J}_h$ is called left pole and $\mathcal{H}_j$ is called right pole of bipolar metric space $(\mathcal{J}_h, \mathcal{H}_j, \mathcal{H})$

and any element of left pole (J_H), right pole (H_J) and $J_H \cap H_J$ is called left element, right element and central element respectively.

Example 1.2.⁶ Let $J_H = \{U_m(\mathbb{R}) : U_m(\mathbb{R}) \text{ is an upper triangular matrix of order } m \text{ over } \mathbb{R}\}$ and $H_J = \{V_m(\mathbb{R}) : V_m(\mathbb{R}) \text{ is a lower triangular matrix of order } m \text{ over } \mathbb{R}\}$.

Define $\mathcal{H} : J_H \times H_J \rightarrow [0, \infty)$ by $\mathcal{H}(P, Q) = \sum_{i,j=1}^m |p_{ij} - q_{ij}|$ for all $P = (p_{ij})_{m \times m} \in U_m(\mathbb{R})$, and $Q = (q_{ij})_{m \times m} \in V_m(\mathbb{R})$. Then, clearly, $(J_H, H_J, \mathcal{H})$ is a bipolar metric space.

Definition 1.3.⁴ Let $(J_H, H_J, \mathcal{H})$ be a bipolar metric space. Then any sequence $(\mathfrak{z}_n) \subseteq J_H$ is called left sequence and is said to be convergent to right element say ' ς ' if $\mathcal{H}(\mathfrak{z}_n, \varsigma) \rightarrow 0$ as $n \rightarrow \infty$. Similarly, a right sequence $(\varsigma_n) \subseteq H_J$ is said to be convergent to a left element say ' $\mathfrak{z}$ ' if $\mathcal{H}(\mathfrak{z}, \varsigma_n) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 1.4.³ Let $(J_{H_1}, H_{J_1}, \mathcal{H}_1)$ and $(J_{H_2}, H_{J_2}, \mathcal{H}_2)$ are two bipolar metric spaces. Let $\mathfrak{T} : J_{H_1} \cup H_{J_1} \rightarrow J_{H_2} \cup H_{J_2}$ be a function:

- (i) If $\mathfrak{T}(J_{H_1}) \subseteq J_{H_2}$ and $\mathfrak{T}(H_{J_1}) \subseteq H_{J_2}$, then $\mathfrak{T}$ is called covariant map and is denoted by

$$\mathfrak{T} : (J_{H_1}, H_{J_1}, \mathcal{H}_1) \rightrightarrows (J_{H_2}, H_{J_2}, \mathcal{H}_2).$$

- (ii) If $\mathfrak{T}(J_{H_1}) \subseteq H_{J_2}$ and $\mathfrak{T}(H_{J_1}) \subseteq J_{H_2}$, then $\mathfrak{T}$ is called contravariant map and is denoted by

$$\mathfrak{T} : (J_{H_1}, H_{J_1}, \mathcal{H}_1) \bowtie (J_{H_2}, H_{J_2}, \mathcal{H}_2).$$

Definition 1.5.³ Let $(J_{H_1}, H_{J_1}, \mathcal{H}_1)$ and $(J_{H_2}, H_{J_2}, \mathcal{H}_2)$ are two bipolar metric spaces.

- (i) A map $\mathfrak{T} : (J_{H_1}, H_{J_1}, \mathcal{H}_1) \rightrightarrows (J_{H_2}, H_{J_2}, \mathcal{H}_2)$ is called left continuous at a point $\mathfrak{z}_0 \in J_{H_1}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mathcal{H}_2(\mathfrak{T}\mathfrak{z}_0, \mathfrak{T}\varsigma) < \epsilon$ whenever $\mathcal{H}_1(\mathfrak{z}_0, \varsigma) < \delta$.
- (ii) A map $\mathfrak{T} : (J_{H_1}, H_{J_1}, \mathcal{H}_1) \rightrightarrows (J_{H_2}, H_{J_2}, \mathcal{H}_2)$ is called right continuous at a point $\varsigma_0 \in H_{J_1}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mathcal{H}_2(\mathfrak{T}\mathfrak{z}, \mathfrak{T}\varsigma_0) < \epsilon$ whenever $\mathcal{H}_1(\mathfrak{z}, \varsigma_0) < \delta$.
- (iii) A map $\mathfrak{T}$ is called continuous, if it is left continuous at each $\mathfrak{z}_0 \in J_{H_1}$ and right continuous at each $\varsigma_0 \in H_{J_1}$.
- (iv) A contravariant map $\mathfrak{T} : (J_{H_1}, H_{J_1}, \mathcal{H}_1) \bowtie (J_{H_2}, H_{J_2}, \mathcal{H}_2)$ is continuous if and only if it is continuous as a covariant map $\mathfrak{T} : (J_{H_1}, H_{J_1}, \mathcal{H}_1) \rightrightarrows (J_{H_2}, H_{J_2}, \mathcal{H}_2)$.

Definition 1.6.³ Let $(J_H, H_J, \mathcal{H})$ be a bipolar metric space.

- (i) A sequence $(\mathfrak{z}_n, \varsigma_n)$ on the set $J_H \times H_J$ is called a bisequence on $(J_H, H_J, \mathcal{H})$.

(ii) If both the sequences $(\mathfrak{z}_n)$ and $(\mathfrak{c}_n)$ converge, then bisequence $(\mathfrak{z}_n, \mathfrak{c}_n)$ is said to be convergent. If both the sequences $(\mathfrak{z}_n)$ and $(\mathfrak{c}_n)$ converge to same point $v \in \mathcal{J}_H \cap H_H$, then this bisequence is said to be biconvergent.

(iii) A bisequence $(\mathfrak{z}_n, \mathfrak{c}_n)$ on $(\mathcal{J}_H, H_H, \mathcal{H})$ is said to be Cauchy bisequence, if for each $\epsilon > 0$

there exists a positive integer $N \in \mathbb{N}$ such that $\mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_m) < \epsilon$ for all $n, m \geq N$.

(iv) A bipolar metric space is said to be complete if every Cauchy bisequence is convergent in this space.

Definition 1.7.⁵ Let $\mathfrak{T} : (\mathcal{J}_H, H_H) \bowtie (\mathcal{J}_H, H_H)$ and $\alpha : \mathcal{J}_H \times H_H \rightarrow [0, \infty)$. Then $\mathfrak{T}$ is said to be α -admissible (contravariant) if

$\alpha(\mathfrak{z}, \mathfrak{c}) \geq 1$ implies $\alpha(\mathfrak{T}\mathfrak{c}, \mathfrak{T}\mathfrak{z}) \geq 1$, for all $(\mathfrak{z}, \mathfrak{c}) \in \mathcal{J}_H \times H_H$.

2. Main Results

Here, we will discuss fixed point results for α -admissible mappings in complete bipolar metric spaces.

Theorem 2.1. Let $(\mathcal{J}_H, H_H, \mathcal{H})$ be a complete bipolar metric space and $\mathfrak{T} : (\mathcal{J}_H, H_H) \bowtie (\mathcal{J}_H, H_H)$ be a contravariant α -admissible mapping. Suppose $\mathcal{Q} : [0, \infty) \rightarrow [0, 1]$ a function and $\{t_n\}$ in $[0, \infty)$ is a sequence which is bounded and $\mathcal{Q}(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ as $n \rightarrow \infty$.

Then if

$$(2.1) \quad (\mathcal{H}(\mathfrak{T}\mathfrak{c}, \mathfrak{T}\mathfrak{z})\alpha(\mathfrak{z}, \mathfrak{T}\mathfrak{z})\alpha(\mathfrak{T}\mathfrak{c}, \mathfrak{c}) + 1)^{\mathcal{H}(\mathfrak{T}\mathfrak{c}, \mathfrak{T}\mathfrak{z})} \leq 2^{\mathcal{Q}(\mathcal{H}(\mathfrak{z}, \mathfrak{c}))\mathcal{H}(\mathfrak{z}, \mathfrak{c})},$$

for all $(\mathfrak{z}, \mathfrak{c}) \in \mathcal{J}_H \times H_H$ and $l \geq 1$.

Let $\mathfrak{T}$ is continuous and $\exists \mathfrak{z}_0 \in \mathcal{J}_H$ such that $\alpha(\mathfrak{z}_0, \mathfrak{T}\mathfrak{z}_0) \geq 1$.

Then, $\mathfrak{T}w = w$.

Proof: Let $\mathfrak{z}_0 \in \mathcal{J}_H$ with $\alpha(\mathfrak{z}_0, \mathfrak{T}\mathfrak{z}_0) \geq 1$. Construct two sequences $(\mathfrak{z}_n)$ and $(\mathfrak{c}_n)$ such that $\mathfrak{T}\mathfrak{z}_n = \mathfrak{c}_n$ and $\mathfrak{T}\mathfrak{c}_n = \mathfrak{z}_{n+1}$.

Then, $(\mathfrak{z}_n, \mathfrak{c}_n)$ is a bisequence on $(\mathcal{J}_H, H_H, \mathcal{H})$.

$$\alpha(\mathfrak{z}_0, \mathfrak{c}_0) = \alpha(\mathfrak{z}_0, \mathfrak{T}\mathfrak{z}_0) \geq 1 \Rightarrow \alpha(\mathfrak{T}\mathfrak{c}_0, \mathfrak{T}\mathfrak{z}_0) = \alpha(\mathfrak{z}_1, \mathfrak{c}_0) \geq 1,$$

$$\alpha(\mathfrak{z}_1, \mathfrak{c}_0) \geq 1 \Rightarrow \alpha(\mathfrak{T}\mathfrak{c}_0, \mathfrak{T}\mathfrak{z}_0) = \alpha(\mathfrak{z}_1, \mathfrak{c}_1) \geq 1,$$

$$\alpha(\mathfrak{z}_1, \mathfrak{c}_1) \geq 1 \Rightarrow \alpha(\mathfrak{T}\mathfrak{c}_1, \mathfrak{T}\mathfrak{z}_1) = \alpha(\mathfrak{z}_2, \mathfrak{c}_1) \geq 1,$$

$$\alpha(\mathfrak{z}_2, \mathfrak{c}_1) \geq 1 \Rightarrow \alpha(\mathfrak{T}\mathfrak{c}_1, \mathfrak{T}\mathfrak{z}_2) = \alpha(\mathfrak{z}_2, \mathfrak{c}_2) \geq 1.$$

By repeating this process, we have

$$(2.2) \quad \alpha(\mathfrak{z}_{n+1}, \mathfrak{c}_n) \geq 1 \quad \text{and} \quad \alpha(\mathfrak{z}_n, \mathfrak{c}_n) \geq 1.$$

for all $n \in \mathbb{N}$.

Now, by the equation (2.1), we have

$$(2.3) \quad 2^{\mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_n)} \leq (\alpha(\mathfrak{z}_n, \mathfrak{H}\mathfrak{z}_n)\alpha(\mathfrak{H}\mathfrak{c}_{n-1}, \mathfrak{c}_{n-1}) + 1)^{\mathcal{H}(\mathfrak{H}\mathfrak{c}_{n-1}, \mathfrak{H}\mathfrak{z}_n)} \\ \leq 2^{\mathbb{Q}(\mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_{n-1}))\mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_{n-1})}.$$

This implies that

$$(2.4) \quad \mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_n) \leq \mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_{n-1}).$$

Again, equation (2.1) implies that

$$(2.5) \quad 2^{\mathcal{H}(\mathfrak{z}_{n+1}, \mathfrak{c}_n)} \leq (\alpha(\mathfrak{z}_n, \mathfrak{H}\mathfrak{z}_n)\alpha(\mathfrak{H}\mathfrak{c}_n, \mathfrak{c}_n) + 1)^{\mathcal{H}(\mathfrak{H}\mathfrak{c}_n, \mathfrak{H}\mathfrak{z}_n)} \\ \leq 2^{\mathbb{Q}(\mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_n))\mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_n)}.$$

This implies that

$$(2.6) \quad \mathcal{H}(\mathfrak{z}_{n+1}, \mathfrak{c}_n) \leq \mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_n).$$

From equations (2.4) and (2.6), we have

$$\mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_n) \leq \mathcal{H}(\mathfrak{z}_{n-1}, \mathfrak{c}_{n-1}) \text{ for all } n \in \mathbb{N}.$$

It follows that the sequence $\{\mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_n)\}$ is a non-increasing sequence of positive real numbers. So, it is convergent and supposes that

$$\lim_{n \rightarrow \infty} \mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_n) = d.$$

Clearly, $d \geq 0$.

Claim: $d = 0$.

Equations (2.3) and (2.6) implies that

$$\frac{\mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_n)}{\mathcal{H}(\mathfrak{z}_{n-1}, \mathfrak{c}_{n-1})} \leq \mathbb{Q}(\mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_{n-1})) \leq 1,$$

This implies that $\lim_{n \rightarrow \infty} \mathbb{Q}(\mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_{n-1})) = 1$.

By applying the property of the function $\mathbb{Q}$, we conclude that

$$(2.7) \quad \lim_{n \rightarrow \infty} \mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_{n-1}) = 0.$$

Similarly, one can prove easily that

$$(2.8) \quad \lim_{n \rightarrow \infty} \mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_n) = 0.$$

Now, we will show that $\{(\mathfrak{z}_n, \mathfrak{c}_n)\}$ is a Cauchy bisequence. If possible suppose that $\{(\mathfrak{z}_n, \mathfrak{c}_n)\}$ is not a Cauchy bisequence. Then there exists $\epsilon > 0$ and sequences $u(\mathfrak{z})$ and $v(\mathfrak{z})$ such that for all positive integers $\mathfrak{z}$, we have

$$v(\mathfrak{z}) > u(\mathfrak{z}) > \mathfrak{z}, \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})}, \mathfrak{c}_{u(\mathfrak{z})}) \geq \epsilon \text{ and } \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})}, \mathfrak{c}_{u(\mathfrak{z})-1}) < \epsilon.$$

By the definition of bipolar metric space and using the above, we have

$$\epsilon \leq \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})}, \mathfrak{c}_{u(\mathfrak{z})}) \leq \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})}, \mathfrak{c}_{u(\mathfrak{z})-1}) + \mathcal{H}(\mathfrak{z}_{u(\mathfrak{z})-1}, \mathfrak{c}_{u(\mathfrak{z})-1}) + \mathcal{H}(\mathfrak{z}_{u(\mathfrak{z})-1}, \mathfrak{c}_{u(\mathfrak{z})})$$

Approaching the limit as $\mathfrak{z} \rightarrow +\infty$ in the above inequality and using equations (2.7) and (2.8), we get

$$(2.9) \quad \lim_{\bar{g} \rightarrow +\infty} \mathcal{H}(\mathfrak{z}_{v(\bar{g})}, \mathfrak{c}_{u(\bar{g})}) = \epsilon.$$

Again by triangle inequality, we have

$$\begin{aligned} \mathcal{H}(\mathfrak{z}_{v(\bar{g})}, \mathfrak{c}_{u(\bar{g})}) &\leq \mathcal{H}(\mathfrak{z}_{v(\bar{g})}, \mathfrak{c}_{v(\bar{g})+1}) + \mathcal{H}(\mathfrak{z}_{v(\bar{g})+1}, \mathfrak{c}_{v(\bar{g})+1}) + \mathcal{H}(\mathfrak{z}_{v(\bar{g})+1}, \mathfrak{c}_{u(\bar{g})}), \\ \mathcal{H}(\mathfrak{z}_{v(\bar{g})+1}, \mathfrak{c}_{u(\bar{g})}) &\leq \mathcal{H}(\mathfrak{z}_{v(\bar{g})+1}, \mathfrak{c}_{v(\bar{g})+1}) + \mathcal{H}(\mathfrak{z}_{v(\bar{g})}, \mathfrak{c}_{v(\bar{g})+1}) + \mathcal{H}(\mathfrak{z}_{v(\bar{g})}, \mathfrak{c}_{u(\bar{g})}) \end{aligned}$$

Taking the limit as $\bar{g} \rightarrow +\infty$, together using equations (2.7) and (2.8), we found that

$$(2.10) \quad \lim_{\bar{g} \rightarrow +\infty} \mathcal{H}(\mathfrak{z}_{v(\bar{g})+1}, \mathfrak{c}_{u(\bar{g})}) = \epsilon.$$

Again by triangle inequality, we have

$$\begin{aligned} \mathcal{H}(\mathfrak{z}_{v(\bar{g})}, \mathfrak{c}_{u(\bar{g})}) &\leq \mathcal{H}(\mathfrak{z}_{v(\bar{g})}, \mathfrak{c}_{v(\bar{g})}) + \mathcal{H}(\mathfrak{z}_{u(\bar{g})+1}, \mathfrak{c}_{v(\bar{g})}) + \mathcal{H}(\mathfrak{z}_{u(\bar{g})+1}, \mathfrak{c}_{u(\bar{g})}), \\ \mathcal{H}(\mathfrak{z}_{u(\bar{g})+1}, \mathfrak{c}_{u(\bar{g})}) &\leq \mathcal{H}(\mathfrak{z}_{u(\bar{g})+1}, \mathfrak{c}_{u(\bar{g})+1}) + \mathcal{H}(\mathfrak{z}_{v(\bar{g})+1}, \mathfrak{c}_{u(\bar{g})+1}) + \mathcal{H}(\mathfrak{z}_{v(\bar{g})+1}, \mathfrak{c}_{v(\bar{g})}) \end{aligned}$$

Taking the limit as $\bar{g} \rightarrow +\infty$, together using equations (2.7)-(2.10), we deduce that

$$(2.11) \quad \lim_{\bar{g} \rightarrow +\infty} \mathcal{H}(\mathfrak{z}_{u(\bar{g})+1}, \mathfrak{c}_{v(\bar{g})}) = \epsilon.$$

From equations (2.1) and (2.11), we have

$$\begin{aligned} 2^{\mathcal{H}(\mathfrak{z}_{v(\bar{g})+1}, \mathfrak{c}_{u(\bar{g})})} &\leq (\alpha(\mathfrak{z}_{u(\bar{g})+1}, \mathfrak{T}\mathfrak{z}_{u(\bar{g})+1})\alpha(\mathfrak{T}\mathfrak{c}_{n(\bar{g})}, \mathfrak{c}_{v(\bar{g})}) + 1)^{\mathcal{H}(\mathfrak{z}_{v(\bar{g})+1}, \mathfrak{c}_{u(\bar{g})})}, \\ &= (\alpha(\mathfrak{z}_{u(\bar{g})+1}, \mathfrak{T}\mathfrak{z}_{u(\bar{g})+1})\alpha(\mathfrak{T}\mathfrak{c}_{v(\bar{g})}, \mathfrak{c}_{v(\bar{g})}) + 1)^{\mathcal{H}(\mathfrak{z}_{v(\bar{g})+1}, \mathfrak{c}_{u(\bar{g})})} \\ &\leq \mathcal{Q}(\mathcal{H}(\mathfrak{z}_{v(\bar{g})}, \mathfrak{c}_{u(\bar{g})})\mathcal{H}(\mathfrak{z}_{u(\bar{g})+1}, \mathfrak{c}_{v(\bar{g})})) \end{aligned}$$

$$\mathcal{H}(\mathfrak{z}_{u(\bar{g})+1}, \mathfrak{c}_{v(\bar{g})}) \leq \mathcal{Q}(\mathcal{H}(\mathfrak{z}_{v(\bar{g})}, \mathfrak{c}_{u(\bar{g})})\mathcal{H}(\mathfrak{z}_{u(\bar{g})+1}, \mathfrak{c}_{v(\bar{g})}))$$

Letting $\bar{g} \rightarrow \infty$ in above inequality, we obtain that

$$\lim_{n \rightarrow \infty} \mathcal{Q}(\mathcal{H}(\mathfrak{z}_{v(\bar{g})}, \mathfrak{c}_{u(\bar{g})})) = 1, \text{ that is}$$

$$\lim_{n \rightarrow \infty} \mathcal{H}(\mathfrak{z}_{v(\bar{g})}, \mathfrak{c}_{u(\bar{g})}) = 0 < \epsilon.$$

a contradiction.

So, $(\mathfrak{z}_n, \mathfrak{c}_n)$ is a Cauchy bisequence.

Since $(\mathcal{J}_\mathfrak{h}, \mathcal{H}_\mathfrak{h}, \mathcal{H})$ is a complete bipolar metric space, so $(\mathfrak{z}_n, \mathfrak{c}_n)$ is convergent and thus biconverges to a point $w \in \mathcal{J}_\mathfrak{h} \cap \mathcal{H}_\mathfrak{h}$ and $\mathfrak{z}_n \rightarrow w, \mathfrak{c}_n \rightarrow w \in \mathcal{J}_\mathfrak{h} \cap \mathcal{H}_\mathfrak{h}$. Since $\mathfrak{T}$ is a continuous map, $\mathfrak{z}_n \rightarrow w$ implies that $\mathfrak{c}_n = \mathfrak{T}\mathfrak{z}_n \rightarrow \mathfrak{T}w$ and $\mathfrak{c}_n \rightarrow w$ implies that $\mathfrak{z}_{n+1} = \mathfrak{T}\mathfrak{c}_n \rightarrow \mathfrak{T}w$. By the uniqueness of the limit, we obtain that $\mathfrak{T}w = w$. Hence w is the fixed point.

Theorem 2.2. Let $(\mathcal{J}_\mathfrak{h}, \mathcal{H}_\mathfrak{h}, \mathcal{H})$ be a complete bipolar metric space and $\mathfrak{T} : (\mathcal{J}_\mathfrak{h}, \mathcal{H}_\mathfrak{h}) \times (\mathcal{J}_\mathfrak{h}, \mathcal{H}_\mathfrak{h}) \rightarrow (\mathcal{J}_\mathfrak{h}, \mathcal{H}_\mathfrak{h})$ be a contravariant α -admissible mapping. Let $\mathcal{Q} : [0, \infty) \rightarrow [0, 1]$ be a function and $\{t_n\}$ is a sequence of positive reals, which is bounded and $\mathcal{Q}(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ as $n \rightarrow \infty$. If

$$(2.12) \quad (\mathcal{H}(\mathfrak{H}\zeta, \mathfrak{H}\mathfrak{z}) + l)^{\alpha(\mathfrak{z}, \mathfrak{H}\mathfrak{z})\alpha(\mathfrak{H}\zeta, \zeta)} \leq \mathbb{Q}(\mathcal{H}(\mathfrak{z}, \zeta))\mathcal{H}(\mathfrak{z}, \zeta) + l,$$

for all $(\mathfrak{z}, \zeta) \in \mathcal{J}_\mathfrak{H} \times \mathcal{H}_\mathfrak{J}$ and $l \geq 1$.

If $\mathfrak{H}$ is continuous and $\exists \mathfrak{z}_0 \in \mathcal{J}_\mathfrak{H}$ such that $\alpha(\mathfrak{z}_0, \mathfrak{H}\mathfrak{z}_0) \geq 1$, Then, $\mathfrak{H}w = w$.

Proof: Let $\mathfrak{z}_0 \in \mathcal{J}_\mathfrak{H}$ with $\alpha(\mathfrak{z}_0, \mathfrak{H}\mathfrak{z}_0) \geq 1$. Construct two sequences $(\mathfrak{z}_n)$ and (ζ_n) such that $\mathfrak{H}\mathfrak{z}_n = \zeta_n$ and $\mathfrak{H}\zeta_n = \mathfrak{z}_{n+1}$. Then, $(\mathfrak{z}_n, \zeta_n)$ is a bisequence on $(\mathcal{J}_\mathfrak{H}, \mathcal{H}_\mathfrak{J}, \mathcal{H})$.

$$\alpha(\mathfrak{z}_0, \zeta_0) = \alpha(\mathfrak{z}_0, \mathfrak{H}\mathfrak{z}_0) \geq 1 \Rightarrow \alpha(\mathfrak{H}\zeta_0, \mathfrak{H}\mathfrak{z}_0) = \alpha(\mathfrak{z}_1, \zeta_0) \geq 1,$$

$$\alpha(\mathfrak{z}_1, \zeta_0) \geq 1 \Rightarrow \alpha(\mathfrak{H}\zeta_0, \mathfrak{H}\mathfrak{z}_0) = \alpha(\mathfrak{z}_1, \zeta_1) \geq 1,$$

$$\alpha(\mathfrak{z}_1, \zeta_1) \geq 1 \Rightarrow \alpha(\mathfrak{H}\zeta_1, \mathfrak{H}\mathfrak{z}_1) = \alpha(\mathfrak{z}_2, \zeta_1) \geq 1,$$

$$\alpha(\mathfrak{z}_2, \zeta_1) \geq 1 \Rightarrow \alpha(\mathfrak{H}\zeta_1, \mathfrak{H}\mathfrak{z}_2) = \alpha(\mathfrak{z}_2, \zeta_2) \geq 1.$$

By repeating this process, we have

$$(2.13) \quad \alpha(\mathfrak{z}_{n+1}, \zeta_n) \geq 1 \quad \text{and} \quad \alpha(\mathfrak{z}_n, \zeta_n) \geq 1.$$

Now, by the equation (2.12), we have

$$(2.14) \quad \begin{aligned} \mathcal{H}(\mathfrak{z}_n, \zeta_n) + l &= \mathcal{H}(\mathfrak{H}\zeta_{n-1}, \mathfrak{H}\mathfrak{z}_n) + l \\ &\leq (\mathcal{H}(\mathfrak{H}\zeta_{n-1}, \mathfrak{H}\mathfrak{z}_n) + l)^{\alpha(\mathfrak{z}_n, \mathfrak{H}\mathfrak{z}_n)\alpha(\mathfrak{H}\zeta_{n-1}, \zeta_{n-1})} \\ &\leq \mathbb{Q}(\mathcal{H}(\mathfrak{z}_n, \zeta_{n-1}))\mathcal{H}(\mathfrak{z}_n, \zeta_{n-1}) + l. \end{aligned}$$

This implies that

$$(2.15) \quad \mathcal{H}(\mathfrak{z}_n, \zeta_n) \leq \mathcal{H}(\mathfrak{z}_n, \zeta_{n-1}).$$

Again, equation (2.12) implies that

$$(2.16) \quad \begin{aligned} \mathcal{H}(\mathfrak{z}_{n+1}, \zeta_n) + l &= \mathcal{H}(\mathfrak{H}\zeta_n, \mathfrak{H}\mathfrak{z}_n) + l \\ &\leq (\mathcal{H}(\mathfrak{H}\zeta_n, \mathfrak{H}\mathfrak{z}_n) + l)^{\alpha(\mathfrak{z}_n, \mathfrak{H}\mathfrak{z}_n)\alpha(\mathfrak{H}\zeta_n, \zeta_n)} \\ &\leq \mathbb{Q}(\mathcal{H}(\mathfrak{z}_n, \zeta_n))\mathcal{H}(\mathfrak{z}_n, \zeta_n) + l. \end{aligned}$$

This implies that

$$(2.17) \quad \mathcal{H}(\mathfrak{z}_{n+1}, \zeta_n) \leq \mathcal{H}(\mathfrak{z}_n, \zeta_n).$$

From equations (2.15) and (2.17), we have

$$\mathcal{H}(\mathfrak{z}_n, \zeta_n) \leq \mathcal{H}(\mathfrak{z}_{n-1}, \zeta_{n-1}) \text{ for all } n \in \mathbb{N}.$$

It follows that the sequence $\{\mathcal{H}(\mathfrak{z}_n, \zeta_n)\}$ is a non-increasing sequence of positive real numbers. So, it is convergent and assumes that

$$\lim_{n \rightarrow \infty} \mathcal{H}(\mathfrak{z}_n, \zeta_n) = d.$$

Clearly, $d \geq 0$.

Claim: $d = 0$.

Equations (2.14) and (2.17) implies that

$$\frac{\mathcal{H}(\mathfrak{z}_n, \zeta_n)}{\mathcal{H}(\mathfrak{z}_{n-1}, \zeta_{n-1})} \leq \mathbb{Q}(\mathcal{H}(\mathfrak{z}_n, \zeta_{n-1})) \leq 1,$$

This implies that $\lim_{n \rightarrow \infty} \mathbb{Q}(\mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_{n-1})) = 1$.

Applying the property of the function $\mathbb{Q}$, we get

$$(2.18) \quad \lim_{n \rightarrow \infty} \mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_{n-1}) = 0.$$

Similarly, one can prove easily that

$$(2.19) \quad \lim_{n \rightarrow \infty} \mathcal{H}(\mathfrak{z}_n, \mathfrak{c}_n) = 0.$$

Now, we will show that $(\mathfrak{z}_n, \mathfrak{c}_n)$ is a Cauchy bisequence. Let us suppose, if possible that $(\mathfrak{z}_n, \mathfrak{c}_n)$ is not a Cauchy bisequence. Then there exists $\epsilon > 0$ and sequences $u(\mathfrak{z})$ and $v(\mathfrak{z})$ such that for all positive integers $\mathfrak{z}$, we have

$$v(\mathfrak{z}) > u(\mathfrak{z}) > \mathfrak{z}, \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})}, \mathfrak{c}_{u(\mathfrak{z})}) \geq \epsilon \text{ and } \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})}, \mathfrak{c}_{u(\mathfrak{z})-1}) < \epsilon.$$

By the definition of bipolar metric space and using above, we have

$$\epsilon \leq \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})}, \mathfrak{c}_{u(\mathfrak{z})}) \leq \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})}, \mathfrak{c}_{u(\mathfrak{z})-1}) + \mathcal{H}(\mathfrak{z}_{u(\mathfrak{z})-1}, \mathfrak{c}_{u(\mathfrak{z})-1}) + \mathcal{H}(\mathfrak{z}_{u(\mathfrak{z})-1}, \mathfrak{c}_{u(\mathfrak{z})})$$

Putting the limit as $\mathfrak{z} \rightarrow +\infty$ in the above inequality and using equations (2.18) and (2.19), we get

$$(2.20) \quad \lim_{\mathfrak{z} \rightarrow +\infty} \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})}, \mathfrak{c}_{u(\mathfrak{z})}) = \epsilon.$$

Again by triangle inequality, we have

$$\begin{aligned} \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})}, \mathfrak{c}_{u(\mathfrak{z})}) &\leq \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})}, \mathfrak{c}_{v(\mathfrak{z})+1}) + \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})+1}, \mathfrak{c}_{v(\mathfrak{z})+1}) + \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})+1}, \mathfrak{c}_{u(\mathfrak{z})}), \\ \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})+1}, \mathfrak{c}_{u(\mathfrak{z})}) &\leq \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})+1}, \mathfrak{c}_{v(\mathfrak{z})+1}) + \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})}, \mathfrak{c}_{v(\mathfrak{z})+1}) + \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})}, \mathfrak{c}_{u(\mathfrak{z})}) \end{aligned}$$

Taking the limit as $\mathfrak{z} \rightarrow +\infty$, together using equations (2.18) and (2.19), we deduce that

$$(2.21) \quad \lim_{\mathfrak{z} \rightarrow +\infty} \mathcal{H}(\mathfrak{z}_{u(\mathfrak{z})+1}, \mathfrak{c}_{v(\mathfrak{z})}) = \epsilon.$$

Again by triangle inequality, we have

$$\begin{aligned} \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})}, \mathfrak{c}_{u(\mathfrak{z})}) &\leq \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})}, \mathfrak{c}_{v(\mathfrak{z})}) + \mathcal{H}(\mathfrak{z}_{u(\mathfrak{z})+1}, \mathfrak{c}_{v(\mathfrak{z})}) + \mathcal{H}(\mathfrak{z}_{u(\mathfrak{z})+1}, \mathfrak{c}_{u(\mathfrak{z})}), \\ \mathcal{H}(\mathfrak{z}_{u(\mathfrak{z})+1}, \mathfrak{c}_{v(\mathfrak{z})}) &\leq \mathcal{H}(\mathfrak{z}_{u(\mathfrak{z})+1}, \mathfrak{c}_{u(\mathfrak{z})+1}) + \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})+1}, \mathfrak{c}_{u(\mathfrak{z})+1}) + \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})+1}, \mathfrak{c}_{v(\mathfrak{z})}) \end{aligned}$$

Taking the limit as $\mathfrak{z} \rightarrow +\infty$, together using equations (2.18)-(2.21), we deduce that

$$(2.22) \quad \lim_{\mathfrak{z} \rightarrow +\infty} \mathcal{H}(\mathfrak{z}_{u(\mathfrak{z})+1}, \mathfrak{c}_{v(\mathfrak{z})}) = \epsilon.$$

From equations (2.12) and (2.22), we have

$$\begin{aligned} \mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})+1}, \mathfrak{c}_{u(\mathfrak{z})+1}) + l &\leq (\mathcal{H}(\mathfrak{z}_{v(\mathfrak{z})+1}, \mathfrak{c}_{u(\mathfrak{z})+1}) + l)^{\alpha(\mathfrak{z}_{u(\mathfrak{z})+1}, \mathfrak{H}\mathfrak{z}_{u(\mathfrak{z})+1})\alpha(\mathfrak{H}\mathfrak{c}_{v(\mathfrak{z})}, \mathfrak{c}_{v(\mathfrak{z})})} \\ &= \mathbb{Q}(\mathcal{H}(\mathfrak{H}\mathfrak{c}_{v(\mathfrak{z})}, \mathfrak{H}\mathfrak{z}_{u(\mathfrak{z})+1}) + l)^{\alpha(\mathfrak{z}_{u(\mathfrak{z})+1}, \mathfrak{H}\mathfrak{z}_{u(\mathfrak{z})+1})\alpha(\mathfrak{H}\mathfrak{c}_{v(\mathfrak{z})}, \mathfrak{c}_{v(\mathfrak{z})})} \\ &\leq \mathbb{Q}(\mathcal{H}(\mathfrak{z}_{u(\mathfrak{z})+1}, \mathfrak{c}_{n(\mathfrak{z})})\mathcal{H}(\mathfrak{z}_{u(\mathfrak{z})+1}, \mathfrak{c}_{v(\mathfrak{z})}) + l \end{aligned}$$

Thus

$$\mathcal{H}(\mathfrak{z}_{v(\bar{z})+1}, \mathfrak{c}_{u(\bar{z})+1}) \leq \mathbb{Q}(\mathcal{H}(\mathfrak{z}_{u(\bar{z})+1}, \mathfrak{c}_{v(\bar{z})})\mathcal{H}(\mathfrak{z}_{u(\bar{z})+1}, \mathfrak{c}_{v(\bar{z})}))$$

Letting $\bar{z} \rightarrow \infty$ in above inequality, we obtain that

$$\lim_{\bar{z} \rightarrow \infty} \mathbb{Q}(\mathcal{H}(\mathfrak{z}_{u(\bar{z})+1}, \mathfrak{c}_{v(\bar{z})})) = 1,$$

that is,

$$\lim_{\bar{z} \rightarrow \infty} \mathcal{H}(\mathfrak{z}_{u(\bar{z})+1}, \mathfrak{c}_{v(\bar{z})}) = 0 < \epsilon.$$

a contradiction.

So, $(\mathfrak{z}_n, \mathfrak{c}_n)$ is a Cauchy bisequence.

Since $(J_{\mathfrak{h}}, H_{\mathfrak{h}}, \mathcal{H})$ is a complete bipolar metric space, so $\{(\mathfrak{z}_n, \mathfrak{c}_n)\}$ is convergent and thus biconverges to a point $w \in J_{\mathfrak{h}} \cap H_{\mathfrak{h}}$ and $\mathfrak{z}_n \rightarrow w, \mathfrak{c}_n \rightarrow w \in J_{\mathfrak{h}} \cap H_{\mathfrak{h}}$.

Since $\mathfrak{T}$ is a continuous map, $\mathfrak{z}_n \rightarrow w$ implies that $\mathfrak{c}_n = \mathfrak{T}\mathfrak{z}_n \rightarrow \mathfrak{T}w$ and $\mathfrak{c}_n \rightarrow w$ implies that $\mathfrak{z}_{n+1} = \mathfrak{T}\mathfrak{c}_n \rightarrow \mathfrak{T}w$. By the uniqueness of the limit, we obtain that $\mathfrak{T}w = w$.

Hence w is the fixed point.

Example 2.3. Let $J_{\mathfrak{h}} = (-\infty, 0]$, $H_{\mathfrak{h}} = [0, \infty)$ and $\mathcal{H} : (-\infty, 0] \times [0, \infty) \rightarrow [0, \infty)$ as $\mathcal{H}(\mathfrak{z}, \mathfrak{c}) = |\mathfrak{z} - \mathfrak{c}|$.

Clearly, $(J_{\mathfrak{h}}, H_{\mathfrak{h}}, \mathcal{H})$ is a complete bipolar metric space.

Define $\mathfrak{T} : ((-\infty, 0], [0, \infty)) \times ((-\infty, 0], [0, \infty))$ as $\frac{-\mathfrak{z}}{3}$.

Clearly, $\mathfrak{T}$ is a contravariant map.

Let $\alpha(\mathfrak{z}, \mathfrak{c}) = 1$ and $\mathbb{Q}t = \frac{1}{2}$.

Now left hand side of equation (2.1) becomes

$$(2.23) \quad (\mathcal{H}(\mathfrak{T}\mathfrak{c}, \mathfrak{T}\mathfrak{z})\alpha(\mathfrak{z}, \mathfrak{T}\mathfrak{z})\alpha(\mathfrak{T}\mathfrak{c}, \mathfrak{c}) + 1)^{\mathcal{H}(\mathfrak{T}\mathfrak{c}, \mathfrak{T}\mathfrak{z})} = 2^{\frac{1}{3}|\mathfrak{z}+\mathfrak{c}|}.$$

And right hand side becomes

$$(2.24) \quad 2^{\mathbb{Q}(\mathcal{H}(\mathfrak{z}, \mathfrak{c}))\mathcal{H}(\mathfrak{z}, \mathfrak{c})} = 2^{\frac{1}{2}|\mathfrak{z}-\mathfrak{c}|}.$$

As $\mathfrak{z}$ is negative real number, so

$$2^{\frac{1}{3}|\mathfrak{z}+\mathfrak{c}|} \leq 2^{\frac{1}{2}|\mathfrak{z}-\mathfrak{c}|}.$$

Hence, equation (2.1) holds.

Since, $\mathfrak{T}$ is continuous and α -admissible.

So, Theorem 2.1 is hold.

Hence, 0 is the fixed point of $\mathfrak{T}$.

Example 2.4. Let $J_{\mathfrak{h}} = (-\infty, 0]$, $H_{\mathfrak{h}} = [0, \infty)$ and $\mathcal{H} : (-\infty, 0] \times [0, \infty) \rightarrow [0, \infty)$ as $\mathcal{H}(\mathfrak{z}, \mathfrak{c}) = |\mathfrak{z} - \mathfrak{c}|$.

Clearly, $(J_{\mathfrak{h}}, H_{\mathfrak{h}}, \mathcal{H})$ is a complete bipolar metric space.

Taking $\mathcal{H} : ((-\infty, 0], [0, \infty)) \times ((-\infty, 0], [0, \infty))$ as $= \frac{-\mathfrak{z}}{4}$.

Clearly, $\mathcal{H}$ is a contravariant map.

Define $\alpha(\mathfrak{z}, \mathfrak{c}) = 1$ and $\mathcal{Q}t = \frac{1}{3}, l \geq 1$.

Now left hand side of equation (2.12) becomes

$$(2.25) \quad (\mathcal{H}(\mathcal{H}\mathfrak{c}, \mathcal{H}\mathfrak{z}) + l)^{\alpha(\mathfrak{z}, \mathcal{H}\mathfrak{z})\alpha(\mathcal{H}\mathfrak{c}, \mathfrak{c})} = \frac{1}{4}|\mathfrak{z} + \mathfrak{c}| + l.$$

And right hand side becomes

$$(2.26) \quad \mathcal{Q}(\mathcal{H}(\mathfrak{z}, \mathfrak{c}))\mathcal{H}(\mathfrak{z}, \mathfrak{c}) + l = \frac{1}{3}|\mathfrak{z} - \mathfrak{c}| + l.$$

As $\mathfrak{z}$ is negative real number, so

$$\frac{1}{4}|\mathfrak{z} + \mathfrak{c}| + l \leq \frac{1}{3}|\mathfrak{z} - \mathfrak{c}| + l.$$

Hence, equation (2.12) holds.

Since, $\mathcal{H}$ is continuous and α -admissible.

So, Theorem 2.1 is hold.

Hence, 0 is the fixed point of $\mathcal{H}$.

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