

Exponentially-fitted Runge-Kutta Method for Volterra Integro-Differential Equations

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(Received May 20, 2024, Accepted June 05, 2024)

Abstract: The present paper is intended to construct the exponentially fitted Runge-Kutta method to solve Volterra integro-differential equation (VIDE). Here, VIDE is first converted into its equivalent integro-differential operator form and assumes that these operators annihilate the pre-assigned set functions $\{e^{\pm\omega t}, te^{\pm\nu t}\}$ with unknown frequencies ω and ν . The optimum values of these frequencies ω and ν are calculated by minimizing the truncation error. To show the utility of the proposed development, these schemes are tested on two examples of VIDE having exponential solution. The comparison of the numerical results with the analytical/exact solution and the best reported results in literature confirms the high accuracy and applicability of the proposed method to the VIDE.

Keywords: Volterra integro-differential equation; Exponential solution; Runge-Kutta method; Exponential fitting; Error estimation; Convergence analysis.

MSC 2020: 65L05, 65R10

1. Introduction

Integral and integro-differential equations are widely used mathematical tool to model variety of physical problems. In modern times, these equations have achieved great popularity among researchers of mathematics and physics to formulate the boundary value problems of electrostatics, gravitation, heat transfer, fluid dynamics, etc. It is also a known fact that

initial value problems and boundary value problems can be converted into integral equations, and sometimes also due to this conversion, we may get some advantage to find its solutions. Among these integral equations, Volterra and Fredholm integral equations have various applications like in glass-forming process¹, nano-hydrodynamics², dropwise condensation³, modeling the competition of tumor cells and immune system⁴, examination of noise term phenomena⁵, and wind ripple in the desert⁶. But, it is challenging to solve Volterra integro-differential equations analytically. So, it is a great need to develop efficient numerical techniques to solve such equations. Although, an ample number of research papers have published on different numerical techniques to solve Volterra integro-differential equations. Some of them have recently formulated e.g., Adomian decomposition method⁷, Lagrange interpolation method⁸, rationalized Haar function method⁹, Legendre matrix method¹⁰, Galerkin method¹¹, Bessel matrix method¹², Runge-Kutta method¹³, spline approximation method¹⁴, and finite difference approximation method¹⁵. In this paper, we shall be focused on the numerical solution of Volterra integro-differential equations. The first order Volterra integro-differential equation (VIDE) is given in the form

$$(1.1) \quad \frac{dy}{dx} = g(x) + \lambda \int_{x_0}^x k(x, s)y(s)ds,$$

or, we may write in functional form as

$$(1.2) \quad \frac{dy}{dx} = f(x, y(x), \int_{x_0}^x k(x, sy(s))ds,$$

subject to initial conditions $y(x_0) = y_0$, where $k(x, s): \{(x, t) \in \mathbb{R}^2 \mid 0 \leq t \leq x \leq X\} \rightarrow \mathbb{R}$ is the kernel and $g: \mathbb{R} \rightarrow \mathbb{R}$ the given function. Assume that the functions $k(x, s)$ and f are given sufficiently smooth to guarantee the existence and uniqueness of the solution¹⁶. However, the standard methods fail to solve the VIDE having exponential or trigonometric solutions efficiently. Because, in order to meet their true solution behaviors, one has to choose a very small step size. So, a dedicated numerical scheme is needed to meet such solution behaviors. Exponential fitting is introduced to handle such VIDE. To solve the initial value problem (IVP) in VIDE, having exponential/trigonometric solution behaviors, several authors have developed exponentially-fitted numerical method e.g. Panternoster¹⁷, Simos¹⁸, and Berghe et. al.¹⁹⁻²⁰ made significant developments in this direction and they have first constructed exponentially and trigonometrically fitted methods to solve IVP. Recently, Tiwari and Pandey²¹⁻²² constructed the exponential fitting of pseudo Runge-Kutta method to solve IVPs.

2. Derivation of the Method

In this section, we first give the brief description of standard Runge-Kutta to solve VIDE and it is called as VIDE Runge-Kutta method (VDRKM). In the next subsection, we formulate its exponential fitting. The VDRKM was originally constructed by Vermiglio¹⁷ which is based on extension of the natural Runge-Kutta method for VIE.

2.1. VIDE Runge-Kutta method (VDRKM)

A variety of research articles has been written to solve VIDE (1-2). Runge-Kutta (RK) method one of them which can be used to solve VIDE (1-2). Here we first give a brief outlook of the VIDE RK method^{16, 23-26}.

Denote $F_n(x) = \int_{x_0}^{x_n} k(x, s)y(s)ds$, $x_n \leq x \leq \bar{x}$ and $\tilde{F}_n(x)$ is the approximation to $F_n(x)$ and it is calculated using some suitable quadrature rule. Then a VIDE-Runge-Kutta (s-stage) method to solve (1.1) and (1.2) takes the following form

$$(2.1) \quad y_{n+1} = y_n + h \sum_{i=1}^s b_i f(x_n + c_i h, Y_i^{[n]}, Z_i^{[n]}),$$

$$(2.2) \quad Y_i^{[n]} = y_n + h \sum_{j=1}^s a_{ij} f(x_n + c_j h, Y_j^{[k]}, Z_j^{[n]}),$$

$$(2.3) \quad Z_i^{[n]} = \tilde{F}_n(x_n + c_i h) + h \sum_{j=1}^s \bar{a}_{ij} k(x_n + d_{ij} h, x_n + c_j h, Y_j^{[n]}).$$

where $c_i = \sum_{j=1}^s a_{ij}$ and $d_{ij} \geq c_j$ whenever $\bar{a}_{ij} \neq 0$. We can call $\tilde{F}_n$ as lag term approximation to the tail $F_n(x) = \int_{x_0}^{x_n} k(x, s, y(s))ds$, $x \in (x_n, x_{n+1})$ and the problem of formulation of the VDRK method is to determine $a_{ij}, \bar{a}_{ij} \neq 0, b_i, d_{ij}$ ($1 \leq i \leq s, 1 \leq j \leq s$).

We sometimes write (1.1) as (in functional form)

$$y' = f\left(x, g(x), \int_{x_0}^x k(x, s, y(s))dt\right),$$

and in the form of system we have

$$(2.4) \quad \begin{bmatrix} y(x) \\ z(x) \end{bmatrix} = \begin{bmatrix} y_0 \\ 0 \end{bmatrix} + \int_{x_0}^x \begin{bmatrix} f(x, y(s), z(s)) \\ k(x, s, y(s)) \end{bmatrix} ds,$$

where (2.4) can be obtained by taking

$$(2.5) \quad z(x) = \int_{x_0}^x k(x, t, y(t))dt.$$

Thus $\frac{dy}{dx} = f(x, y(x), z(x))$

$$\Rightarrow \int_{x_0}^x dy = \int_{x_0}^x f(x, y(s), z(s))ds$$

and integrating we get

$$(2.6) \quad y(x) = y_0 - \int_{x_0}^x f(x, y(t), z(s)) ds.$$

If $a_{ij} = 0$ for $i \leq j$ then system (2.1)-(2.3) is called explicit VIDE-RK method. The first approach to compute tail (lag term) approximation was proposed by Bellen et al.²⁵ and reanalyzed by D-Conte et al.²⁶. The approach of computing $\tilde{F}_n(x)$ is based on natural continuous extension of the numerical solution by the interpolation formula of degree $d \leq p$.

$$(2.7) \quad y(x_n + \theta h) = \sum_{j=1}^s w_j(\theta) Y_j^{[n]},$$

where $w_j(\theta)$ are polynomial of degree d with $F\left[\frac{p}{z}\right] \leq d \leq \min\{s-1, p\}$ which gives the tail approximation as

$$(2.8) \quad \tilde{F}_n(x) = f(x) + h \sum_{l=0}^n \sum_{j=0}^s \gamma_j k(x, x_{l-1} + \xi_j h, Y_i^{[l-1]}).$$

where γ_j are weights and ξ_j are (nodes) $j=1, 2, 3, \dots, s$ which corresponds to a quadrature rule of order $\geq p$. To compute the tail approximation $\tilde{F}_n(x)$ we many use the quadrature formula, some time, we many use same RK method.

2.2 Exponentially fitted VIDE-Runge-Kutta method (ef-VDRKM)

To construct exponentially fitting of VIDE-RK method, we follow the approach of Berghe¹⁹. For the other related exponentially fitting procedures, we can see Ixaru and Berghe's²⁰. We assume the kernel $k(x, y(s))$ is independent of s . like in Cante et al.²⁶, Bellen et al.²⁵ and Brunner²⁴, we can take VIDE (1.2) in the following simpler form

$$(2.9) \quad y'(x) = y(x) + \int_0^x k(x, y(s)) ds.$$

We rewrite the VDRK method (given in section 2.1) in the following form: The internal stages are

$$(2.10) \quad Y_i^{[n]} = y_n + h \sum_{j=1}^s a_{ij} f(x_n + c_i h, Y_j^{[n]}, Z_j^{[n]}),$$

and

$$(2.11) \quad Z_i^{[n]} = \tilde{F}_n(x_n + c_i h) + h \sum_{j=1}^s \bar{a}_{ij} k(x_n + d_{ij} h, Y_j^{[n]}),$$

whereas the external stage (objective) of the solution is

$$(2.12) \quad y_{n+1} = y_n + h \sum_{i=1}^s b_i f(x_n + c_i h, Y_i^{[n]}, Z_i^{[n]}).$$

The tail approximation $\tilde{F}_n(x)$ and corresponding to the tail $F_n(x)$ is given by

$$(2.13) \quad \bar{F}_n(x) = h \sum_{l=1}^n \sum_{j=1}^\mu v_j k(x, Y_j^{[l-1]}).$$

To compute tail approximation in (2.13) we use the same ef- VDRK method as in external stage of the solution given in equation (2.1). We first define the linear integral operator associated to the external stage (3) as

$$(2.14) \quad L(y(x); h) \cong y(x+h) - y(x) - h \sum_{i=1}^s b_i f(x_n + c_i h, Y_i^{[n]}, Z_i^{[n]})$$

$$\text{Or} \quad L(y(x); h) \cong y(x+h) - y(x) - h \sum_{i=1}^s b_i y'(x_n + c_i h).$$

The linear integral operators associated with the integral stages (2.2) and (2.3) are

$$(2.15) \quad L_i(y(x); h) \cong y(x + c_i h) - y(x) - h \sum_{j=0}^s a_{ij} y'(x + c_j h).$$

These operators L and L_i have following properties. We may define $L: C(\mathbb{R}) \rightarrow \mathbb{R}$ associated with the internal stage of the numerical scheme and $L_i: C(\mathbb{R}) \rightarrow \mathbb{R}$ associated with the internal stages of the numerical scheme. These operators L and L_i are linear as if satisfy

$$L[c_1 y_1(x) + c_2 y_2(x)] = c_1 L(y_1(x)) + c_2 L(y_2(x)).$$

And similar argument can be shown function L_i here $y_1(x), y_2(x) \in C^\infty(\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$.

For the exponentially fitting approach, we assume L should be operated on $y(x) = x^r e^{\mu x}$, $\mu \in \mathbb{R}$, $r \geq 0$. We define μ moment of L of order r as

$$(2.16) \quad E_r = L[x^r e^{\mu x}; h]_{x=0}$$

The following relations are satisfied (for the detailed description see Ixaru and Berghe's²⁷)

Lemma 2.1: $L[e^{\mu x}; h] = e^{\mu x} E_0$ where $E_0 = L[e^{\mu x}; h]_{x=0}$.

Proof: consider $L[e^{\mu(x-a)}]$ and using the linearity property are can show that

$$(2.17) \quad L[e^{\mu(x-a)}; h] = e^{-\mu a} L[e^{\mu x}; h]$$

Multiplying both sides by $e^{+\mu x}$ and then putting $a = x$ we get

$$L[e^{\mu x}; h]_{x=0} = L[e^{\mu x}; h] e^{\mu x}$$

$$\text{or,} \quad L[e^{\mu x}; h] = E_0 e^{\mu x}.$$

Lemma 2.2: The higher order μ moments can be obtained by

$$E_r = \frac{\partial^r E_0}{\partial \mu^r}, \quad r = 1, 2, 3, \dots$$

Proof: Differentiating (2.16) r times with respect to μ we get

$$L\left[\frac{\partial^r e^{\mu x}}{\partial \mu^r}; h\right] = \sum_{k=0}^r {}^r C_k \frac{\partial^k e^{\mu x}}{\partial \mu^k} \cdot \frac{\partial^{r-k} E_0}{\partial \mu^{r-k}}.$$

Denote $\frac{\partial^k e^{\mu x}}{\partial \mu^k} = x^k e^{\mu x}$ and taking $x = 0$ we get

$$E_r = \frac{\partial^r E_0}{\partial \mu^r}, r = 1, 2, 3, \dots$$

Corollary: Successive moments can be evaluated by

$$E_r(\mu) = \frac{\partial E_{r-1}}{\partial \mu}, r = 1, 2, 3, \dots$$

2.2.1 Order Conditions for Evaluation of Coefficients

Here, we derive the order conditions for ef-VDRKM and the approach of the derivation is similar to the work on exponential fitting of various methods given by Ixaru and Berghe's²⁷.

To evaluate the coefficients for exponentially fitted VDRKM, we define the classic moment of $L[y(x); h]$ as $L[y(x); h] = L[x^r; h]_{n=0}$ and denote h^l as a dimension of the operator L .

Then $L_r[y(x); h] = E_r[h = 0]n = h^{l+r} E_r^*(z = 0)$ where $z = \mu h$ a dimension-less quantity. Thus $L_m[y(x); h] = h^{l+m} L_r^*(a)$.

$L^*(a) = E^*(z = \mu h = 0, h)$ is dimension less. The following theorem is the basis for the evaluation of the coefficients of ef-VDRKM.

Theorem 2.1: If $y(x)$ is infinitely differentiable, then

$$L[y(x); h] = \sum_{r=0}^{\infty} \frac{1}{r!} L_r[y^{(r)}(x); h]$$

Proof: Proof can be established using Taylor's series approximation of $y(x)$ around the point a . For details are can see²⁷. The following properties are satisfied for the operator L .

Lemma 2.3: The conditions $L[x^r e^{\mu x}; h] = 0$ for $r = 0, 1, 2, 3, \dots, m-1$ imply that $L_r^*(\mu h, a) = 0$ for $r = 0, 1, 2, 3, \dots, m-1$ and vice versa, where $z = \mu h$.

Proof: Proof can be seen in²⁷.

Again,

$$(2.18) \quad \bar{L}_i(y(x); k_i h) \cong \int_{x_0}^x k(x, t, y(t)) dt - h \sum_{j=0}^s \bar{a}_{ij} k(x + d_{ij} h, x + c_j h, y(x + c_j h))$$

where $i = 1, 2, 3, \dots, s$ and in the case of explicit VIDE-RK method we may take $s = i-1$.

To calculate the values of weights b_i and coefficients $a_{ij}, \bar{a}_{ij}, d_{ij}, c_j$, we use the Lemma 2.1-2.3 and Theorem 2.1. so, we assume the integral operators (2.14), (2.15) and (2.16) annihilate the set of functions $S = \{1, x, y(x) = e^{\pm \mu x}, k(x, y(x)) = e^{\pm u x} y(x)\}$, i.e., $L(1, h) = 0$ and $L(x; h) = 0$ gives $x + h - x - h \sum b_i = 0$ which implies

$$(2.19) \quad \sum_{i=1}^s b_i = 1,$$

and $L_i(1, h) = 0, L_i(x_i, h) = 0$ gives

$$(2.20) \quad \sum_{i=1}^s a_{ij} = c_i. \quad (\text{consistency condition})$$

Equation (2.20) is the consistency condition which is similar to the standard RK method.

Restricting for $s = 2$ and for explicit version of ef-VDR method, we must have $a_{11} = a_{12} = 0, a_{22} = 0$.

Taking $\bar{a}_{ij} = a_{ij} \Rightarrow \bar{a}_{11} = \bar{a}_{12} = \bar{a}_{22} = 0$.

And if the internal stage operator L_i annihilate the set of functions S then

$$L_i(e^{\mu x}, e^{\nu x} y(x), h) = 0 \text{ gives}$$

$$(2.21) \quad \frac{1}{\nu} \left(-e^{c_2 h \nu} - e^{c_2 h(\mu+\nu)} + a_{21} e^{c_1 h(\mu+\nu)} h \mu \right) = 0.$$

Setting $L_i(e^{\pm \mu x}) = 0$, we get

$$(2.22) \quad -1 + e^{\pm h \mu} \mp b_1 e^{\pm c_1 h \mu} h \mu \pm b_2 e^{\pm c_2 h \mu} h \mu = 0.$$

Solving the system of equations (24), the weights in the external stage can be obtained as.

$$b_1 = \frac{e^{-\omega(1-c_1)}(-1+e^\omega)(e^\omega - e^{2c_2\omega})}{(e^{2c_1\omega} - e^{2c_2\omega})\omega}, b_2 = \frac{e^{-\omega(1-c_2)}(1-e^\omega)(e^\omega - e^{2c_1\omega})}{(-e^{2c_1\omega} + e^{2c_2\omega})\omega}.$$

where $\omega = h\mu$ is called frequency, these weights can be converted into the weights of standard RK method (2-stage explicit) under the limit $h\mu = \omega \rightarrow 0$, i.e. under the limit $h\mu \rightarrow 0, b_1 = 1 - \frac{1}{2c_2}, b_2 = \frac{1}{2c_2}$ with c_2 as free parameter (node).

On the other hand, solving the equation (23) the coefficient a_{21} can be obtained as

$$a_{21} = \frac{e^{c_2\theta - c_1(\theta+\omega)}(-1+e^{c_2\omega})}{\omega}, \text{ where } \theta = h\nu, \omega = h\mu,$$

and $a_{21} = c_2 e^{-c_1\theta}$ under the limit $\omega \rightarrow 0$.

Thus $a_{21} = c_2$ for $c_1 = 0$.

Again, we have same consistency condition as of standard RK method. Also, the coefficient in other internal stage $Z_i^{[n]}$ in $\bar{a}_{21} = a_{21}$. From the values of a_{21}, b_1, b_2 and $\bar{a}_{21}$ it is clear that they are frequency (μ and ν) dependent. To compute the values of these unknown frequencies, we minimize the truncation error (like in Berghe¹⁹⁻²⁰).

2.2.2 Error

Let $y(x_{n+1})$ denote the exact solution at $x = x_{n+1}$ and y_{n+1} be the solution by ef-VDRKM then local truncation error is given by

$$TE_{n+1} = |y_{n+1} - y(x_{n+1})|,$$

$$= \left| y_n + h \sum_{i=1}^2 b_i f \left(x_n + c_i h, Y_i^{[n]}, Z_i^{[n]} \right) - f(x, y(s), z(s)) \right|$$

where $z(x) = \int_0^x k(x, y(s)) ds$ and $Y_i^{[n]}$ and $Z_i^{[n]}$ are given in (2.2)-(2.3).

Using Taylor's approximation (like in the case of the standard RK method) it can be observed that $TE_{n+1} = O(h^3)$ i.e., the proposed ef-VDRKM is of order two. In the similar way as made by Ixaru and Berghes²⁵ the optimum values of frequencies μ and ν can be obtained by minimizing the expression of T.E (2.23).

3. Numerical experiments

Here we consider two VIDE of order 1 and apply the VDRKM and ef-VDRKM (proposed in sections (2.1) and (2.2)). The numerical results are reported in Tables 1 and 2.

Consider the general first-order VIDE

$$(2.23) \quad y'(x) = g(x) + \lambda \int_{x_0}^x k(x, s) y(s) ds, \quad x_0 \leq x \leq X.$$

subject to the initial condition: $y(x_0) = y_0$.

For various values of the kernel $k(x, s)$ and $g(x)$ we consider the following two different problems.

Example 3.1: For the first numerical example of VIDE, taking $g(x) = 1 + x$ and $k(x, s) = (x - s)y(s)$ with $\lambda = 1$ and the initial condition $y(0) = 1$ in equation (25) and solve the resulting equation by the proposed schemes ef-VDRKM and VDRKM. The true solution of the problem is $y(x) = e^x$. The numerical results are compared with the true solution and error is given in Table 1. From Table 1, it can be observed that both the methods are efficient and produce significantly small error, but ef-VDRKM is most efficient.

We implement the ef-VDRKM (Section 2.1) and standard VDRKM (VRK) (Section 2.2) to solve numerically the VIDE (25) for the step size $h = \frac{X - x_0}{N}$.

The absolute values of end point errors are compared in Table 1.

We denote $e_n = |y(X) - y_N|$ as absolute endpoint error, where y_N denotes the end point numerical solution and $y(X)$ as endpoint true solution.

$cdy_N = -\log_{10}(|y(X) - y_N|)$ correct digits in endpoint solution obtained by taking steps N , and $\tilde{p} = \frac{(cdy_N - cdy_{N/2})}{\log_{10}(2)}$ as estimated order of the method.

Table 1: Comparison of absolute endpoint error $|y(x_N) - y_N|$ and order of the methods for Example 1 for various step sizes ' h ' (frequencies $\omega=1$, $\nu=1$)

Step-Size h	VDRKM			ef-VDRKM		
	$ y(X) - y_N $	cdy_N	$\tilde{p}$	$ y(X) - y_N $	cdy_N	$\tilde{p}$
0.01	1.89×10^{-5}	4.72		4.37×10^{-6}	5.35	
0.005	4.79×10^{-6}	5.31	1.985	1.10×10^{-6}	5.95	1.987
0.0025	1.19×10^{-6}	5.92	1.998	2.77×10^{-7}	6.55	1.990
0.00125	2.99×10^{-7}	6.52	1.999	6.93×10^{-8}	7.15	2.000
0.000625	7.51×10^{-8}	7.12	1.995	1.73×10^{-8}	7.76	1.998

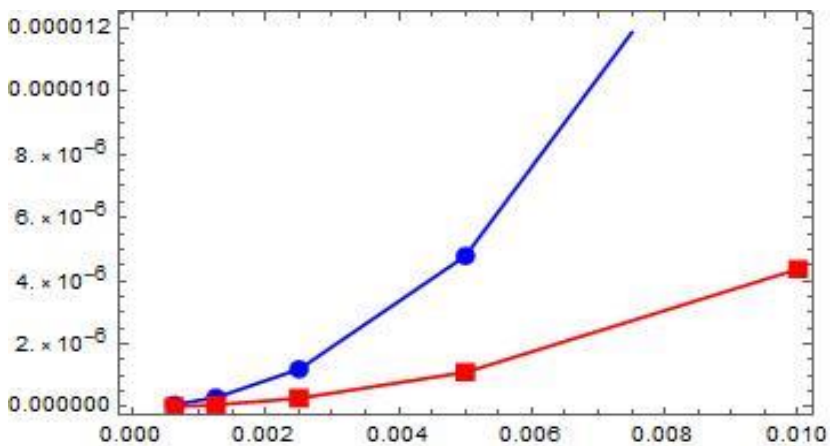


Fig 1: Comparison of absolute endpoint error $|y(x_N) - y_N|$ vs. various stop sizes ' h ' for Example 3.1

Example 3.2: For the second numerical example of VIDE, we consider $g(x) = 2 + 4x$ and $k(x, s) = (x - s)y(s)$ with $\lambda = 8$ and the initial condition $y(0) = 1$ in equation (25) and solve the resulting VIDE by both the proposed schemes VDRKM and ef-VDRKM. The true solution of the problem is $y(x) = e^{2x}$. The numerical results are compared with the true solution and the error is given in Table 2. From Table 2, it can be observed that both the methods are efficient and produce significantly small errors, but ef-VDRKM is more efficient than standard VDRKM. Nearly more than one (approx. 1.25) decimal digits are corrected in the solution if we proceed through ef-VDRKM. This fact directly reveals that the ef-VDRKM is cost-efficient.

Table 2: Comparison of absolute endpoint error $|y(x_N) - y_N|$ and order of the methods for Example 3.2 for various step sizes 'h' (frequencies $\omega = 1, \nu = 1$).

Step-Size h	VDRKM			ef-VDRKM		
	$ y(X) - y_N $	cdy_N	$\tilde{p}$	$ y(X) - y_N $	cdy_N	$\tilde{p}$
0.01	2.60×10^{-4}	3.58		1.18×10^{-5}	4.93	
0.005	6.65×10^{-5}	4.18	1.993	3.18×10^{-6}	5.49	1.851
0.0025	1.65×10^{-5}	4.78	1.996	8.34×10^{-7}	6.07	1.931
0.00125	4.10×10^{-6}	5.38	1.998	2.13×10^{-7}	6.67	1.966
0.000625	1.03×10^{-6}	5.98	1.992	5.39×10^{-8}	7.26	1.983

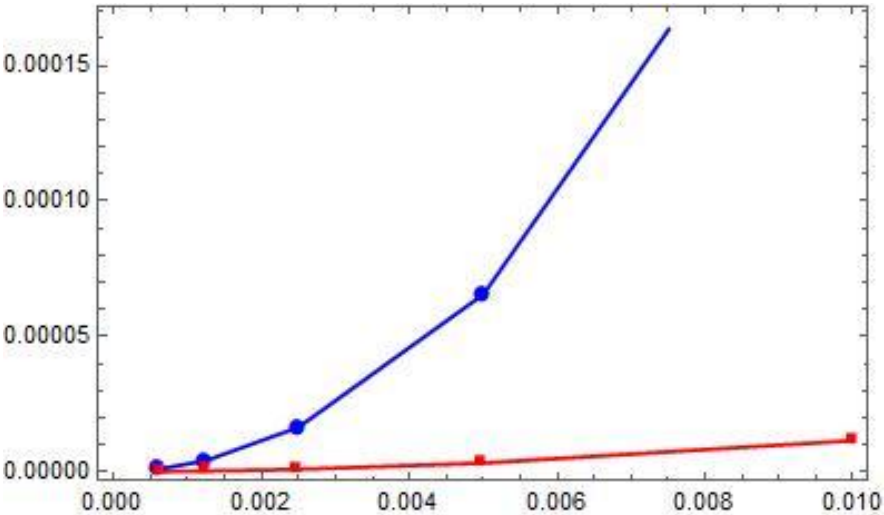


Fig 2: Comparison of absolute endpoint error $|y(x_N) - y_N|$ vs. various stop sizes 'h' for Example 3.2

5. Conclusion

In this paper, we have successfully proposed the exponential-fitting of Volterra-integro Runge-Kutta method (ef-VDRKM) for the numerical solution of Volterra integro-differential equations of order 1. The proposed method is tested on two VIDEs of order one. The results are reported in Tables 1 and 2. In Tables 1 and 2 the comparison of end point errors for varying step sizes is made. Also, we have calculated the estimated order of the method and found that there will be no report of order reduction. From these tables, it can be concluded that ef-VDRKM produces smaller errors than the standard VDRKM. This shows the superiority and cost efficiency of the

exponentially fitted VDRKM. The construction of the Nyström type method to solve VIDE of order 2 or higher order would be an attractive future aim of the research work in this direction.

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