

# Riemannian Manifolds with Sub-biharmonic Functions

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**Abstract:** The object of the present paper is to investigate biharmonic and sub-biharmonic functions on Riemannian manifolds.

## 1. Introduction

By a closed Riemannian manifold  $M$ , we mean the smooth connected and compact orientable Riemannian manifold of dimension  $n \geq 2$  with the compact orientable boundary  $S$  and the Riemannian metric  $g_{ji}$ . Let  $K_{kji}^h$ ,  $K_{ji}$  and  $r$  denote the components of the Riemannian curvature tensor, Ricci tensor and the scalar curvature of  $M$  respectively. As usual  $\nabla$  denotes the covariant differentiation on  $M$ .

The Laplacian of a smooth function  $f$  on  $M$ , that is,  $\Delta f$  on  $M$ , is defined by

$$\Delta f = g^{ji} \nabla_j \nabla_i f,$$

where  $\Delta = g^{ji} \nabla_j \nabla_i$  and  $g^{ji}$  are the components of the contravariant metric tensor  $g$  on  $M$ . The second order Laplacian of  $f$  on  $M$ , that is,  $\Delta^2 f$  on  $M$ , is given by

$$\Delta^2 f = \Delta(\Delta f) = g^{lj} g^{ki} \nabla_l \nabla_j \nabla_k \nabla_i f.$$

A smooth function  $f$  on  $M$  is said to be harmonic or biharmonic in  $M$  according as it satisfies

$$\Delta f = 0 \quad \text{or} \quad \Delta^2 f = 0$$

everywhere in  $M$ . By a sub-biharmonic function  $f$  in  $M$ , we mean

$$\Delta^2 f \geq 0 \quad (\text{or } \Delta^2 f \leq 0)$$

everywhere in  $M$ . We denote the normal derivative of  $f$  with respect to the  $N^i$ , the components of the unit normal  $N$  to the boundary  $S$ , by  $g_{ji}(\nabla^j f)N^i$ .

The main purpose of this paper is to establish the following theorems:

**Theorem 1.** *For a closed Riemannian manifold  $M$ , if the scalar curvature of  $M$  is such that the normal derivative of  $\Delta r$  is zero on  $S$  and  $R(Dr, N) = 0$  on  $S$ , then*

$$(1.1) \quad \int_V \nabla^4 r dV = 0,$$

where  $V$  is the region (or volume) inside  $M$ ,  $Dr$  is the vector field associated with the differential 1-form  $dr$  with the components  $\nabla_i r$  and  $\nabla^4 r$  is given by (2.12).

**Theorem 2.** *A smooth function  $f$  which is sub-biharmonic everywhere inside the region of the closed Riemannian manifold  $M$  and constant on the boundary  $S$  must be constant everywhere inside that region.*

**Theorem 3.** *Two smooth functions which are biharmonic everywhere inside the region of the closed Riemannian manifold  $M$  and identical on the boundary  $S$  must be identical everywhere inside that region.*

**Theorem 4.** *If two smooth functions  $f$  and  $g$  which are sub-biharmonic everywhere inside the region of the closed Riemannian manifold  $M$  and harmonic on the boundary  $S$ , and whose normal derivatives are same on the boundary  $S$ , must differ at most by an additive constant inside that region.*

## 2. Proofs of our Main Theorems

For a smooth vector field  $X$  on  $M$  with local components  $X^i$ , the divergence of  $X$  is defined by

$$\text{div} X = g^{ji} \nabla_j X_i,$$

where  $X_i = g_{ji} X^j$ . In this paper, we use the Einstein summation conventions, that is, the sum is over all the repeated indices over the range  $1, 2, 3, \dots, n$ . The raising and lowering of a tensor are performed using  $g^{ji}$  and  $g_{ji}$  respectively.

The Ricci identity in local coordinates is given by

$$(2.1) \quad \nabla_k \nabla_j \omega_i - \nabla_j \nabla_k \omega_i = -K_{kji}^h \omega_h,$$

where  $\omega_i$  are the local components of the 1-form on  $M$ . From the second Bianchi identity<sup>1</sup>, we have

$$(2.2) \quad \nabla_t K_k^t = \frac{1}{2} \nabla_k r,$$

where  $K_k^t = g^{it} K_{ki}$ . We now quote the following Stoke's theorem<sup>1</sup> which is needed to prove our main theorems.

**Theorem A:** For a closed Riemannian manifold  $M$  with the boundary  $S$ , we have

$$\int_V (\operatorname{div} X) dV = \int_S g(X, N) dS$$

or in local coordinates

$$(2.3) \quad \int_V (\nabla_i X^i) dV = \int_S (X_i N^i) dS,$$

where  $dS$  is the surface element of  $S$ .

In (2.3), if we set  $X^i = \nabla^i f$ , then we have

$$(2.4) \quad \int_V \Delta f dV = \int_S g_{ji} (X^j, N^i) dS.$$

If the normal derivative of  $f$  is zero, that is,  $g_{ji} (\nabla^j f) N^i = 0$  on the boundary  $S$ , then (2.4) reduces to

$$(2.5) \quad \int_V \Delta f dV = 0.$$

If we replace  $f$  by  $\Delta f$  in (2.5), we find

$$(2.6) \quad \int_V \Delta^2 f dV = 0.$$

Consider

$$(2.7) \quad \nabla_i [(\nabla_j r) K^{ji}] = (\nabla_i \nabla_j r) K^{ij} + \frac{1}{2} (\nabla_i r) (\nabla^i r),$$

wherein we have used (2.2). Now taking the volume integral over  $V$  on both sides of (2.7), we have

$$(2.8) \quad \int_V [\nabla_i \{(\nabla_j r) K^{ji}\}] dV = \int_V [(\nabla_i \nabla_j r) K^{ij} + \frac{1}{2} (\nabla_i r) (\nabla^i r)] dV.$$

Further, applying Theorem A to the left hand side of the expression (2.8), we find

$$(2.9) \quad \int_V [\nabla_i \{(\nabla_j r) K^{ij}\}] dV = \int_S K^{ij} (\nabla_j r) N_i dS.$$

Comparing (2.8) and (2.9), we obtain

$$(2.10) \quad \int_S K^{ji} (\nabla_j r) N_i dS = \int_V [(\nabla_i \nabla_j r) K^{ij} + \frac{1}{2} (\nabla_i r) (\nabla^i r)] dV$$

or in the index free notations,

$$\int_V \left[ g(\nabla^2 r, R) + \frac{1}{2} |dr|^2 \right] dV = \int_S R(Dr, N) dS,$$

where  $g(\nabla^2 r, R) = (\nabla_i \nabla_j r) K^{ji}$ . If  $R(Dr, N) = 0$  on the boundary  $S$ , then

$$(2.11) \quad \int_V \left[ g(\nabla^2 r, R) + \frac{1}{2} |dr|^2 \right] dV = 0.$$

For the sake of compactness, we denote

$$(2.12) \quad \nabla^4 r = \nabla_i \nabla_j \nabla^i \nabla^j r.$$

We now prove our main theorems.

**Proof of Theorem 1:** Replacing  $\omega_i$  by  $\nabla_i r$  in the Ricci identity (2.1), we have

$$(2.13) \quad \nabla_k \nabla_j \nabla_i r - \nabla_j \nabla_k \nabla_i r = -K_{kji}^h \nabla_h r.$$

Now taking the covariant derivative on both sides of (2.13) with respect to the index  $l$ , we find

$$(2.14) \quad \nabla_l \nabla_k \nabla_j \nabla_i r - \nabla_l \nabla_j \nabla_k \nabla_i r = -K_{kji}^h \nabla_l \nabla_h r + (\nabla_l K_{kji}^h)(\nabla_h r).$$

Transvecting with  $g^{lj} g^{ki}$  on both sides of (2.14) and then simplifying using (2.2), it is easy to see that

$$(2.15) \quad \nabla_i \nabla_j \nabla^i \nabla^j r - \Delta^2 r = K^{lh} \nabla_l \nabla_h r + \frac{1}{2} (\nabla_i r)(\nabla^i r).$$

In the index free notations, (2.15) may be written as

$$(2.16) \quad \nabla^4 r = \Delta^2 r + g(\nabla^2 r, R) + \frac{1}{2} |dr|^2.$$

Taking the volume integral over  $V$  on both sides of (2.16) and then using (2.6) and (2.11), we obtain (1.1).

This finishes the proof.  $\square$

**Proof of Theorem 2:** Since  $f$  is the sub-biharmonic inside the region  $V$  of  $M$  and constant on the boundary  $S$ , it follows from (2.6) that

$$\Delta^2 f = 0$$

everywhere in  $M$ . Thus sub-biharmonic function  $f$  must be biharmonic in  $V$ . If  $g$  is another function on  $M$ , then consider,

$$\nabla^i [f \nabla_i g - g \nabla_i f] = f \Delta g - g \Delta f.$$

Applying Theorem A, we find

$$(2.17) \quad \int_S [f \nabla_i g - g \nabla_i f] N^i dS = \int_V [f \Delta g - g \Delta f] dV.$$

If we set  $g = \Delta f$  in integral equation (2.17) and then using the fact that  $f$  is constant on  $S$ , we find

$$\int_V f \Delta^2 f dV = \int_V (\Delta f)^2 dV.$$

From which, we conclude that  $f$  is biharmonic inside the region  $V$  if and only if it is harmonic inside that region. Hence a function  $f$  which is sub-biharmonic everywhere inside the region  $V$  must be harmonic inside that region. A function which is harmonic everywhere inside the region  $V$  and constant on the boundary  $S$  must be constant inside that region, follows from the following integral equation:

$$\int_V [f\Delta f + |\nabla f|^2] dV = \int_S f (\nabla_i f) N^i dS.$$

We therefore, finally, conclude that on a closed manifold the functions which are sub-biharmonic in  $V$  are constant functions.

This completes the proof.  $\square$

**Proof of Theorem 3:** Since two functions  $f$  and  $g$  are identical on  $S$ ,  $f = g$  on  $S$ , therefore replacing  $f$  by  $h$  and  $g$  by  $\Delta h$  in (2.17), we find

$$\int_V h \Delta^2 h dV = \int_V (\Delta h)^2 dV.$$

From which we conclude that  $h$  is harmonic in  $V$  if and only if it is biharmonic in  $V$ . Thus  $h$  is zero in  $V$ , that is,  $f$  and  $g$  agree in  $V$ .

Hence the proof is completed.  $\square$

**Proof of Theorem 4:** Using the similar arguments as in Theorem 2, we can prove that for two functions  $f$  and  $g$  are harmonic in  $V$ , therefore  $f - g$  is harmonic in  $V$ . Further, since the normal derivatives of these two functions are the same on  $S$ , the normal derivatives of  $f - g$  is zero on  $S$ . Thus  $f - g$  must be constant in  $V$ .

Hence the proof is established.  $\square$

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## Reference

1. K. Yano, *Integral formulas in Riemannian geometry*, Marcel Dekker Inc. New York, 1970.