

Certain Identities of the Lie - derivative of Tensors in Generalized $\mathcal{BK}$ –Fifth Recurrent Finsler Space

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Abstract: This study employs advanced tensor calculus techniques to investigate the properties of the tensorial derivative in generalized fifth-order recurrent Finsler spaces. By systematically analyzing the underlying geometric structure, we derive identities that provide new insights into the relationships between various tensorial quantities. Our results demonstrate the effectiveness of these techniques in exploring complex geometric structures. This paper introduces a new identity connecting the tensors in generalized fifth recurrence Finsler space for Cartan's fourth curvature tensor in the sense of Berwald by using Lie-derivative. We prove that the Lie - derivative and the Berwald covariant derivative of the fifth order for some curvature and torsion tensors are commutative under certain conditions. We have shown the Lie - derivative for some tensors behave as fifth recurrent and we obtain various identities on Lie - derivative in $GBK - 5RF_n$.

Keywords: Finsler space F_n , generalized $\mathcal{BK}$ –fifth recurrent space, Berwald covariant derivative of fifth order, Lie - derivative.

Subject Classification: 53B40, 53C60.

1. Introduction and Preliminaries

Tensor calculus is a powerful tool for studying the geometry of manifolds. A Lie-derivative evaluate the rate of change of a vector field or a tensor field along the flow of another vector or tensor field. The Lie-derivative in generalized fifth recurrence Finsler space for Cartan's fourth curvature tensor in sense of Berwald introduced by AL-Qashbari and Baleedi¹. Gouin² introduced some remarks on the Lie-derivative. Opondo³ studied the Lie-

derivative on W -curvature tensor in recurrent and birecurrent Finsler space. The Lie - derivative and its application was investigated⁴⁻⁵.

The concept of generalized recurrence, birecurrence and trirecurrence properties for various curvature tensors in sense of Berwald have been discussed⁶⁻¹⁸. The generalized recurrent Finsler spaces of higher orders have been studied¹⁹⁻²².

Let us explore the infinitesimal transformation point given²¹

$$(1.1) \quad x^{-i} = x^i + v^i(x)\varepsilon.$$

where ε is an infinitesimal point constant and $v^i(x)$ is a contravariant vector field independent of directional arguments and dependent on positional coordinates x^i only and $v^i(x) \neq 0$. Infinitesimal method is a tool that leads to Lie-derivatives. The symbol L_v denote the Lie-differentiation operator with respect to the transformation (1.1). The Lie-derivative of a vector field x^i in sense of Berwald is given by

$$(1.2) \quad L_v x^i = v^j \mathcal{B}_j x^i - x^j \mathcal{B}_j v^i + (\partial_j x^i) \mathcal{B}_s v^j y^s.$$

The Lie - derivative of a general mixed tensor field $T_j^i(x, \dot{x})$, a general mixed tensor field $T_j^i(x, \dot{x})$ and the connection coefficient $G_{jk}^i(x, \dot{x})$ are expressed in the form

$$(1.3) \text{ a) } L_v T_j^i = v^m \mathcal{B}_m T_j^i - T_j^m \mathcal{B}_m v^i + T_m^i \mathcal{B}_j v^m + (\partial_m T_j^i) \mathcal{B}_s v^m y^s$$

$$\text{ b) } L_v T_{jkh}^i = v^m \mathcal{B}_m T_{jkh}^i - T_{jkh}^m \mathcal{B}_m v^i + T_{mkh}^i \mathcal{B}_j v^m + T_{jmh}^i \mathcal{B}_k v^m + T_{jkm}^i \mathcal{B}_h v^m + \partial_m T_{jkh}^i \mathcal{B}_r v^m y^r.$$

and

$$\text{ c) } L_v G_{jk}^i = \mathcal{B}_j \mathcal{B}_k v^h + H_{jkh}^i v^h + G_{ijk}^l \mathcal{B}_s v^l y^s.$$

The commutation formula involving Lie-derivatives and other derivatives for tensors T_j^i and T_{jk}^i are given by

$$(1.4) \quad L_v(\partial_l T_j^i) - \partial_l(L_v T_j^i) = 0$$

and

$$(1.5) \quad L_v(\mathcal{B}_m T_{jk}^i) - \mathcal{B}_m(L_v T_{jk}^i) = T_{jk}^l L_v G_{lm}^i - T_{lk}^i L_v G_{jm}^l - T_{jl}^i L_v G_{km}^l - \partial_l T_{jk}^i L_v G_{sm}^l y^s,$$

respectively.

The vector y^i is Lie-invariant³ i.e.

$$(1.6) \quad L_v y^i = 0.$$

An n -dimensional Finsler space F_n equipped with the metric function $F(x, y)$ satisfying the request conditions²³⁻²⁶. The metric function F , the metric tensor g_{ij} and the Kronecker delta δ_h^i are given by

$$\begin{aligned}
 (1.7) \text{ a) } g_{ij} y^i y^j &= F^2, \quad \text{b) } g_{ij} y^j = y_i, \quad \text{c) } g_{hj} = \dot{\partial}_h y_j, \quad \text{d) } y_i y^i = F^2, \\
 \text{e) } g_{ij} g^{ik} &= \delta_j^k = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases} \quad \text{f) } \delta_k^i y_i = y_k, \quad \text{g) } \delta_k^i y^k = y^i, \\
 \text{h) } g^{ik} y_k &= y^i \quad \text{i) } \delta_h^j = \dot{\partial}_h y^j \quad \text{and} \quad \text{j) } \delta_i^i = n.
 \end{aligned}$$

The metric tensor, Cartan's connection parameters and Berwald's connection parameters are symmetric in their lower indices and they are positive homogeneous of degree zero in the directional arguments.

Berwald's covariant derivative of the vectors y^i and y_i vanish identically, i.e.

$$(1.8) \quad \text{a) } \mathcal{B}_k y^i = 0 \quad \text{and} \quad \text{b) } \mathcal{B}_k y_i = 0.$$

The Cartan's fourth curvature tensor K_{jkh}^i and Cartan's third curvature tensor R_{jkh}^i , are skew-symmetric in their last two lower indices and satisfy the following relations

$$\begin{aligned}
 (1.9) \quad \text{a) } H_{kh}^i &= K_{jkh}^i y^j = R_{jkh}^i y^j, \quad \text{b) } H_{jkh}^i = K_{jkh}^i + y^m (\dot{\partial}_j K_{m kh}^i), \\
 \text{c) } K_{jkh}^i &= R_{jkh}^i - C_{js}^i H_{kh}^s, \quad \text{d) } K_{ijkh} = g_{rj} K_{ikh}^r \\
 \text{e) } H_{jkh}^i - K_{jkh}^i &= P_{jk}^i P_{rh}^i - P_{jhk}^i - P_{jh}^r P_{rk}^i, \\
 \text{f) } R_{ijhk} &= K_{ijhk} + C_{ijm} K_{rhk}^s y^m, \quad \text{g) } R_{ijkh} = g_{rj} R_{ikh}^r.
 \end{aligned}$$

Ricci tensor K_{jk} , curvature vector K_j and curvature scalar K are given by

$$(1.10) \quad \text{a) } K_{jki}^i = K_{jk}, \quad \text{b) } K_{jk} y^k = K_j \quad \text{and} \quad \text{c) } K_{jk} g^{jk} = K.$$

Ricci tensor R_{jk} , deviation tensor R_h^i and curvature scalar R are given by

$$\begin{aligned}
 (1.11) \quad \text{a) } R_{jkr}^r &= R_{jk}, \quad \text{b) } R_{jkh}^i g^{jk} = R_h^i, \quad \text{c) } R_{jk} g^{jk} = R \quad \text{and} \\
 \text{d) } R_{jk} &= K_{jk} + C_{js}^r H_{rk}^s.
 \end{aligned}$$

The curvature tensor of Berwald H_{jkh}^i , $h(v)$ -torsion tensor H_{kh}^i are skew-symmetric in their last two lower indices and satisfy the following relations

$$\begin{aligned}
 (1.12) \quad \text{a) } \dot{\partial}_j H_{kh}^i &= H_{jkh}^i, \quad \text{b) } \dot{\partial}_k H_h^i = H_{kh}^i, \quad \text{c) } H_{ji}^i y_i = 0 \quad \text{and} \\
 \text{d) } y^r \dot{\partial}_j \dot{\partial}_r H_{kh}^i &= 0.
 \end{aligned}$$

The Ricci tensor H_{jk} , deviation tensor H_h^i and curvature scalar H satisfy the following relations

$$\begin{aligned}
 (1.13) \quad \text{a) } H_{kh}^i y^k &= H_h^i, \quad \text{b) } H_k^i y^k = 0, \quad \text{c) } H_{jk} = H_{jkr}^r, \quad \text{d) } H_{kr}^r = H_k \\
 \text{e) } y_i H_k^i &= 0, \quad \text{f) } K_{jk} y^j = H_k \quad \text{g) } K_j y^j = (n-1)H \quad \text{and} \\
 \text{h) } H_r^r &= (n-1)H.
 \end{aligned}$$

A Finsler space whose Berwald connection parameter G_{kh}^i is independent of y^i is called an *affinely connected space (Berwald space)*. Thus, an affinely connected space is characterized by one of the equivalent conditions

$$(1.14) \quad a) \mathcal{B}_k g_{ij} = 0 \quad \text{and} \quad b) \mathcal{B}_k g^{ij} = 0.$$

Let us explore a generalized $BK - 5^{th}$ recurrent Finsler space satisfying the condition

$$(1.15) \quad \begin{aligned} \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K_{jkh}^i &= a_{sqlnm} K_{jkh}^i + b_{sqlnm} (\delta_h^i g_{jk} - \delta_k^i g_{jh}) \\ &\quad - c_{sqlnm} (\delta_h^i C_{jkn} - \delta_k^i C_{jhn}) - d_{sqlnm} (\delta_h^i C_{jkl} - \delta_k^i C_{jhl}) \\ &\quad - e_{sqlnm} (\delta_h^i C_{jkq} - \delta_k^i C_{jhq}) - 2b_{qlnm} y^r \mathcal{B}_r (\delta_h^i C_{jks} - \delta_k^i C_{jhs}). \end{aligned}$$

The above space satisfies the following relations

$$(1.16) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H_{kh}^i = a_{sqlnm} H_{kh}^i + b_{sqlnm} (\delta_h^i y_k - \delta_k^i y_h).$$

$$(1.17) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H = a_{sqlnm} H + b_{sqlnm} F^2.$$

$$(1.18) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H_k = a_{sqlnm} H_k + b_{sqlnm} (n-1) y_k.$$

$$(1.19) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H_h^i = a_{sqlnm} H_h^i + b_{sqlnm} (\delta_h^i F^2 - y^i y_h).$$

Furthermore,

$$(1.20) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K_{jk} = a_{sqlnm} K_{jk},$$

$$(1.21) \quad \begin{aligned} \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m R_{jkh}^i &= a_{sqlnm} R_{jkh}^i + b_{sqlnm} (\delta_h^i g_{jk} - \delta_k^i g_{jh}) \\ &\quad - c_{sqlnm} (\delta_h^i C_{jkn} - \delta_k^i C_{jhn}) - d_{sqlnm} (\delta_h^i C_{jkl} - \delta_k^i C_{jhl}) \\ &\quad - e_{sqlnm} (\delta_h^i C_{jkq} - \delta_k^i C_{jhq}) - 2b_{qlnm} y^r \mathcal{B}_r (\delta_h^i C_{jks} - \delta_k^i C_{jhs}), \end{aligned}$$

if and only if

$$(1.22) \quad b_{sqlnm} g_{jk} - c_{sqlnm} C_{jkn} - d_{sqlnm} C_{jkl} - e_{sqlnm} C_{jkq} - 2b_{qlnm} y^r \mathcal{B}_r C_{jks} = 0.$$

$$(1.23) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (C_{jt}^i H_{kh}^t) - a_{sqlnm} (C_{jt}^i H_{kh}^t) = 0.$$

Taking the Lie-derivative of both sides of (1.7f) and using (1.7e), we get

$$(1.24) \quad L_v y_k = 0.$$

Taking the Lie-derivative of both sides of (1.7b) and using (1.24) and (1.6), we get

$$(1.25) \quad L_v g_{ij} = 0.$$

Similarly, taking the Lie-derivative of both sides of (1.7h) and using (1.24) and (1.6), we get

$$(1.26) \quad L_v g^{ik} = 0.$$

Using (1.7h) in [(1.9)a], we get

$$K_{jkh}^i g^{jk} y_k = R_{jkh}^i g^{jk} y_k .$$

Using (1.11b) in above equation, we get

$$K_{jkh}^i g^{jk} y_k = R_h^i y_k .$$

Since $y_k \neq 0$, so above equation can be written as

$$(1.27) \quad K_{jkh}^i g^{jk} = R_h^i .$$

In this paper, we apply tensor calculus to the analysis of generalized fifth-order recurrent Finsler spaces. Our goal is to derive new identities involving the tensorial derivative that can be used to characterize these spaces more fully. By employing a combination of algebraic and geometric methods, we are able to obtain significant results.

2. A Lie-derivative of Tensors in $GBK - 5RF_n$

Let us explore $GBK - 5RF_n$ for which whose Cartan's fourth curvature tensor K_{jkh}^i is defined as (1.15). By applying (1.3b) on the Cartan's fourth curvature tensor K_{jkh}^i , we have

$$L_v K_{jkh}^i = v^m \mathcal{B}_m K_{jkh}^i - K_{jkh}^m \mathcal{B}_m v^i + K_{mkh}^i \mathcal{B}_j v^m + K_{jmh}^i \mathcal{B}_k v^m + K_{jkm}^i \mathcal{B}_h v^m + \dot{\partial}_m K_{jkh}^i \mathcal{B}_r v^m y^r .$$

Using (1.8a) and (1.7i,e) when $r \neq m$ in above equation, we get

$$(2.1) \quad L_v K_{jkh}^i = v^m \mathcal{B}_m K_{jkh}^i - K_{jkh}^m \mathcal{B}_m v^i + K_{mkh}^i \mathcal{B}_j v^m + K_{jmh}^i \mathcal{B}_k v^m + K_{jkm}^i \mathcal{B}_h v^m$$

Using (1.7e) in (1.15), taking the Lie-derivative of both sides of result equation, we get

$$(2.2) \quad L_v (\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K_{jkh}^i) = a_{sqlnm} L_v K_{jkh}^i + K_{jkh}^i L_v a_{sqlnm} .$$

Taking Berwald covariant derivative of fifth order for (2.1) with respect to x^m, x^n, x^l, x^q and x^s , we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (L_v K_{jkh}^i) = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m [v^m \mathcal{B}_m K_{jkh}^i - K_{jkh}^m \mathcal{B}_m v^i + K_{mkh}^i \mathcal{B}_j v^m + K_{jmh}^i \mathcal{B}_k v^m + K_{jkm}^i \mathcal{B}_h v^m] .$$

From (2.2) and above equation, we get

$$(2.3) \quad L_v (\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K_{jkh}^i) = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (L_v K_{jkh}^i) ,$$

if and only if

$$(2.4) \quad a_{sqlnm} L_v K_{jkh}^i + K_{jkh}^i L_v a_{sqlnm} = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m [v^m \mathcal{B}_m K_{jkh}^i - K_{jkh}^m \mathcal{B}_m v^i + K_{mkh}^i \mathcal{B}_j v^m + K_{jmh}^i \mathcal{B}_k v^m + K_{jkm}^i \mathcal{B}_h v^m] .$$

Thus, we conclude

Theorem 2.1. *In $GBK - 5RF_n$, the Lie - derivative for Cartan's fourth curvature tensor K_{jkh}^i and Berwald covariant derivative of the fifth order are commutative if and only if (2.4) holds.*

Using (2.3) in (2.2), we get

$$(2.5) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (L_v K_{jkh}^i) = a_{sqlnm} (L_v K_{jkh}^i)$$

if and only if $L_v a_{sqlnm} = 0$. Thus, we conclude

Corollary 2.1. *In $GBK - 5RF_n$, the Lie - derivative for Cartan's fourth curvature tensor K_{jkh}^i behaves as fifth recurrent if and only if $L_v a_{sqlnm} = 0$, [provided (2.4) holds].*

By applying (1.3b) on the Cartan's third curvature tensor R_{jkh}^i , using (1.8a) and (1.7 i,e) when $r \neq m$, we get

$$(2.6) \quad L_v R_{jkh}^i = v^m \mathcal{B}_m R_{jkh}^i - R_{jkh}^m \mathcal{B}_m v^i + R_{mkh}^i \mathcal{B}_j v^m \\ + R_{jmh}^i \mathcal{B}_k v^m + R_{jkm}^i \mathcal{B}_h v^m.$$

Using (1.7e) in (1.21), and taking the Lie-derivative of both sides of result equation, we get

$$(2.7) \quad L_v (\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m R_{jkh}^i) = a_{sqlnm} L_v R_{jkh}^i + R_{jkh}^i L_v a_{sqlnm}$$

if and only if (1.23) holds. Taking Berwald covariant derivative of fifth order for (2.6) with respect to x^m, x^n, x^l, x^q and x^s , we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (L_v R_{jkh}^i) = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m [v^m \mathcal{B}_m R_{jkh}^i - R_{jkh}^m \mathcal{B}_m v^i \\ + R_{mkh}^i \mathcal{B}_j v^m + R_{jmh}^i \mathcal{B}_k v^m + R_{jkm}^i \mathcal{B}_h v^m].$$

From (2.7) and above equation, we get

$$(2.8) \quad L_v (\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m R_{jkh}^i) = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (L_v R_{jkh}^i),$$

if and only if

$$(2.9) \quad a_{sqlnm} L_v R_{jkh}^i + R_{jkh}^i L_v a_{sqlnm} = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m [v^m \mathcal{B}_m R_{jkh}^i \\ - R_{jkh}^m \mathcal{B}_m v^i + R_{mkh}^i \mathcal{B}_j v^m + R_{jmh}^i \mathcal{B}_k v^m + R_{jkm}^i \mathcal{B}_h v^m],$$

provided (1.23) holds. Thus, we conclude

Theorem 2.2. *In $GBK - 5RF_n$, the Lie - derivative for Cartan's third curvature tensor R_{jkh}^i and Berwald covariant derivative of the fifth order are commutative [provided (1.23) and (2.9) hold].*

Using (2.8) in (2.7), we get

$$(2.10) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (L_v R_{jkh}^i) = a_{sqlnm} (L_v R_{jkh}^i)$$

if and only if $L_v a_{sqlnm} = 0$, provided (1.24) and (2.9) hold.

Thus, we conclude

Corollary 2.2. *In $GBK - 5RF_n$, the Lie - derivative for Cartan's third curvature tensor R_{jkh}^i behaves as fifth recurrent if and only if $L_v a_{sqlnm} = 0$, [provided (1.23) and (2.9) hold].*

By applying [(1.3)b] on the $h(v)$ -torsion tensor H_{kh}^i , using (1.8a) and [(1.8)i,e] when $r \neq m$, we get

$$(2.11) \quad L_v H_{kh}^i = v^m \mathcal{B}_m H_{kh}^i - H_{kh}^m \mathcal{B}_m v^i + H_{mh}^i \mathcal{B}_k v^m + H_{km}^i \mathcal{B}_h v^m.$$

Using (1.7e) in (1.16), taking the Lie - derivative of both sides of result equation, we get

$$(2.12) \quad L_v (\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H_{kh}^i) = a_{sqlnm} L_v H_{kh}^i + H_{kh}^i L_v a_{sqlnm}.$$

Taking Berwald covariant derivative of fifth order for (2.11) with respect to x^m, x^n, x^l, x^q and x^s , we get

$$\begin{aligned} & \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (L_v H_{kh}^i) \\ &= \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m [v^m \mathcal{B}_m H_{kh}^i - H_{kh}^m \mathcal{B}_m v^i + H_{mh}^i \mathcal{B}_k v^m + H_{km}^i \mathcal{B}_h v^m]. \end{aligned}$$

From (2.12) and above equation, we get

$$(2.13) \quad L_v (\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H_{kh}^i) = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (L_v H_{kh}^i),$$

If and only if

$$\begin{aligned} (2.14) \quad & a_{sqlnm} L_v H_{kh}^i + H_{kh}^i L_v a_{sqlnm} \\ &= \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m [v^m \mathcal{B}_m H_{kh}^i - H_{kh}^m \mathcal{B}_m v^i + H_{mh}^i \mathcal{B}_k v^m + H_{km}^i \mathcal{B}_h v^m]. \end{aligned}$$

Thus, we conclude

Theorem 2.3. *In $GBK - 5RF_n$, the Lie - derivative for $h(v)$ -torsion tensor H_{kh}^i and Berwald covariant derivative of the fifth order are commutative if and only if (2.14) holds.*

Using (2.13) in (2.12), we get

$$(2.15) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (L_v H_{kh}^i) = a_{sqlnm} (L_v H_{kh}^i)$$

if and only if $L_v a_{sqlnm} = 0$. Thus, we conclude

Corollary 2.3. *In $GBK - 5RF_n$, the Lie - derivative for $h(v)$ -torsion tensor H_{kh}^i behaves as fifth recurrent if and only if $L_v a_{sqlnm} = 0$, [provided (2.14) holds].*

Now, contracting the indices i and h in (2.15), and using (1.13d), we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (L_v H_k) = a_{sqlnm} (L_v H_k)$$

if and only if $L_v a_{sqlnm} = 0$. Thus, we conclude

Theorem 2.4. In $GBK - 5RF_n$, the Lie - derivative for the vector H_k behaves as fifth recurrent if and only if $L_v a_{sqlnm} = 0$, [povided (2.14) holds].

Taking the Lie - derivative of both sides of [(1.9) a], we get

$$L_v H_{kh}^i = L_v (K_{jkh}^i y^j).$$

Using (1.6) in above equation, we get

$$L_v H_{kh}^i = y^j L_v K_{jkh}^i.$$

Using (2.1) in above equation, we get

$$L_v H_{kh}^i = v^m \mathcal{B}_m H_{kh}^i - H_{kh}^m \mathcal{B}_m v^i + K_{m kh}^i \mathcal{B}_j v^m y^j + H_{mh}^i \mathcal{B}_k v^m + H_{km}^i \mathcal{B}_h v^m.$$

In view of above equation and (2.11), we get

$$K_{m kh}^i \mathcal{B}_j v^m y^j = 0.$$

Since $K_{m kh}^i \neq 0$, then above equation can be written as

$$\mathcal{B}_j v^m y^j = 0.$$

Using (1.9a) in above equation, since $y^j \neq 0$, we get

$$(2.16) \quad \mathcal{B}_j v^m = 0.$$

Now, by applying (1.3b) on the Berwald curvature tensor H_{jkh}^i , using (2.16), we have

$$(2.17) \quad L_v H_{jkh}^i = v^m \mathcal{B}_m H_{jkh}^i.$$

Taking the Lie - derivative of both sides of (1.9b), using (1.6), we get

$$L_v H_{jkh}^i = L_v K_{jkh}^i + L_v (\partial_j K_{m kh}^i y^m).$$

Using (1.9a) in above equation, we get

$$(2.18) \quad L_v H_{jkh}^i = L_v K_{jkh}^i + L_v (\partial_j H_{kh}^i).$$

Using (2.1), (2.16) and (2.17) in above equation, we get

$$L_v (\partial_j H_{kh}^i) = v^m \mathcal{B}_m (H_{jkh}^i - K_{jkh}^i) = v^m \mathcal{B}_m (\partial_j H_{kh}^i).$$

Multiplying above equation by n , we get

$$n L_v (\partial_j H_{kh}^i) = v^m \mathcal{B}_m (\partial_j n H_{kh}^i).$$

Using [(1.7) j,i] in right side of above equation, we get

$$n L_v (\partial_j H_{kh}^i) = v^m \mathcal{B}_m (\partial_j \partial_r y^r H_{kh}^i).$$

Since y^r is independent of y^j , so above equation can be written as

$$n L_v (\partial_j H_{kh}^i) = v^m \mathcal{B}_m (y^r \partial_j \partial_r H_{kh}^i).$$

Using (1.12d) in above equation, we get

$$(2.19) \quad L_v(\partial_j H_{kh}^i) = 0.$$

Now, taking Berwald covariant derivative of fifth order for (2.19) with respect to x^m, x^n, x^l, x^q and x^s , using (2.18), we get

$$(2.20) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m L_v H_{jkh}^i = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m L_v K_{jkh}^i.$$

Thus, we conclude

Theorem 2.5. *In $GBK - 5RF_n$, the Berwald covariant derivative of fifth order for the Lie - derivative for the Berwald curvature tensor H_{jkh}^i and Cartan's fourth curvature tensor K_{jkh}^i are equal.*

By applying (1.5) on the $h(v)$ -torsion tensor H_{kh}^i , using and (1.7i,e) when $s \neq l$, we get

$$(2.21) \quad L_v(\mathcal{B}_m H_{kh}^i) - \mathcal{B}_m(L_v H_{kh}^i) = H_{kh}^l L_v G_{lm}^i - H_{lh}^i L_v G_{km}^l - H_{kl}^i L_v G_{hm}^l.$$

Multiplying above equation by y_i , and using [(1.12)c], we get

$$y_i [L_v(\mathcal{B}_m H_{kh}^i) - \mathcal{B}_m(L_v H_{kh}^i)] = H_{kh}^l L_v G_{lm}^i.$$

Again, multiplying above equation by y_l , and using (1.12c), since $y_l \neq 0$, we get

$$(2.22) \quad L_v(\mathcal{B}_m H_{kh}^i) - \mathcal{B}_m(L_v H_{kh}^i) = 0.$$

Which can be written as

$$(2.23) \quad L_v(\mathcal{B}_m H_{kh}^i) = \mathcal{B}_m(L_v H_{kh}^i).$$

Multiplying (2.21) by y_l , using [(1.12)c] and (2.22), then taking Berwald covariant derivative of fifth order for the resulting equation with respect to x^m, x^n, x^l, x^q and x^s , we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (H_{lh}^i L_v G_{km}^l) = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (-H_{kl}^i L_v G_{hm}^l).$$

Since the $h(v)$ -torsion tensor H_{kh}^i is skew-symmetric in its two lower indices, so above equation can be written as

$$(2.24) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (H_{lh}^i L_v G_{km}^l) = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (H_{lk}^i L_v G_{hm}^l).$$

Thus, we conclude

Theorem 2.6. *In $GBK - 5RF_n$, the Berwald covariant derivative of fifth order for the tensor $(H_{lh}^i L_v G_{km}^l)$ is symmetric in its two lower indices h and k .*

Using (2.16) and [(1.3)c] in (2.24), since $v^r \neq 0$, we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (H_{lh}^i H_{kmr}^l) = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (H_{lk}^i H_{hmr}^l).$$

Multiplying above equation by n and using (1.7 j,i) and (1.12a) in right side of the resulting equation, we get

$$n \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (H_{lh}^i H_{kmr}^l) = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (y^r \dot{\partial}_h \dot{\partial}_r H_{lk}^i H_{mr}^l).$$

Using [(1.12)d] in above equation, then contacting i and h , using (1.14d) in the resulting equation, we get

$$(2.25) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (H_l H_{kmr}^l) = 0.$$

Thus, we conclude

Theorem 2.7. *In $GBK - 5RF_n$, the Berwald covariant derivative of fifth order for the tensor $(H_{lh}^i L_v G_{km}^l)$ is vanishing.*

By applying [(1.3)a] on the deviation tensor R_h^i , using (2.16), we get

$$L_v R_h^i = v^m \mathcal{B}_m R_h^i.$$

Using [(1.11)b] and (1.27) in left side of above equation, we get

$$g^{jk} L_v R_{jkh}^i = v^m \mathcal{B}_m R_h^i.$$

Using (2.6) and (2.16) in above equation, since $v^m \neq 0$, we get

$$g^{jk} \mathcal{B}_m R_{jkh}^i = \mathcal{B}_m R_h^i.$$

Taking Berwald covariant derivative of fourth order for above equation with respect to x^n, x^l, x^q and x^s , using (1.14b), (1.21), (1.7e) and (1.11b), we get

$$(2.26) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m R_h^i = a_{sqlnm} R_h^i$$

if and only if (1.23) holds. Thus, we conclude

Theorem 2.8. *In $GBK - 5RF_n$, the deviation tensor R_h^i behaves as fifth recurrent if and only (1.23) holds good.*

By applying (1.4) on the deviation tensor H_h^i , using (2.16), since $v^m \neq 0$, we get

$$\mathcal{B}_m (\dot{\partial}_l H_h^i) - \dot{\partial}_l (\mathcal{B}_m H_h^i) = 0$$

Using (1.12b) in above equation, we get

$$\mathcal{B}_m (H_{lh}^i) = \dot{\partial}_l (\mathcal{B}_m H_h^i).$$

Multiplying above equation by y^r and using [(1.7)i] and [(1.8)a], then taking Berwald covariant derivative of fourth order for result equation with respect to x^n, x^l, x^q and x^s , we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (H_{lh}^i y^r) = \delta_l^r (\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H_h^i).$$

Using (1.19) in above equation, we get

$$(2.27) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (H_{lh}^i y^r) = \delta_l^r (a_{sqlnm} H_h^i)$$

if and only if

$$(2.28) \quad b_{sqlnm} (\delta_h^i F^2 - y^i y_h) = 0.$$

Contracting the indices i and h in (2.27) and using [(1.13)d,h], we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (H_l y^r) = (n-1) \delta_l^r (a_{sqlnm} H).$$

Contracting the indices r and l in above equation and using [(1.7)j], we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (H_r y^r) = n(n-1) (a_{sqlnm} H).$$

Which can be written as

$$(2.29) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (H_r y^r) = a_{sqlnm} (H_r y^r)$$

if and only if $H_r y^r = n(n-1)H$.

Thus, we conclude

Theorem 2.9. *In $GBK - 5RF_n$, the tensor $(H_r y^r)$ behaves as fifth recurrent if and only if $H_r y^r = n(n-1)H$, [provided (2.28) holds].*

By applying [(1.3)b] on the Ricci tensor H_{jk} , using (2.16), we get

$$L_v H_{jk} = v^m \mathcal{B}_m H_{jk}.$$

Using [(1.13)c] and [(1.9)a,b] in left side of above equation, we get

$$L_v H_{jkr}^r = L_v K_{jkr}^r + L_v (\dot{\partial}_j H_{kr}^r) = v^m \mathcal{B}_m H_{jk}.$$

Contracting the indices i and h in equations (2.1) and (2.19), using the resulting equations, (2.16) and [(1.10)a] in above equation, we get

$$v^m \mathcal{B}_m H_{jk} = v^m \mathcal{B}_m K_{jk}$$

Taking Berwald covariant derivative of fourth order for the above equation with respect to x^n , x^l , x^q and x^s , using (2.16), we get

$$(2.30) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H_{jk} = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K_{jk}.$$

Thus, we conclude

Theorem 2.10. *In $GBK - 5RF_n$, the Berwald covariant derivative of fifth order for the Ricci tensors H_{jk} and K_{jk} are equal.*

By applying [(1.3)b] on curvature vector H_k , using (2.16), we get

$$L_v H_k = v^m \mathcal{B}_m H_k.$$

By applying [(1.3)b] on Ricci tensor K_{jk} then using the resulting equation, (2.16), (1.13f) and (1.6) in the left side of above equation, we get

$$y^j \mathcal{B}_m K_{jk} = \mathcal{B}_m H_k.$$

Multiplying above equation by g^{jk} , using [(1.14)b] and [(1.10)c], we get

$$y^j \mathcal{B}_m K = g^{jk} \mathcal{B}_m H_k.$$

Taking Berwald covariant derivative of fourth order for above equation with respect to x^n , x^l , x^q and x^s , using [(1.8)a], [(1.14)b] and (1.18), we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K y^j = a_{sqlnm} (g^{jk} H_k)$$

if and only if

$$(2.31) \quad b_{sqlnm}(n-1)y_k = 0.$$

Which can be written as

$$(2.32) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (Ky^j) = a_{sqlnm} (Ky^j)$$

if and only if $Ky^j = g^{jk} H_k$. Thus, we conclude

Theorem 2.11. *In $GBK - 5RF_n$, the tensor (Ky^j) behaves as fifth recurrent if and only $Ky^j = g^{jk} H_k$, [provided (2.31) holds].*

By applying [(1.3)b] on Ricci tensor R_{jk} , using (2.16), we get

$$L_v R_{jk} = v^m \mathcal{B}_m R_{jk}.$$

By applying [(1.3)b] on Ricci tensor K_{jk} then using the resulting equation, (2.16) and [(1.12)d] in the left side of above equation, we get

$$L_v C_{js}^r H_{rk}^s = v^m \mathcal{B}_m R_{jk} - v^m \mathcal{B}_m K_{jk}.$$

Taking Berwald covariant derivative of fourth order for above equation with respect to x^n, x^l, x^q and x^s , using [(1.11)d] and (2.16), we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n L_v (C_{js}^r H_{rk}^s) = v^m \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (C_{js}^r H_{rk}^s).$$

By applying [(1.3)b] on tensor $(C_{js}^i H_{rk}^s)$, using the resulting equation and (2.16) in above equation, we get

$$(2.33) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n L_v (C_{js}^r H_{rk}^s) = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n L_v (C_{js}^r H_{rk}^s).$$

Thus, we conclude

Theorem 2.12. *In $GBK - 5RF_n$, the Lie - derivative for tensor $(C_{js}^i H_{rk}^s)$ behaves as fourth recurrent.*

Multiplying [(1.9)c] by g^{jk} and using [(1.11)b] and (1.27), we get

$$C_{js}^i H_{hk}^s = 0.$$

Taking the Lie - derivative for above equation, using [(1.11)d], we get

$$v^m \mathcal{B}_m R_{jk} - v^m \mathcal{B}_m K_{jk} = 0.$$

Taking Berwald covariant derivative of fourth order for above equation with respect to x^n, x^l, x^q and x^s , using (2.16), we get

$$(2.34) \quad \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m R_{jk} = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K_{jk}.$$

Thus, we conclude

Theorem 2.13. *In $GBK - 5RF_n$, the Berwald covariant derivative of fifth order for the Ricci tensors R_{jk} and K_{jk} are equal.*

From Theorem 2.10 and Theorem 2.13, we conclude

Corollary 2.4. *In $GBK - 5RF_n$, the Berwald covariant derivative of fifth order for the Ricci tensors R_{jk} and H_{jk} are equal.*

In view of (2.30) and (1.20), we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H_{jk} = a_{sqlnm} H_{jk}$$

if and only if $H_{jk} = K_{jk}$.

Similarly, in view of (2.34) and (1.20), we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m R_{jk} = a_{sqlnm} R_{jk}$$

if and only if $R_{jk} = K_{jk}$. Thus, we conclude

Corollary 2.5. *In $GBK - 5RF_n$, the Ricci tensors H_{jk} and R_{jk} behave as fifth recurrent if and only if each of them equals the Ricci tensor K_{jk} [provided (1.22) holds].*

3. Conclusions

This paper investigates the properties of the Lie derivative in generalized fifth-order recurrent Finsler spaces. We established conditions which the Lie derivative and Berwald covariant derivative of the fifth order commute for certain curvature and torsion tensors. Also, we proved that the Lie derivative behaves as a fifth recurrent operator for Cartan's fourth curvature tensor and the $h(v)$ -torsion tensor under specific conditions. The relationships between the Lie derivative of various tensors and Berwald covariant derivative of the fifth order have been discussed. These results contribute to a deeper understanding of how the Lie derivative interacts with geometric structures in generalized fifth-order recurrent Finsler spaces.

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