

Common Fixed Point Results in Super Metric Spaces

Priya Shahi

St. Andrew's College of Arts, Science and Commerce,

St. Dominic Road, Bandra (W), Mumbai, India

E-mail: priyashahi.research@gmail.com

(Received April 26, 2023, Accepted June 08, 2023)

Abstract: Recently, Karapinar and Khojasteh¹ proposed a new structure, namely the Super Metric Spaces, a revolutionary generalisation of metric spaces. In the framework of super metric spaces, we present a common fixed point result in this paper. To support this research, we consider an illustrative example.

Keywords: Metric space; fixed point; common fixed point; super metric space.

Mathematics Subject Classification (2000): 54H25, 47H10, 54E50.

1. Introduction

The Banach contraction principle², one of the basic theorems of metric fixed-point theory, has many applications in classical functional analysis as well as a variety of other domains, especially mathematical economics and psychology. Throughout the past many years, by reducing the contraction criteria and expanding the class of underlying metric space, writers have extended this result³⁻¹⁶.

A substantial amount of effort in the theory of fixed points has recently been focused on translating traditional or current results of metric fixed point theory from metric spaces to specific generalised metric spaces; for a recent survey, see Van An *et al.*⁷. Many generalisations of traditional metric spaces have surfaced in recent years. A b-metric space was first described by Czerwik¹⁷ in 1993. The fixed point theory subject in such spaces has since been covered in a number of papers^{4,13-14,18-20}, as well as any references therein. Hitzler and Seda²¹ in 2000 outlined the concept of dislocated metric spaces, where a point's self-distance need not be equal to zero.

Very recently, Karapinar and Khojasteh¹ proposed a new structure known as super metric space in order to remove the fixed point theory's congestion. Thereafter, Karapinar and Fulga²² studied in the framework of the super

metric space some rational contractions. Following this direction of research, we aim to investigate some common fixed point results in relation to super metric space, a fascinating generalisation of metric space.

2. Preliminaries

We begin with definition of super metric spaces well followed by some of their relevant properties.

Definition 2.1.¹ A nonempty set P shall exist. We state that $m: P \times P \rightarrow [0, +\infty)$ is super-metric or super metric if

- (i) if $m(p, q) = 0$, then $p = q$ for all $p, q \in P$,
- (ii) $m(p, q) = m(q, p)$, for all $p, q \in P$,
- (iii) there is a value $s \geq 1$ such that distinct sequences $(p_n), (q_n) \subset P$ exist for all $q \in P$ with $m(p_n, q_n) \rightarrow 0$ when n tends to infinity, such that

$$\lim_{n \rightarrow \infty} \sup m(q_n, q) \leq s \lim_{n \rightarrow \infty} \sup m(p_n, q)$$

Then, we call (P, m) a super metric space.

Definition 2.2.¹ Let (P, m) be a super metric space and $\{p_n\}$ a sequence in P . When $m(p_n, p) \rightarrow 0$ as n tends to ∞ , we say that $\{p_n\}$ converges to the point p in P .

Definition 2.3.¹ Let (P, m) be a super metric space and $\{p_n\}$ a sequence in P . The sequence (p_n) is said to be Cauchy, if and only if, $\lim_{n \rightarrow \infty} \sup \{m(p_n, p_m) : m > n\} = 0$.

Definition 2.4.¹ Let (P, m) be a super metric space and $\{p_n\}$ a sequence in P . The space (P, m) is said to be complete, if and only if, each Cauchy sequence converges in P .

Proposition 2.1.²² A convergent sequence's limit on a super metric space is distinct.

3. Main Results

With the proper tools, we can now illustrate our following main discovery:

Theorem 3.1. Suppose (P, m) is a complete super-metric space, $0 < \alpha < 1$, and suppose $k, l: P \rightarrow P$ are a pair of commutative mappings. Assume

- (i) $m(kp, kq) \leq \alpha.m(lp, lq)$, for all $p, q \in P$,
- (ii) The mappings k and l possess the property of weak compatibility, that is,

$$kp = lp, p \in P \Rightarrow klp = lkp.$$

(iii) $k(P) \subseteq l(P)$ and $l(P)$ is closed.

Then a unique common fixed point exists between the mappings k and l .

Proof: Let $p_0 \in P$ be any one of the elements such that $q_0 = kp_0$. From the fact that $k(P) \subseteq l(P)$, there exists $p_1 \in P$ such that

$$q_0 = kp_0 = lp_1.$$

Again, in view of $k(P) \subseteq l(P)$, there exists $p_2 \in P$ such that

$$q_1 = kp_1 = lp_2.$$

By induction, we can create two sequences $\{p_n\}, \{q_n\} \subset P$ by continuing this process defined by

$$q_n = kp_n = lp_{n+1}, \quad n \geq 0.$$

If $lp_0 = lp_1$, then p_0 is the coincidence point and this finishes the proof. So, assume that $lp_0 \neq lp_1$. Thus $m(lp_0, lp_1) > 0$. Therefore, we can assert without losing generality that $lp_n \neq lp_{n+1}$. So, $m(lp_n, lp_{n+1}) > 0, \forall n$ belonging to N . Consequently,

$$\begin{aligned} m(kp_n, kp_{n+1}) &\leq \alpha \cdot m(lp_n, lp_{n+1}) \\ &\leq \alpha^2 \cdot m(lp_{n-1}, lp_n) \\ &\vdots \\ &\leq \alpha^n m(lp_1, lp_0), \end{aligned}$$

Limiting the above inequality by both sides, we get

$$\lim_{n \rightarrow \infty} m(lp_n, lp_{n+1}) = 0$$

Now, imagine that, $a, b \in N$ and $a > b$. If $q_b = q_a$, we have

$$k^b p_0 = k^a p_0 \Rightarrow l(k^b p_0) = l(k^a p_0) \Rightarrow k^{a-b}(l(k^b p_0)) = l(k^b p_0)$$

Thus, we obtain that the point $l(k^b p_0)$ fixes the mapping k^{a-b} .

Also

$$k^b p_0 = k^a p_0 \Rightarrow k(k^b p_0) = k(k^a p_0) \Rightarrow k^{a-b}(k(k^b p_0)) = k(k^b p_0)$$

Thus, we obtain that the point $k(k^b p_0)$ fixes the mapping k^{a-b} .

Therefore, $k(k^b p_0) = l(k^b p_0)$ thereby proving that $k^b p_0$ is a coincidence point of the mappings k and l . Without losing the ability to generalize, we can assume that $q_b \neq q_a$. So,

$$\limsup_{b \rightarrow \infty} m(lp_n, lp_{n+2}) \leq s \limsup_{b \rightarrow \infty} m(lp_{n+1}, lp_{n+2})$$

Thus, since $\limsup_{b \rightarrow \infty} m(lp_b, lp_{b+2}) = 0$, we have

$$\limsup_{b \rightarrow \infty} m(lp_b, lp_{b+3}) \leq s \limsup_{b \rightarrow \infty} m(lp_{b+2}, lp_{b+3}) = 0.$$

Inductively, we get

$$\lim_{b \rightarrow \infty} \sup \{ m(lp_b, lp_a) : a > b \} = 0.$$

Therefore, $\{lp_a\}$ is a Cauchy sequence. The super metric space (P, d) completeness property implies that there is a r belonging to P such that

$$\lim_{b \rightarrow \infty} q_b = \lim_{b \rightarrow \infty} l p_b = \lim_{b \rightarrow \infty} k p_b = r.$$

The closed property of the set $l(P)$ implies the existence of some $x \in P$ such that $y = gx$. Let us claim $m(kz, lz) = 0$. On the contrary, assume that $m(kz, lz) > 0$. Note that

$$m(lp_{b+1}, kz) = m(kp_b, kz) \leq \alpha.m(lp_b, lz) \rightarrow 0 \quad (\text{as } b \rightarrow \infty).$$

Thus, $\lim m(lp_{b+1}, fz) = 0$. If there $N > 0$ such that for all $b > N$, $lp_{b+1} = lz$, which implies in view of (6) that $m(lz, kz) = 0$ and so the coincidence point of the mappings is z . Otherwise, suppose that for all $b \in \mathbb{N}$, $lp_b \neq lz$. Thus, we have

$$m(lz, kz) \leq s \lim_{b \rightarrow \infty} \sup m(lp_{b+1}, kz),$$

and hence $m(lp, kp) = 0$, which leads to a contradiction. Consequently, $x = gz = fz$.

We will now demonstrate that the common fixed point of the mappings is x . According to the weak compatibility feature of the mappings, we infer that $klz = lkz$, i.e.,

$$kx = lx.$$

Now, we claim that $m(x, kx) = 0$. On contrary, assume that $m(x, kx) > 0$. So,

$$m(x, kx) = m(kz, kx) \leq \alpha.m(lz, lx) = \alpha.m(x, kx) < m(x, kx),$$

which leads to a contradiction thereby implying that $x = kx = lx$. The distinctiveness of the common fixed point will now be demonstrated. Assume that $m(x, o)$ is bigger than zero and that o is another fixed point that k and l share. So

$$m(x, o) = m(fx, fo) \leq \alpha.m(lx, lo) = \alpha.m(x, o) < m(x, o),$$

which is a contradiction. In light of this, we deduce that the mappings have a singular common fixed point.

Example 3.1. Let $P = [1, 9]$ and define

$$m(p, q) = \begin{cases} pq, & \text{if } p \neq q \\ 0, & \text{if } p = q. \end{cases}$$

Let $(p_b), (q_b)$ be two distinct sequences such that $m(p_b, q_b) \rightarrow 0$ as $n \rightarrow \infty$. Since the sequences are distinct, we have $m(p_b, q_b) = p_b q_b \rightarrow 0$, and we can choose $q_b \rightarrow 0$ and $p_b \rightarrow u$ as $b \rightarrow \infty$, where $q \in P$. Moreover, for any $q \in P$

$$\begin{aligned}
\lim_{b \rightarrow \infty} \sup m(q_b, q) &= \lim_{b \rightarrow \infty} \sup q_b \cdot q \\
&= 0 \\
&\leq s \lim_{b \rightarrow \infty} \sup m(p_n, q) \\
&= \lim_{b \rightarrow \infty} \sup p_b q \\
&= u \cdot q
\end{aligned}$$

and it concludes that (P, m) is a super metric space. Now consider $k: P \rightarrow P$ as follows

$$k(p) = \begin{cases} 1, & \text{if } p \neq 9, \\ \frac{1}{8}, & \text{if } p = 9. \end{cases}$$

Choose $\alpha = \frac{1}{2}$, $p \neq 9$ and $q = 9$, we have

$$m(kp, kq) = m\left(1, \frac{1}{8}\right) = \frac{1}{8} \leq \frac{1}{2} \cdot (2p - 1)(2q - 1),$$

because $1 \leq p, q \leq 9$. Since the other scenarios are simple, the mappings share a single common fixed point $p = 1$.

4. Conclusions

In recent years, several results in fixed point theory have been rediscovered or have overlapped with those already known in the field of theory of fixed points. Additionally, comparable versions have been published as a result of some untrue presumptions. The squeeze on theory is the primary cause of these circumstances. For extending the theory concerning fixed points, the notion of super metric space is perhaps a very strong choice. A fixed point theorem for this novel structure is presented in this study. The metric fixed point theory's congestion, in our opinion, will be prioritized by a thorough analysis of this structure.

References

1. E. Karapinar and F. Khojasteh; Super Metric Spaces, *Filomat* in press.
2. S. Banach; Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae* 3 (1922), 133-181.
3. T. Abdeljawad, E. Karapinar and K. Tas; A generalized contraction principle with control functions on partial metric spaces, *Comput. Math. Appl.* 63 (2012), 716-719.
4. M. Akkouchi; Common fixed point theorems for two self-mappings of a b-metric space under an implicit relation, *Hacet. J. Math. Stat.* 40 (6)

- (2011), 805–810.
5. B. Alqahtani, A. Fulga and E. Karapinar; Sehgal Type Contractions on b-Metric Space, *Symmetry* 10 (2018), 560.
 6. B. Alqahtani, A. Fulga, E. Karapinar and V. Rakocevic; Contractions with rational in- equalities in the extended b-metric space, *J. Inequal. Appl.* 2019 (2019), 220.
 7. T. V. An, N. Van Dung, Z. Kadelburg and S. Radenovic; Various generalizations of metric spaces and fixed point theorems, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RAC- SAM* 109(1) (2015), 175-198.
 8. H. Aydi, C.M. Chen and E. Karapinar; Interpolative C'iric' Reich Rus type contractions via the Branciari distance, *Mathematics* 7 (2019), 84.
 9. M. Jleli and B. Samet; A new generalization of the Banach contraction principle, *J. Inequal. Appl.* 38 (2014).
 10. E. Karapinar; Recent advances on metric fixed point theory: A review, *Appl. Comput. Math.* in press.
 11. E. Karapinar; A note on a rational form contraction with discontinuities at fixed points, *Fixed Point Theory* 21 (2020), 211-220.
 12. E. Karapinar, D. O' Regan and B. Samet; On the existence of fixed points that belong to the zero set of a certain function, *Fixed Point Theory Appl.* 2015 (2015) 1-14.
 13. W. Kirk, N. Shahzad; b-Metric spaces, *Fixed Point Theory in Distance Spaces*, (2014) 113-131.
 14. B. Popovic, S. Radenovic, S. Shukla; Fixed point results to tvs-cone b-metric spaces, *Gulf J. Math.* 1 (2013) 51–64.
 15. B. Samet, C. Vetro and P. Vetro; Fixed point theorem for α - ψ contractive type mappings, *Nonlinear Anal.* 75 (2012) 2154-2165.
 16. D. Wardowski; Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.* 94 (2012).
 17. S. Czerwik; Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostrav.* 1 (1993), 5-11.
 18. V. Berinde; Sequences of operators and fixed points in quasi metric spaces, *Stud. Univ. Babes-Bolyai Math.* 41 (1996), 23-27.
 19. M. Boriceanu, M. Bota and A. Petrusel; Multivalued fractals in b-metric spaces, *Cent. Eur.J. Math.* 8(2) (2010), 367-377.
 20. S. Czerwik, K. Dlutek and S.L. Singh; Round-off stability of iteration procedures for set- valued operators in b-metric spaces, *J. Natur. Phys. Sci.* 11 (2007), 87-94.
 21. P. Hitzler and A.K. Seda; Dislocated topologies, *J. Electr. Eng.* 51(12) (2000), 3-7.
 22. E. Karapinar and A. Fulga; Contraction in Rational Forms in the Framework of Super Metric Spaces, *Mathematics* 10 (2022), 3077.