

Invariant Submanifolds of a Manifold Admitting

$F(a_1, a_2, a_3, \dots, a_n)$ - Structure

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Abstract: The purpose of this paper is to study invariant submanifolds of an n -dimensional manifold M endowed with a structure satisfying $a_n F^n + a_{n-1} F^{n-1} + a_{n-2} F^{n-2} + \dots + a_2 F^2 + a_1 F = 0$, where n is a fixed positive integer greater than 2. The case when n is odd (≥ 3) has been considered in this paper. We have shown that an invariant submanifold $\bar{M}$, embedded in the manifold M in such a way that the complementary distribution $D\bar{M}$ is never tangential to the invariant submanifold $\Phi(\bar{M})$, admits an almost complex structure. Some theorems regarding the integrability conditions of the structure have also been proved.

Key Words: Invariant Submanifolds, $F(a_1, a_2, a_3, \dots, a_n)$ -Structure, Distributions, Nijenhuis Tensor, Integrability Conditions.

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1. Introduction

Let F be a non-zero tensor field of type $(1, 1)$ and of class C^∞ on an n -dimensional manifold M such that

$$(1.1) \quad a_n F^n + a_{n-1} F^{n-1} + a_{n-2} F^{n-2} + \dots + a_2 F^2 + a_1 F = 0,$$

where n is a fixed positive integer greater than 2 and $a_1, a_2, a_3, \dots, a_n$ are scalars. Such structure on M is called an $F(a_1, a_2, a_3, \dots, a_n)$ - structure of rank r , if the rank of F is constant everywhere and equal to r .

Let the operators on M be defined as follows³

$$(1.2) \quad l = -\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1}, \quad m = l_n + \frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1}$$

where l_n denotes the identity operator on M . For the operators defined by (1.2), we have

$$(1.3) \quad l + m = I_n, \quad l^2 = l, \quad m^2 = m, \quad lm = ml = 0.$$

For F satisfying (1.1), there exist complementary distributions D_l and D_m corresponding to the projection operators l and m respectively. If $\text{rank}(F) = \text{constant}$ on M , and equal to r , then $\dim D_l = r$ and $\dim D_m = (n-r)$.

The following results for such structure follow easily:

$$(1.4a) \quad Fl = lF = F, \quad Fm = mF = 0,$$

$$(1.4b) \quad \left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right) l = -l, \left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right) m = 0.$$

Thus $\left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right)^{1/2}$ acts on D_l as an almost complex structure operator and on D_m as a null operator.

2. Invariant submanifolds

Let $\bar{M}$ be a differentiable manifold embedded differentiably as a submanifold in an n -dimensional C^∞ Riemannian manifold M with $F(a_1, a_2, a_3, \dots, a_n)$ -structure and we denote the embedding by $\Phi: \bar{M} \rightarrow M$. Let us denote the Jacobian map of Φ by $B: T(\bar{M}) \rightarrow T(M)$ where $T(\bar{M})$ and $T(M)$ are tangent bundles of $\bar{M}$ and M respectively. $T(\bar{M}, M)$ is the set of all vectors tangent to the submanifold $\Phi(\bar{M})$. It is well known that $B: T(\bar{M}) \rightarrow T(\bar{M}, M)$ is an isomorphism².

Let X and Y be two C^∞ vector fields tangent to $\Phi(\bar{M})$. Let BX and BY be the local extensions of X and Y . The restriction $[BX, BY]_{\bar{M}}$ is determined independently of the choice of these local extensions BX and BY . Therefore we can define

$$(2.1) \quad [X, Y] = [BX, BY]_{\bar{M}}$$

Since B is an isomorphism, it is easy to see that $[BX, BY] = B[X, Y]$ for all $X, Y \in T(\bar{M})$. We denote by G the Riemannian metric tensor of M and put

$$(2.2) \quad g(X, Y) = G(BX, BY) \text{ for all } X, Y \in T(\bar{M}),$$

where g is induced in $\bar{M}$.

Definition 2.1: We say that $\bar{M}$ is an invariant submanifold of M if

- (i) the tangent space $T_p(\Phi(\bar{M}))$ of the submanifold $\Phi(\bar{M})$ is invariant by the linear mapping F at each point p of $\Phi(\bar{M})$,
- (ii) for each $X \in T(\bar{M})$, we have

$$(2.3) \quad \left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right)^{1/2} (BX) = B \left(\left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right)^{1/2} X \right),$$

where f is a $(1, 1)$ -tensor field defined in $\bar{M}$.

Let $\bar{M}$ be an invariant submanifold, then we have

$$(2.4a) \quad F(BX) = B(fX),$$

$$(2.4b) \quad \left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right)^{1/2} (BX) = B \left(\left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right)^{1/2} X \right).$$

There are two cases for any invariant submanifold $\overline{M}$ which are as follows:

Case 1. The distribution D_m is never tangential to $\Phi(\overline{M})$.

Case 2. The distribution D_m is always tangential to $\Phi(\overline{M})$.

Let us consider Case 1 and assume that no vector field of the type mBX where $BX \in T(\Phi(\overline{M}))$, is tangential to $\Phi(\overline{M})$.

Theorem 2.1: Let M admits $F(a_1, a_2, a_3, \dots, a_n)$ -structure and $\overline{M}$ is the invariant submanifold of M . If the distribution D_m is never tangential to $\Phi(\overline{M})$ and f is the induced (1,1) tensor field on $\overline{M}$ then

$$\left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right)^{1/2} \text{ gives an almost complex structure on } \overline{M}.$$

Proof : Operating (2.4b) by $\left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right)^{1/2}$, we obtain

$$(2.5) \quad \left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right)^{1/2} \left(\left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right)^{1/2} (BX) \right) \\ = \left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right)^{1/2} \left(B \left(\left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right)^{1/2} X \right) \right).$$

Making use of (2.4a) in (2.5), we get

$$(2.6) \quad \left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right) (BX) = B \left(\left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right) X \right).$$

In order to show that vector fields of the type BX belong to the distribution D_b , we suppose that $m(BX) \neq 0$, then using (1.2), we have

$$(2.7) \quad m(BX) = \left(I_n + \frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right) (BX) \\ = BX + \left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right) (BX).$$

This, in view of (2.6), becomes

$$(2.8) \quad m(BX) = BX + B \left(\left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right) X \right)$$

$$= B \left[X + \left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right) X \right].$$

This contrary to our assumption, shows that $m(BX)$ is tangential to $\Phi(\overline{M})$. Thus $m(BX) = 0$.

Also, in view of (1.4b), (1.3) and (2.6), we obtain

$$\begin{aligned} B \left(\left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right) X \right) &= \left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right) (BX) \\ &= -l(BX) = -(I_n - m)BX = -BX + m(BX). \end{aligned}$$

Thus

$$(2.9) \quad B \left(\left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right) X \right) = -BX.$$

Since B is an isomorphism, we get

$$(2.10) \quad \left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right) = -I_n.$$

Let $\mathfrak{J}(M)$ be the ring of real valued differentiable functions on M , and let $\chi(M)$ be the module of derivatives of $\mathfrak{J}(M)$. Then $\chi(M)$ is Lie algebra over the real numbers and the elements of $\chi(M)$ are called vector fields. M is equipped with $(1,1)$ -tensor field F which is a linear map

$$(2.11) \quad F : \chi(M) \rightarrow \chi(M)$$

Let M be of degree n and n be a positive odd integer greater than 2. Then we consider a positive definite Riemannian metric with respect to which D_l and D_m are orthogonal so that

$$(2.12) \quad G(BX, BY) = G(HBX, HBY) + G(mBX, BY)$$

$$\text{where } H = \left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right)^{1/2} \text{ for all } BX, BY \in \chi(M).$$

Definition 2.2: The induced metric g defined by (2.2) is Hermitian if the following is satisfied :

$$(2.13) \quad g(HX, HY) = g(X, Y)$$

$$\text{where } H = \left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right)^{1/2}.$$

Theorem 2.2: If $\overline{M}$ is an invariant submanifold of $F(a_1, a_2, \dots, a_n)$ -structure manifold M such that distribution D_m is never tangential to $\Phi(\overline{M})$ and the Riemannian metric G on M is defined by (2.12) then the induced metric g of $\overline{M}$ defined by (2.2) is Hermitian.

Proof. In view of (2.2) and (2.13), we obtain

$$\begin{aligned}
 (2.14) \quad & g \left(\left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right)^{1/2} X, \left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right)^{1/2} Y \right) \\
 &= G \left(B \left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right)^{1/2} X, B \left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right)^{1/2} Y \right)
 \end{aligned}$$

Applying (2.4) and (2.12) in (2.14), we get

$$\begin{aligned}
 (2.15) \quad & g \left(\left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right)^{1/2} X, \left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right)^{1/2} Y \right) \\
 &= G \left(\left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right)^{1/2} BX, \left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right)^{1/2} BY \right) \\
 &= G(BX, BY) - G(mBX, BY)
 \end{aligned}$$

Since the distribution D_m is never tangential to $\Phi(\overline{M})$, on using (2.2) we get

$$\begin{aligned}
 (2.16) \quad & g \left(\left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right)^{1/2} X, \left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right)^{1/2} Y \right) \\
 &= G(BX, BY) = g(X, Y)
 \end{aligned}$$

Now, we consider the second case and assume that the distribution D_m is always tangential to $\Phi(\overline{M})$. In view of Case 2, we have $m(BX) = BX^*$, where $X^* \in T(\overline{M})$ for some $X^* \in T(\overline{M})$.

We define (1,1)- tensor fields $\overline{m}$ and $\overline{l}$ in $\overline{M}$ as follows :

$$(2.17a) \quad \overline{l} = -\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1}, \quad \overline{m} = I + \frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1}$$

$$(2.17b) \quad \overline{m} X = X^*, \quad m(BX) = B(\overline{m} X).$$

Theorem 2.3: We have

$$(2.18) \quad B(\overline{l} X) = l(BX)$$

Proof: In view of (2.17a), equation (2.18) assumes the following form:

$$\begin{aligned}
 (2.19) \quad & B(\overline{l} X) = B \left(- \left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right) X \right) \\
 &= -B \left(\left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1} \right) X \right).
 \end{aligned}$$

Making use of (2.6) and (2.15) in (2.19), we get

$$(2.20) \quad B(\overline{l} X) = - \left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1} \right) (BX) = l(BX).$$

Theorem 2.4: For $\bar{l}$ and $\bar{m}$ satisfying (2.17a), we have

$$(2.21) \quad \bar{l} + \bar{m} = I, \quad \bar{l}^2 = \bar{l}, \quad \bar{m}^2 = \bar{m}$$

Proof: From (1.3), we have $l+m = I_n$, which can be written as $(l+m)BX=BX$. Thus we have

$$(2.22) \quad lBX + mBX = BX$$

which in view of (2.17b) and (2.18), becomes

$$(2.23) \quad B(\bar{l}X) + B(\bar{m}X) = B(\bar{l} + \bar{m})X = BX.$$

Therefore $\bar{l} + \bar{m} = I$, since B is an isomorphism. Proof of the other relations follows in the similar manner.

Theorem 2.4 shows that $\bar{l}$ and $\bar{m}$ defined by (2.17a), are complementary projection operators on $\bar{M}$.

Theorem 2.5: If $\bar{M}$ is an invariant submanifold of $F(a_1, a_2, \dots, a_n)$ -structure manifold M such that distribution D_m is always tangential to $\bar{M}$, then there exists an induced $f(a_1, a_2, \dots, a_n)$ -structure manifold which admits a similar Riemannian metric g satisfying

$$(2.24) \quad g(X, Y) = g(hX, hY) + g(\bar{m}X, Y).$$

Proof: From (2.4b), we get

$$(2.25) \quad B\left(\left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1}\right)^{1/2} X\right) = \left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} + \dots + a_2 F}{a_1}\right)^{1/2} (BX).$$

Further

$$(2.26) \quad B(fX) = F(BX),$$

which in view of (1.1) and (2.4a), yields

$$(2.27) \quad B(fX) = B\left(-\left(\frac{a_n f^n + a_{n-1} f^{n-1} + \dots + a_2 f^2}{a_1}\right) X\right).$$

This shows that f defines an $f(a_1, a_2, \dots, a_n)$ -structure satisfying

$$(2.28) \quad a_n f^n + a_{n-1} f^{n-1} + \dots + a_2 f^2 + a_1 f = 0.$$

In consequence of (2.2), (2.4b) and (2.12), we obtain

$$(2.29) \quad \begin{aligned} g(hX, hY) + g(\bar{m}X, Y) &= G(BhX, BhY) + G(B\bar{m}X, BY) \\ &= G(HBX, HBY) + G(mBX, BY) \\ &= G(BX, BY) \end{aligned}$$

$$\text{where } h = \left(\frac{a_n f^{n-1} + a_{n-1} f^{n-2} + \dots + a_2 f}{a_1}\right)^{1/2},$$

which in view of the fact that B is an isomorphism, gives

$$(2.30) \quad g(hX, hY) + g(\bar{m}X, Y) = g(X, Y).$$

3. Integrability Conditions

The Nijenhuis tensor N of the type (1,2) of F satisfying (1.1) in M is given by⁴

$$(3.1) \quad N(BX, BY) = [F(BX), F(BY)] - F[F(BX), BY] - F[BX, F(BY)] + F^2[BX, BY]$$

Now we prove the following:

Theorem 3.1: *The Nijenhuis tensor N and $\bar{N}$ of M and $\bar{M}$ satisfy the relation*

$$(3.2) \quad N(BX, BY) = B \bar{N}(X, Y).$$

Proof: From (3.1), we get

$$N(BX, BY) = [F(BX), F(BY)] - F[F(BX), BY] - F[BX, F(BY)] + F^2[BX, BY]$$

which in view of (2.4a), becomes

$$(3.4) \quad \begin{aligned} N(BX, BY) &= [BfX, BfY] - F[BfX, BY] - F[BX, BfY] + F^2[BX, BY] \\ &= B[fX, fY] - FB[fX, Y] - FB[X, fY] + F^2B[X, Y] \\ &= B[fX, fY] - Bf[fX, Y] - Bf[X, fY] + Bf^2[X, Y] \\ &= B \bar{N}(X, Y) \end{aligned}$$

Theorem 3.2: *The following identities hold:*

$$(3.5) \quad \begin{aligned} B \bar{N}(\bar{l}X, \bar{l}Y) &= N(lBX, lBY), \quad B \bar{N}(\bar{m}X, \bar{m}Y) = N(mBX, mBY), \\ B \{ \bar{m} \bar{N}(X, Y) \} &= m N(BX, BY). \end{aligned}$$

Proof: The proof of (3.5) follows from theorem 3.1 and equations (1.4a), (2.4b), (2.17b) and (3.2).

For f satisfying (2.28), there exists complementary distributions D_l and D_m corresponding to the projection operators $\bar{l}$ and $\bar{m}$ in $\bar{M}$ given by (2.17a). Then in view of the integrability condition of $f(a_1, a_2, \dots, a_n)$ -structure the following theorems hold.

Theorem 3.3: *If D_l and D_m are integrable in M , then D_l and D_m are also integrable in $\bar{M}$.*

Theorem 3.4: *If $F(a_1, a_2, \dots, a_n)$ -structure is integrable in M , the induced structure is also integrable in $\bar{M}$.*

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