

Conversion of Indirect Input Data into Good Quality Direct Input Data for Mathematical Models of Blood Flow in an Animal CVS

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Abstract: The aim of this paper is to convert flow rate profiles of animal CVS [McDonald¹, Pedley² and Milnor³] i.e. indirect input data to pressure gradient (direct input data) using finite difference implicit technique in inverse Womersley problem. Further pressure gradient profiles are converted into mathematical form in terms of Fourier series and shown through Figures 1 to 9. Fourier coefficients of pressure gradient profiles are given through Tables-1 to 9. The computer program is prepared using the finite difference implicit technique and is presented through Appendix.

Keywords: Pressure gradient profiles, animal cardio-vascular system (CVS), blood, flow rate profiles, Inverse Womersley problem.

2000 Mathematics Subject Classification No.: **76ZXX**

1. Introduction

Many mathematical models [Womersley⁴, Sud and Sekhon⁵⁻⁷, Chaturani and Palanisamy⁸⁻¹⁰ etc.] have been presented on pulsatile flow of blood. The input data in these models is quite different from the actual data. The actual input data is either scarce or is in the form which is difficult to use.

It is noticed that many mathematical models [Sud and Sekhon⁵⁻⁷, Chaturani and Palanisamy⁸⁻¹⁰] required the pressure gradient profiles as the input data. A search of the literature shows the existence of the pressure gradient profiles for very few locations in animal cardiovascular system. The direct data is so scarce that many researchers are even unaware of it. For instance, Sud and Sekhon⁵⁻⁶ needed the pressure gradient profiles (direct data) for their mathematical models; perhaps, they could not locate any; so they obtained pressure gradient profile from average flow rate (by an incorrect method). Many more

examples of this type can be given. Though the profiles for pressure-time, velocity-time and flow rate-time are available for many locations. But these profiles cannot be used directly as the input data, in the above-mentioned mathematical models. Hence this type of data will be referred as indirect data.

Womersley⁴ gave a method for the calculation of velocity, flow rate in arteries for known pressure gradient. Tsangaris and Stergiopoulos¹¹ discussed the inverse Womersley problem for pulsatile flow in a straight rigid tube. They converted given flow rate as a function of time into the pressure gradient profiles along the pipe axis numerically. Since the basic equations were non-linear, therefore they obtained numerical solutions by using a finite difference technique.

The aim of the present paper is to convert flow rate profiles into pressure gradient profiles numerically and to present these profiles into mathematical form, which could be directly used in mathematical models of blood flow in animal CVS as input data.

2. Formulation of the Problem

Assuming blood as a viscous incompressible and Newtonian fluid, the tube wall to be rigid and its length be very long compared to its diameter, flow to be symmetric about tube axis (z-axis), then the governing equation of motion is given by

$$(1) \quad \rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial z} + \mu_f \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right).$$

The non-dimensional form of the equation (1) can be written as

$$(2) \quad \alpha^2 \frac{\partial u^*}{\partial t^*} = -\frac{\partial p^*}{\partial z^*} + \frac{1}{r^*} \frac{\partial}{\partial r^*} \left[r^* \frac{\partial u^*}{\partial r^*} \right],$$

where $u^* = \frac{u}{\omega R}$, $r^* = \frac{r}{R}$, $t^* = t \omega$, $p^* = \frac{p}{\mu \omega}$, $z^* = \frac{z}{R}$,

$\alpha = R \sqrt{(\omega/\mu) \rho}$ is the Womersley parameter, ω is the angular frequency of the pulse, μ_f is the viscosity coefficient, R is tube radius and ρ is the density of the blood.

The initial and boundary conditions in non-dimensional form are

$$(3) \quad \frac{\partial u^*}{\partial r^*}(0, t^*) = 0, \quad u^*(1, t^*) = 0 \quad \text{and} \quad u^*(r^*, 0) = V^*(0)$$

where asterisk (*) denotes the non-dimensional parameters.

3. Method of Solution

Integrating along the radius of the pipe, we get

$$(4) \quad \alpha^2 \frac{\partial}{\partial t^*} \int_0^1 2r^* u^*(r^*, t^*) dr^* = -\frac{\partial p^*}{\partial z^*} \int_0^1 2r^* dr^* + 2 \left(\frac{\partial}{\partial r^*} \int_0^1 r^* \frac{\partial u^*}{\partial r^*} dr^* \right),$$

which can be written as

$$(5) \quad \alpha^2 \frac{\partial}{\partial t^*} \left(2 \int_0^1 r^* u^*(r^*, t^*) dr^* \right) = - \frac{\partial p^*}{\partial z^*} + 2 \frac{\partial u^*}{\partial r^*} \Big|_{r^*=1}.$$

The non-dimensional wall shear τ_w^* and flow rate $V^*(t)$ can be defined as

$$(6) \quad \tau_w^* = \frac{\tau_w}{\mu \omega} = \frac{\partial u^*}{\partial r^*} \Big|_{r^*=1},$$

$$(7) \quad V^*(t) = \frac{V}{\pi R^3 \omega} = 2 \int_0^1 r^* u^*(r^*, t^*) dr^*.$$

Using equations (6) and (7) into the equation (5), we get

$$(8) \quad \alpha^2 \frac{dV^*}{dt^*} = - \frac{\partial p^*}{\partial z^*} + 2 \tau_w^*.$$

Eliminating the pressure gradient term from equations (2) and (8), we get

$$(9) \quad \alpha^2 \frac{\partial u^*}{\partial t^*} = \alpha^2 \frac{dV^*}{dt^*} - 2 \frac{\partial u^*}{\partial r^*} \Big|_{r^*=1} + \frac{1}{r^*} \frac{\partial}{\partial r^*} \left[r^* \frac{\partial u^*}{\partial r^*} \right].$$

The equation (9), a parabolic partial differential equation with the initial and boundary conditions (3), is solved using a finite difference implicit technique [Tsangaris and Stergiopoulos¹¹].

The equidistant grid is used i.e. N equally spaced points along the radius (spatial axis) and NT equally spaced points along the period T . Using first order finite difference approximations for the time derivatives of the velocity and flow rate and second order approximation for the spatial derivatives of the velocity, the governing equation (9) can be written as

$$(10) \quad \alpha^2 \frac{u_i^{j+1} - u_i^j}{\Delta t} = \alpha^2 \frac{V^{j+1} - V^j}{\Delta t} - 2 \frac{3u_N^{j+1} - 4u_{N-1}^{j+1} + u_{N-2}^{j+1}}{2h} \\ + \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} + \frac{1}{r_i} \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2h},$$

where N denotes point at the wall, h is the spatial step $1/(N-1)$, Δt is the time step and

$$r_i = (i-1)h.$$

The above system of equations can be written in matrix form as

$$(11) \quad A_{ik} u_k = D_i,$$

where A_{ik} is a non-symmetric quadratic matrix. The elements of the rows $A_{1,k}$ and $A_{N,k}$ are derived from the boundary conditions (3).

The system of the equations (11) is solved by using Gauss elimination method and iteration method [Kreyszig¹²] to obtain the velocity profiles. After getting velocity profile, the pressure gradient can be obtained by

$$(12) \quad -\frac{\partial p^*}{\partial z^*} = \alpha^2 \frac{dV^*}{dt^*} - 2 \frac{\partial V^*}{\partial r^*} \Big|_{r^*=1}$$

A computer program in C++ language is developed for above-mentioned procedure and presented in Appendix.

4. Flow Rate Profiles of Animals

Available experimental flow rate profiles of animals in literature are given below:

1. Flow rate profile in ascending aorta of a dog [Milnor³],
2. Flow rate profile in main pulmonary artery of a dog [Milnor³],
3. Flow rate profile in renal artery of a dog [Milnor³],
4. Flow rate profile in femoral artery of a dog [Milnor³],
5. Flow rate profile in arteriole of a dog [Milnor³],
6. Flow rate profile in inferior vena cava of a dog [Milnor³],
7. Flow rate profile in terminal pulmonary vein of a dog [Milnor³],
8. Flow rate profile in left circumflex coronary artery of a dog [Pedley²],
9. Flow rate profile in femoral artery of a dog [McDonald¹].

Above nine flow rate profiles for eight different locations of dog's CVS are converted into pressure gradient profiles.

5. Results and Discussion

In the available literature [McDonald¹, Pedley² and Milnor³], the nine flow rate profiles are located. These indirect data (flow rate profiles) are converted into direct data (pressure gradient profiles) with the help of computer program, developed here in C++ language and presented through Appendix. The mathematical forms of pressure gradient profiles are also obtained and presented through Figures 1 to 9. It is observed that except at sharp points of the profiles, the values obtained from mathematical model, developed here, and the pressure gradient profiles obtained from experimental flow rate profiles through computer program, have an error of the order of 10%. Fourier series coefficients and values of pressure gradient profiles are given through Tables-1 to 9 in M.K.S. unit.

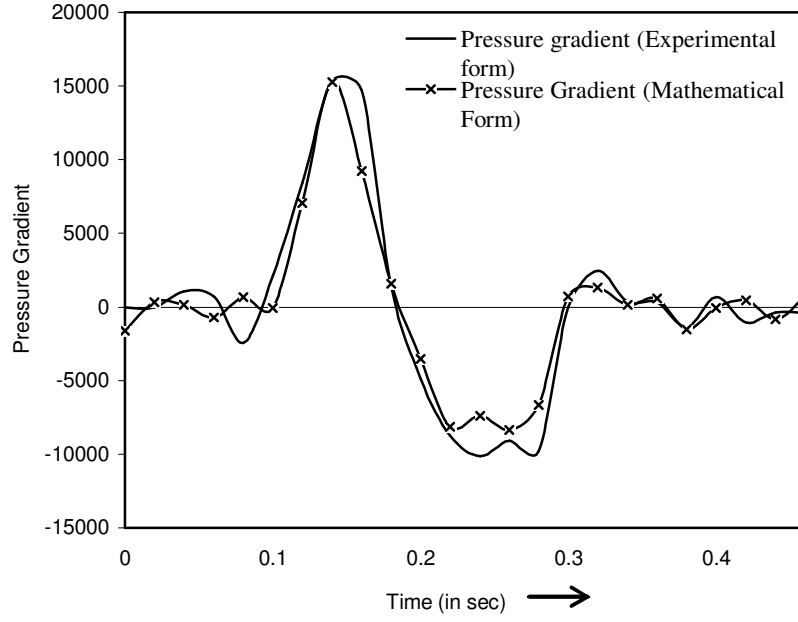


Fig-1: Comparison of pressure gradient profile (direct data), obtained from flow rate profile (indirect data) [Milnor³], with the mathematical form of pressure gradient profile for Ascending aorta of a dog.

Table-1. The values of Fourier coefficients A_n and B_n ($\text{kg } m^{-2} s^{-2}$) for the pressure gradient profile.

$A_0 = -25.3889$		
n	A_n	B_n
1	-224.7653	4644.1034
2	39.4979	-5599.0495
3	359.3647	2336.4234
4	-1756.9440	501.7115
5	1001.5976	-590.8469
6	263.4640	224.0376
7	-1346.4898	-169.6519

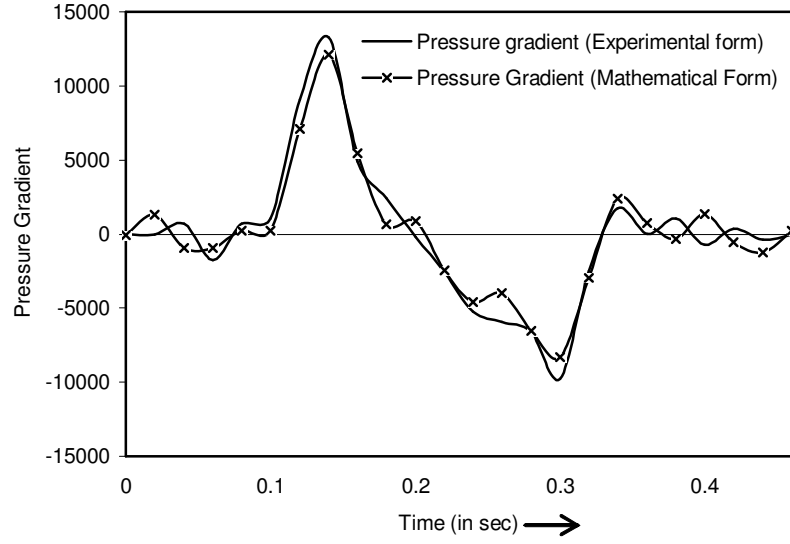


Fig. 2. Comparison of pressure gradient profile (direct data), obtained from flow rate profile (indirect data) [Milnor³], with the mathematical form of pressure gradient profile for main pulmonary artery of a dog.

Table-2. The values of Fourier coefficients A_n and B_n ($\text{kg m}^{-2} \text{s}^{-2}$) for the pressure gradient profile.

$A_0 = -14.1918$		
n	A_n	B_n
1	-1040.8042	4298.6187
2	1652.1349	-3211.8622
3	1524.2082	-141.4443
4	-1793.6114	140.3832
5	-574.2571	-504.5952
6	562.1237	1646.7077
7	-391.3191	97.3025

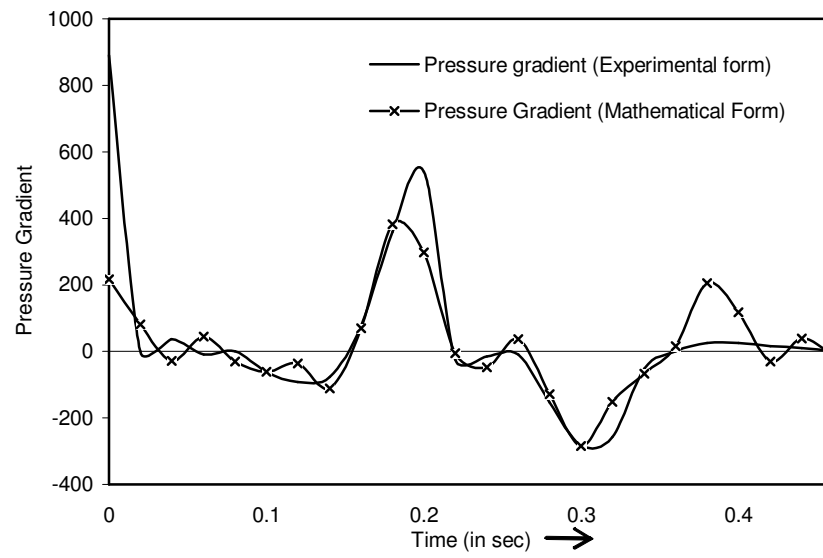


Fig. 3. Comparison of pressure gradient profile (direct data), obtained from flow rate profile (indirect data) [Milnor³], with the mathematical form of pressure gradient profile for renal artery of a dog.

Table-3. The values of Fourier coefficients A_n and B_n ($\text{kg m}^{-2} \text{s}^{-2}$) for the pressure gradient profile.

$A_0 = 10.9827$		
n	A_n	B_n
1	-48.3401	49.6844
2	158.3857	32.4721
3	-15.8461	-3.6605
4	68.8233	-54.4580
5	-42.4270	30.0229
6	65.0838	-0.3763
7	19.7012	10.1088

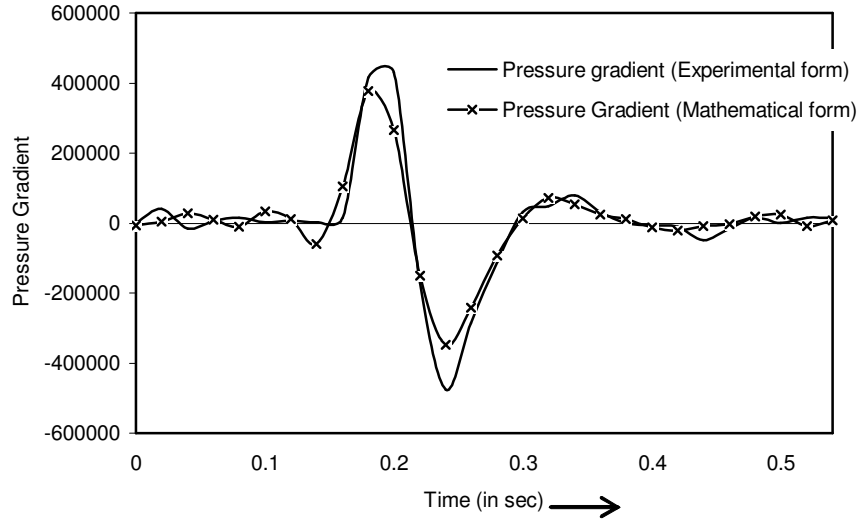


Fig. 4. Comparison of pressure gradient profile (direct data), obtained from flow rate profile (indirect data) [Milnor³], with the mathematical form of pressure gradient profile for femoral artery of a dog.

Table-4. The values of Fourier coefficients A_n and B_n ($\text{kg m}^{-2} \text{s}^{-2}$) for the pressure gradient profile.

$A_0 = 2939.7568$		
n	A_n	B_n
1	14261.7287	43388.9319
2	-41073.9531	-82203.1226
3	37168.2721	114779.0040
4	-39340.2290	-82525.6547
5	45143.6938	47793.8296
6	-42642.0518	-5319.5727
7	17536.4016	-12320.8698

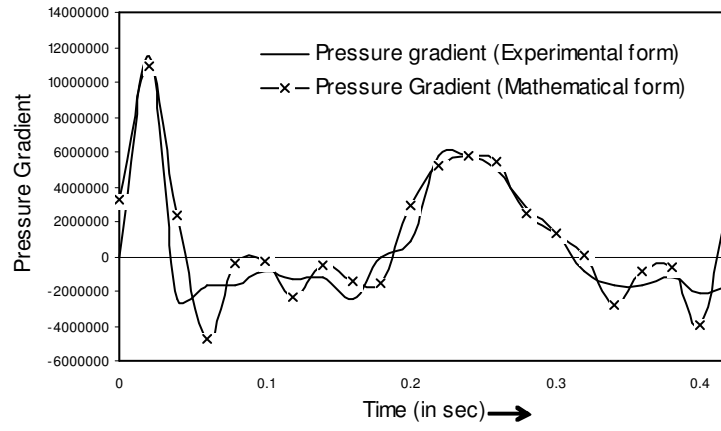


Fig. 5. Comparison of pressure gradient profile (direct data), obtained from flow rate profile (indirect data) [Milnor³], with the mathematical form of pressure gradient profile for arteriole of a dog.

Table-5. The values of Fourier coefficients A_n and B_n ($\text{kg m}^{-2} \text{s}^{-2}$) for the pressure gradient profile.

$A_0 = 1225151.9770$		
n	A_n	B_n
1	-312712.0947	-549398.0914
2	1382525.1648	1289137.9361
3	633660.2044	-43631.4241
4	507217.3806	891790.1938
5	-19748.1839	731335.9586
6	-186594.8865	985458.1254
7	-501244.3246	724744.9682

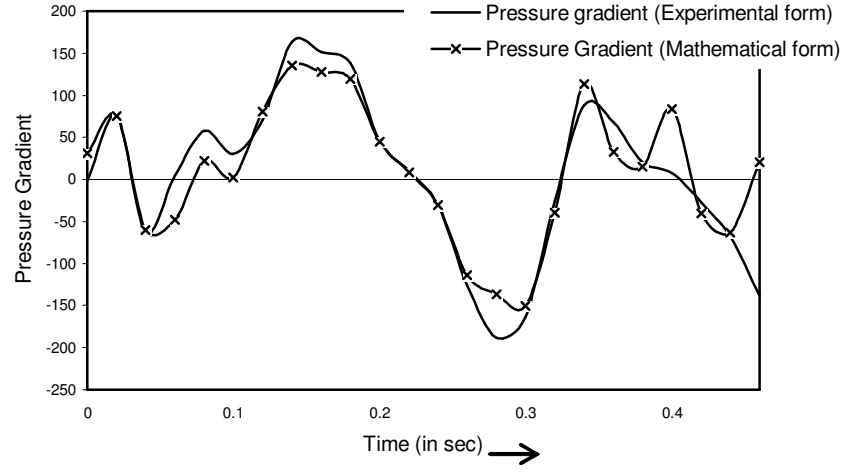


Fig. 6. Comparison of pressure gradient profile (direct data), obtained from flow rate profile (indirect data) [Milnor³], with the mathematical form of pressure gradient profile for inferior vena cava of a dog.

Table-6. The values of Fourier coefficients A_n and B_n ($\text{kg m}^{-2} \text{s}^{-2}$) for the pressure gradient profile.

$A_0 = 11.1039$		
n	A_n	B_n
1	-22.7651	60.4619
2	65.0421	-58.8245
3	6.8591	-19.3701
4	-18.4574	-2.4647
5	-2.4065	20.0700
6	-1.9436	26.7610
7	-6.1193	27.0940

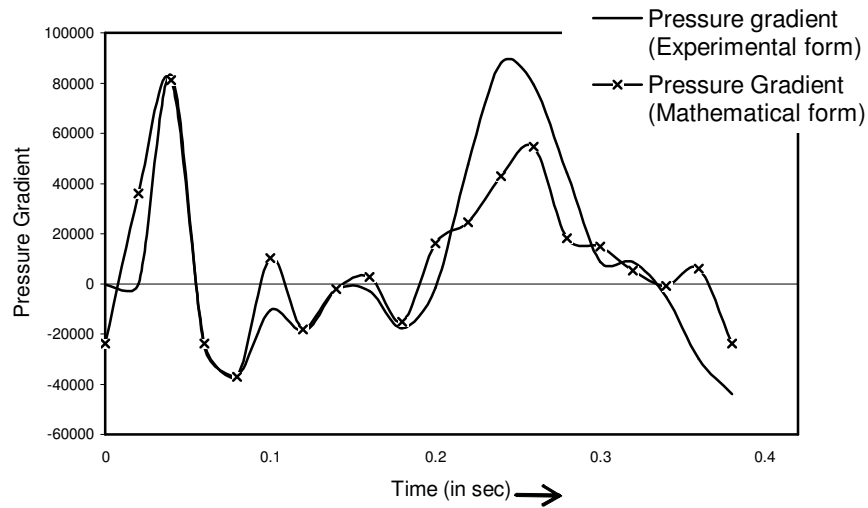


Fig. 7. Comparison of pressure gradient profile (direct data), obtained from flow rate profile (indirect data) [Milnor³], with the mathematical form of pressure gradient profile for terminal pulmonary vein of a dog.

Table-7. The values of Fourier coefficients A_n and B_n ($\text{kg m}^{-2} \text{s}^{-2}$) for the pressure gradient profile.

$A_0 = 10119.5593$		
n	A_n	B_n
1	-568.8897	-14532.0545
2	1640.9997	18407.7707
3	7461.6693	10596.6236
4	-11525.3869	9438.8397
5	-13958.3744	5896.4351
6	-12987.0592	-4297.3577
7	-3927.2532	-15789.0581

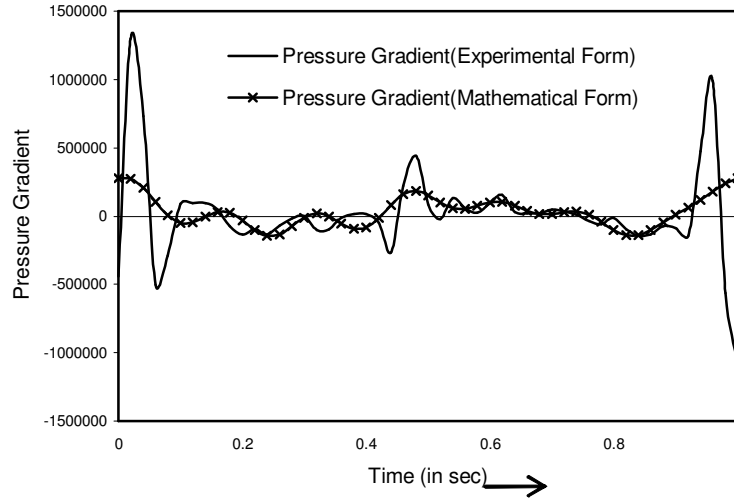


Fig. 8. Comparison of pressure gradient profile (direct data), obtained from flow rate profile (indirect data) [Pedley²], with the mathematical form of pressure gradient profile for left circumflex coronary artery of a dog.

Table-8. The values of Fourier coefficients A_n and B_n ($\text{kg m}^{-2} \text{s}^{-2}$) for the pressure gradient profile.

$A_0 = 29096.7247$		
n	A_n	B_n
1	7857.8630	-19863.1186
2	105479.2288	35470.2638
3	45938.8456	394.3600
4	50500.2185	-20705.7513
5	4870.7072	-14153.7395
6	31336.0142	-17320.3298
7	3999.0176	51830.5810

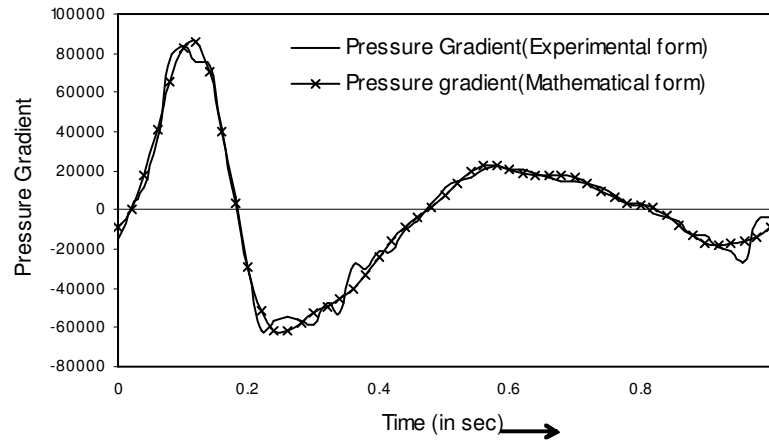


Fig. 9. Comparison of pressure gradient profile (direct data), obtained from flow rate profile (indirect data) [McDonald¹], with the mathematical form of pressure gradient profile for femoral artery of a dog.

Table-9. The values of Fourier coefficients A_n and B_n ($\text{kg m}^{-2} \text{s}^{-2}$) for the pressure gradient profile.

$A_0 = -195.9208$		
n	A_n	B_n
1	8660.2803	-12878.5475
2	13743.6308	31859.7996
3	-12276.5574	19612.5620
4	-14071.3264	4715.4507
5	-7650.1794	-5408.9905
6	-485.4051	-5136.0169
7	3023.5844	-2749.2707

6. Conclusions

In this paper, indirect data (flow rate profiles) are converted into direct data (pressure gradient profiles) by using a finite difference implicit technique suggested by Tsangaris and Stergiopoulos¹¹. The inverse Womersley problem for pulsatile flow in a straight rigid tube has been considered. Nine flow rate profiles for animals for eight different locations are converted into pressure gradient profiles (Figures 1 to 9 and Tables-1 to 9). A computer program in C++ Language is developed to convert indirect data into direct data using finite difference implicit technique. The developed program is presented through Appendix.

But the obtained pressure gradient profiles (direct data), which are in digital form, are not useful for the mathematical models. So these digital data and the geometrical forms of pressure gradients are further converted into mathematical form (Fourier series), which can be readily used as input data for the mathematical models in question. Efforts are made to obtain sufficient accuracy (error up to 10%), except at few sharp points of the profiles.

Thus a good quality direct input data (pressure gradient profiles) are obtained for mathematical models of pulsatile flow of blood. By using these good quality direct input data, the better results would be obtained in the problems of mathematical models of pulsatile flow of blood.

Efforts are being made to locate direct or indirect data for other important locations for an animal CVS and to convert it into a form, which could be readily used for mathematical models. In literature, most of the experimental data is available in the form of velocity-time profiles and pressure-time profiles. There is a need to develop a mathematical method to convert this type of indirect data into direct data. Use of such data as input for mathematical models may help to carry out better influences.

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Appendix

```

/* Program to convert indirect data (flow rate profiles) into direct data (pressure gradient
profile*/
# include <stdio.h>
# include <math.h>
# include <conio.h>
# include <iostream.h>
void main ()
{
int IP1,I,T,K,NN,J,JJ,KP1,E;
float RS, C, SUM, AB, BIG, QUOT, r, CH, PW;
float A[52][52], U[52][52], V[52], X[52], TAU[52], PG[52];
const double h=0.02; const double alpha=8.113; const double dt=0.02;
A[1][1]= -3.0; A[1][2]= 4.0;
A[1][3]= -1.0; A[51][51]= 1.0;
A[48][49]=-dt*h*(95.0-94.0*h);
A[49][49]=96.0*h*(pow((h*alpha),2.0)+2.0*dt*(1.0+h));
A[50][49]=dt*h*(98.0*h-97.0); A[49][50]=-dt*h*(97.0+384.0*h);
A[50][50]=98.0*h*(pow((h*alpha),2.0)+2.0*dt*(1.0-2.0*h));
A[50][51]=-dt*h*(99.0-294.0*h);
clrscr();
for (J=0; J<=50; J=J+1.0){
cout<<"enter value of flow rate V for j="<<J<<"\t";
cin>>V[J];

$$V[J] = V[J] / \pi R^3 \omega ;$$

for (I=2; I<=49; I=I+1){
r=(I-1.0)*h;
A[I][I-1]=dt*(h-2.0*r);
A[I][51]=6.0*dt*h*r;}
for (I=2; I<=48; I=I+1){
r=(I-1.0)*h;
A[I][I]=2.0*r*(pow((h*alpha),2.0)+2.0*dt);
A[I][50]=-8.0*dt*h*r;}
for (I=2; I<=47; I=I+1){
r=(I-1.0)*h;
A[I][I+1]=-dt*(h+2.0*r);
A[I][49]=2.0*dt*h*r;}
}
}

```

Again :

```

for (T=0; T<=49; T=T+1){
for (I=2; I<=50; I=I+1){
r=(I-1.0)*h;
A[I][52]=2.0*pow((h*alpha),2.0)*r*(V[T+1]-V[T]+U[I][T+1]);}
for (K=1;K<=50;K=K+1){
BIG=abs(A[K][K]);
JJ=K;
KP1=K+1;
for (I=KP1; I<=51; I=I+1){
AB=abs(A[I][K]);
if (AB != 0.0){if (BIG != 0.0){
if ((BIG-AB)<=0.0){
BIG=AB;
JJ=I;}}
if (BIG == 0.0){
BIG=AB;
JJ=I;}}
for (J=K; J<=52; J=J+1){
C=A[J][J];
A[J][J]=A[K][J];
A[K][J]=C;}
for (I=KP1; I<=51; I=I+1){
if (A[K][K]!=0.0){
QUOT=A[I][K]/A[K][K];
for (J=KP1; J<=52; J=J+1){
A[I][J]=A[I][J]-QUOT*A[K][J];}}
for (I=KP1; I<=51; I=I+1){
A[I][K]=0.0;}}
if (A[51][51]!=0.0){
U[51][T+2]=A[51][52]/A[51][51];}
for (NN=1; NN<=50; NN=NN+1){
SUM=0.0; I=51-NN; IP1=I+1;
for (J=IP1; J<=51; J=J+1){
SUM=SUM+A[I][J]*U[J][T+2];}
if (A[I][I]!=0.0){
U[I][T+2]=(A[I][52]-SUM)/A[I][I];}}
for (I=1;I<=51; I=I+1){
X[I]=abs(U[I][51]-U[I][1]);

```

```
if (X[I]>0.001){  
for (I=1; I<=51; I=I+1){  
U[I][1]=U[I][51];}  
goto Again;}}  
clrscr();  
cout<<"\n\tpressure Gradient\n";  
for(J=2; J<=51; J=J+1){  
TAU[J]=(3.0*U[51][J]-4.0*U[50][J]+U[49][J])/(2.0*h);  
PG[J]=alpha*alpha*(V[J-1]-V[J-2])/dt-2.0*TAU[J];  
E=J-1;  
cout<<E<<"\t"<<PG[J]<<"\n";}  
getch ();  
}
```