

Composition Operators on the Bergman Space over the Upper Half Plane

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Abstract: In this paper we study the nature of composition operators on the Bergman space over the upper half plane H^+ in the complex plane C . We discuss conditions for the composition operator to be bounded, compact and Hilbert-Schmidt. Moreover, we have shown that if $\phi: H^+ \rightarrow H^+$ is analytic and has angular derivative at some point of the real line then C_ϕ cannot be compact.

In this paper we study the nature of composition operators on the Bergman space over the upper half plane in the complex plane C . Let H^+ denote the upper half plane in C and suppose that

$$L_a^2(H^+) = \{f: H^+ \rightarrow C \text{ be analytic and } \iint_{H^+} |f(x+iy)|^2 dx dy < \infty\}.$$

We call $L_a^2(H^+)$ as the Bergman space over the upper half plane. The space $L_a^2(H^+)$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \iint_{H^+} f(x+iy) \overline{g(x+iy)} dx dy$$

for $f, g \in L_a^2(H^+)$. The function $(e_n)_{n=0}^\infty$ defined by

$$e_n(z) = \frac{2}{\sqrt{\pi}} \sqrt{n+1} \left(\frac{z-i}{z+i} \right)^n \frac{1}{(z+i)^2}, z \in H^+,$$

serves as an orthonormal basis for $L_a^2(H^+)$.

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1. Boundedness and Norm estimates

Let $\phi: H^+ \rightarrow H^+$ be a function such that $f \circ \phi \in L_a^2(H^+)$, $f \in L_a^2(H^+)$, so that the operator C_ϕ defined by $C_\phi f = f \circ \phi$ is a bounded linear operator on $L_a^2(H^+)$. In this section we want to obtain a condition on the inducing function ϕ so that C_ϕ becomes a bounded linear operator and by the process we get some upper and lower bounds for the norms of C_ϕ .

In this connection we note that if $\phi: D \rightarrow D$ is an analytic function then $f \circ \phi \in L_a^2(D)$ if $f \in L_a^2(D)$ and $f \circ \phi \in H^2(D)$ if $f \in H^2(D)$. In other words, C_ϕ is defined on $L_a^2(D)$ or $H^2(D)$ if the function ϕ is analytic. However, in the case of $L_a^2(H^+)$ such assertion is not true. The following example proves that C_ϕ is not defined on $L_a^2(H^+)$ even if the inducing function ϕ from H^+ into H^+ is analytic.

Example 1.1: Let the function $\phi: H^+ \rightarrow H^+$ be defined by $\phi(z) = \frac{1+z}{1-z}$, $z \in H^+$.

Let us take $f(z) = \frac{2}{(z+i)^2}$, $z \in H^+$. Note $f \in L_a^2(H^+)$. Now, $|(f \circ \phi)(z)|^2 = \left| \frac{1-z}{i+z} \right|^2$. It is clear that $\lim_{|z| \rightarrow \infty} |(f \circ \phi)(z)|^2 = 1$. Indeed, we can find $r_0 > 0$, such that

$$|(f \circ \phi)(z) - 1| < \frac{1}{2} \text{ for } |z| > r_0,$$

that is,

$$\frac{1}{2} < |(f \circ \phi)(z)| < \frac{3}{2} \text{ for } |z| > r_0.$$

Hence,

$$\iint_{H^+} |(f \circ \phi)(x+iy)|^2 dx dy \geq \frac{1}{2} \iint_{H^+} dx dy = \infty.$$

This shows that $f \circ \phi$ is not in $L_a^2(H^+)$. The above example proves our assertion that C_ψ cannot be defined on $L_a^2(H^+)$ for any analytic function ψ on H^+ . However, it can be seen from the following theorems that C_ψ becomes a bounded linear operator under certain restriction on the inducing functions ψ . It is very interesting to note from Theorem 1.2 that the boundedness of C_ψ depends on the boundedness of $M_\beta C_\phi$, where M_β is the multiplication operator with symbol β and C_ϕ is a composition operator on $L_a^2(D)$.

Theorem 1.1: Let $\phi: H^+ \rightarrow H^+$ be a one-to-one conformal mapping with

$$\inf_{z \in H^+} |\phi'(z)| = d > 0.$$

Then C_ϕ is a bounded linear operator on $L_a^2(H^+)$ with the norm $\|C_\phi\| \leq \frac{1}{d}$.

Proof: Substituting $w = u + iv = \phi(z)$, $z = x + iy \in H^+$, we have

$$\begin{aligned} \|f\|^2 &= \iint_{H^+} |f(u + iv)|^2 du dv \\ &\geq \iint_{\phi(H^+)} |f(u + iv)|^2 du dv \\ &= \iint_{H^+} |f(\phi(z))|^2 |\phi'(z)|^2 dx dy \\ &\geq d^2 \iint_{H^+} |f(\phi(z))|^2 dx dy \\ &= d^2 \|C_\phi f\|^2. \end{aligned}$$

Hence, $\|C_\phi f\| \leq \frac{1}{d} \|f\|$ which implies that $\|C_\phi\| \leq \frac{1}{d}$.

Suppose that h is the linear fractional transformation defined by

$$(1.1) \quad h(z) = w = \frac{z - i}{z + i}, \quad z \in H^+.$$

The function h maps the upper half plane into the unit disk. Let L be its inverse defined by

$$(1.2) \quad L(w) = h^{-1}(w) = i \frac{1 + w}{1 - w}, \quad w \in D.$$

Lemma 1.1: The function $f \in L_a^2(H^+)$ if and only if $(f \circ L)L' \in L_a^2(D)$.

Proof: Notice that $f \in L_a^2(H^+)$ if and only if

$$\iint_{H^+} |f(u + iv)|^2 du dv < \infty.$$

This is true if and only if

$$\iint_D |f(L(x + iy))|^2 |L'(x + iy)|^2 dx dy < \infty$$

and the lemma follows.

With the above notation and definition we proceed to prove the following theorem.

Theorem 1.2: Let the map $\phi: D \rightarrow D$ be analytic and L be the linear fractional transformation from D into H^+ defined by (1.2). Suppose $\psi(z) = L \circ \phi \circ L^{-1}(z)$, so that ψ is an analytic map from H^+ into H^+ . Then C_ψ defines a bounded linear operator on $L_a^2(H^+)$ if and only if $M_\beta C_\phi$ is a bounded operator on $L_a^2(D)$, where $\beta(z) = \left(\frac{1 - \phi(z)}{1 - z} \right)^2$.

Proof: To prove the theorem, let $f \in L_a^2(H^+)$ and put $g = f \circ \psi$. Then by

Lemma 1.1, $g \in L_a^2(H^+)$ if and only if $(g \circ L)L' \in L_a^2(D)$. If $z = x + iy$, this is true if and only if $\iint_D |(f \circ L \circ \phi)(z)|^2 \frac{4}{|1-z|^4} dx dy < \infty$, that is, if and only if

$$(1.3) \quad \iint_D \frac{|(f \circ L \circ \phi)(z)|^2}{|1-\phi(z)|^4} \frac{|1-\phi(z)|^4}{|1-z|^4} dx dy < \infty.$$

Now if we write $F(z) = (f \circ L)(z)L'(z)$ then $F \in L_a^2(D)$ and hence $F \circ \phi \in L_a^2(D)$. So we can write the condition (1.3) as $\iint_D |F(\phi(z))|^2 \frac{|1-\phi(z)|^4}{|1-z|^4} dx dy < \infty$. But the last condition is equivalent to $\|M_\beta C_\phi F\| < \infty$. This completes the proof of the theorem.

2. Compactness of C_ϕ

In this section we want to study the Hilbert-Schmidtness and compactness of the composition operators on $L_a^2(H^+)$. In the following theorem we obtain a necessary and sufficient condition for C_ϕ to be Hilbert-Schmidt.

Theorem 2.1: Let C_ϕ be a bounded linear operator on $L_a^2(H^+)$. Then C_ϕ

is Hilbert-Schmidt on $L_a^2(H^+)$ if and only if $\iint_{H^+} \frac{dx dy}{|\operatorname{Im} \phi(z)|^2} < \infty$.

Proof: Let $e_n : H^+ \rightarrow C, n \in \mathbb{Z}_+$ be given by $e_n(z) = \frac{2}{\sqrt{\pi}} \sqrt{n+1} \left(\frac{z-i}{z+i} \right)^n \frac{1}{(z+i)^2}$.

We know that (e_n) forms an orthonormal basis for $L_a^2(H^+)$. Then C_ϕ is Hilbert-Schmidt if and only if $\sum_{n=0}^{\infty} \|C_\phi e_n\|^2 < \infty$. That is, if and only if $\sum_{n=0}^{\infty} \|e_n \circ \phi\|^2 < \infty$. But we have, for $z = x + iy$,

$$\begin{aligned} \sum_{n=0}^{\infty} \|e_n \circ \phi\|^2 &= \sum_{n=0}^{\infty} \frac{4(n+1)}{\pi} \iint_{H^+} \left| \frac{\phi(z)-i}{\phi(z)+i} \right|^{2n} \frac{dx dy}{|\phi(z)+i|^4} \\ &= \frac{4}{\pi} \iint_{H^+} \sum_{n=0}^{\infty} (n+1) \left| \frac{\phi(z)-i}{\phi(z)+i} \right|^{2n} \frac{dx dy}{|\phi(z)+i|^4} \end{aligned}$$

$$\begin{aligned}
&= \frac{4}{\pi} \iint_{H^+} \frac{1}{\left(1 - \left|\frac{\phi(z)-i}{\phi(z)+i}\right|^2\right)^2} \frac{dx dy}{|\phi(z)+i|^4} \\
&= \frac{4}{\pi} \iint_{H^+} \frac{dx dy}{(|\phi(z)+i|^2 - |\phi(z)-i|^2)^2} \\
&= \frac{4}{\pi} \iint_{H^+} \frac{dx dy}{16|\operatorname{Im} \phi(z)|^2} \\
&= \frac{1}{4\pi} \iint_{H^+} \frac{dx dy}{|\operatorname{Im} \phi(z)|^2}.
\end{aligned}$$

Hence C_ϕ is Hilbert-Schmidt if and only if $\iint_{H^+} \frac{dx dy}{|\operatorname{Im} \phi(z)|^2} < \infty$.

The next theorem gives a characterisation of compact composition operators on $L_a^2(H^+)$. Before that we want to prove the following inequality.

Lemma 2.1: If $f \in L_a^2(H^+)$ and $z \in H^+$ then

$$(2.1) \quad |f(z)| \leq \frac{\|f\|}{2|\operatorname{Im} z|}.$$

Proof: We know that ^{1,2} if $g \in L_a^2(D)$ then $|g(w)| \leq \frac{\|g\|}{1-|w|^2}$, $w \in D$.

By Lemma 1.1, we have $f \in L_a^2(H^+)$ if and only if $g(w) = (f \circ L)(w)L'(w)$
 $= f\left(i \frac{1+w}{1-w}\right) \frac{2i}{(1-w)^2} \in L_a^2(D)$. Hence

$$2|f(z)| \leq \frac{|1-w|^2}{1-|w|^2} \|f\| = \frac{\left|1 - \frac{z-i}{z+i}\right|^2}{1 - \left|\frac{z-i}{z+i}\right|^2} \|f\| = \frac{\|f\|}{|\operatorname{Im} z|}$$

and the result follows.

Theorem 2.2: A composition operator C_ϕ on $L_a^2(H^+)$ is compact if and only if for every norm bounded sequence (f_n) in $L_a^2(H^+)$ which converges to zero uniformly on compact subsets of H^+ , the image sequence $C_\phi f_n$ converges to zero in norm.

Proof: Let C_ϕ be compact and (f_n) be a sequence of norm bounded functions in $L_a^2(H^+)$ converging uniformly to zero on every compact subset of H^+ .

We shall show that $\|C_\phi f_n\|$ converges to zero. To do so we prove that the sequence (f_n) converges weakly to zero. That is,

$$\lim_{n \rightarrow \infty} \iint_{H^+} f_n(z) \overline{g(z)} dx dy = 0$$

for every $g \in L_a^2(H^+)$ and $z = x + i y$.

Let $\sup_n \|f_n\| = M_1 < \infty$ and $g \in L_a^2(H^+)$. Since H^+ is locally compact for $\varepsilon > 0$, we can find a compact subset K of H^+ such that

$$\left(\iint_{K^c} |g(z)|^2 dx dy \right)^{\frac{1}{2}} < \frac{\varepsilon}{2M_1}.$$

Here K^c denotes the complement of K in H^+ . Further since g is analytic in H^+ , it is bounded on every compact subset of its domain. Let $\sup_{z \in K} |g(z)| < M_2 < \infty$. We take the case when the two dimensional Lebesgue measure of K is positive. Now since f_n converges uniformly to zero on every compact subset of H^+ , for $\varepsilon > 0$, we can find an n_0

such that $|f_n(z)| < \frac{\varepsilon}{2m(K)M_2}$ for $n > n_0$ and $z \in K$, where $m(K)$ is the two dimensional

Lebesgue measure of the set K . Hence for $n > n_0$,

$$\begin{aligned} \left| \iint_{H^+} f_n(z) \overline{g(z)} dx dy \right| &\leq \left| \iint_K f_n(z) \overline{g(z)} dx dy \right| + \left| \iint_{K^c} f_n(z) \overline{g(z)} dx dy \right| \\ &\leq \frac{\varepsilon}{2m(K)M_2} M_2 m(K) + \left(\iint_{K^c} |f_n(z)|^2 dx dy \right)^{\frac{1}{2}} \left(\iint_{K^c} |g(z)|^2 dx dy \right)^{\frac{1}{2}} \\ &\leq \frac{\varepsilon}{2} + \left(\iint_{H^+} |f_n(z)|^2 dx dy \right)^{\frac{1}{2}} \frac{\varepsilon}{2M_1} \\ &= \frac{\varepsilon}{2} + \|f_n\| \frac{\varepsilon}{2M_1} \\ &\leq \frac{\varepsilon}{2} + M_1 \frac{\varepsilon}{2M_1} = \varepsilon. \end{aligned}$$

For the case when $m(K) = 0$, the proof is even simpler. Thus we have shown that f_n converges to zero weakly. As C_ϕ is compact, $\|C_\phi f_n\|$ converges to zero. The converse is proved by usual normal family argument. We can show that a norm bounded family is a normal family by using the inequality (2.1).

We shall define the angular derivative of a function ϕ in the upper half plane as follows:

Definition 2.1: Let $\phi: H^+ \rightarrow H^+$ be holomorphic. We say that ϕ has an angular derivative at a point x_0 if

$$\lim_{y \rightarrow 0} \frac{\phi(x_0 + i y) - \phi(x_0)}{i y}$$

exists.

Remark: This under the Cayley transform reduces to the angular derivative $\lim_{x \rightarrow 1} \frac{\phi(1) - \phi(x)}{1 - x}$ as defined by Caratheodery³. We shall try to get an analogue of the result of Shapiro and Taylor⁴ for $L_a^2(H^+)$.

Lemma 2.2: Let $f_\alpha : D \rightarrow C$ be defined by

$$f_\alpha(z) = \frac{\sqrt{1-\alpha}}{(1+z)^{\frac{\alpha+1}{2}}}$$

for $0 < \alpha < 1$. Then $f_\alpha \in L_a^2(D)$ and $\|f_\alpha\| \leq M_\alpha$, where M_α is a constant depending on α .

Proof: Let $z = re^{i\theta}$ and $z \in D$. Then

$$\begin{aligned} \left| f_\alpha(re^{i\theta}) \right|^2 &= \frac{1-\alpha}{\left(|1+z|^2 \right)^{\frac{\alpha+1}{2}}} \\ &= \frac{1-\alpha}{\left[1+r^2+2r \cos \theta \right]^{\frac{\alpha+1}{2}}} \\ &= \frac{1-\alpha}{\left[(1-r)^2 + 4r \cos^2 \frac{\theta}{2} \right]^{\frac{\alpha+1}{2}}}. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{-\pi}^{\pi} \left| f_\alpha(re^{i\theta}) \right|^2 d\theta &= \int_{-\pi}^{\pi} \frac{1-\alpha}{\left[(1-r)^2 + 4r \cos^2 \frac{\theta}{2} \right]^{\frac{\alpha+1}{2}}} d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2(1-\alpha)}{\left[(1-r)^2 + 4r \cos^2 t \right]^{\frac{\alpha+1}{2}}} dt \\ &= 4(1-\alpha) \int_0^{\frac{\pi}{2}} \frac{dt}{\left[(1-r)^2 + 4r \cos^2 t \right]^{\frac{\alpha+1}{2}}} \end{aligned}$$

$$\begin{aligned}
&= 4(1-\alpha) \int_0^{\frac{\pi}{2}} \frac{d\theta}{\left[(1-r)^2 + 4r \sin^2 \theta \right]^{\frac{\alpha+1}{2}}} \\
&\leq 4(1-\alpha) \int_0^{\frac{\pi}{2}} \frac{d\theta}{\left[(1-r)^2 + \frac{16r\theta^2}{\pi^2} \right]^{\frac{\alpha+1}{2}}} \quad \left(\text{as } \frac{2\theta}{\pi} < \sin \theta < \theta \text{ for } 0 \leq \theta \leq \frac{\pi}{2} \right) \\
&= 4(1-\alpha) \int_0^{\frac{\pi}{2(1-r)}} \frac{(1-r) dt}{\left[(1-r)^2 + \frac{16r(1-r)^2 t^2}{\pi^2} \right]^{\frac{\alpha+1}{2}}} \\
&= \frac{4(1-\alpha)}{(1-r)^\alpha} \int_0^{\frac{\pi}{2(1-r)}} \frac{dt}{\left[1 + \frac{16rt^2}{\pi^2} \right]^{\frac{\alpha+1}{2}}} \\
&= \frac{4(1-\alpha)}{(1-r)^\alpha} \int_0^{\frac{\pi}{2(1-r)}} \frac{dt}{r^{\frac{\alpha+1}{2}} \left[\frac{1}{r} + \frac{16t^2}{\pi^2} \right]^{\frac{\alpha+1}{2}}} \\
&\leq \frac{4(1-\alpha)}{r^{\frac{\alpha+1}{2}} (1-r)^\alpha} \int_0^\infty \frac{dt}{\left[1 + \frac{16t^2}{\pi^2} \right]^{\frac{\alpha+1}{2}}} \\
&\leq \frac{4(1-\alpha)}{r^{\frac{\alpha+1}{2}} (1-r)^{\frac{\alpha+1}{2}}} \int_0^\infty \frac{dt}{\left[1 + \frac{16t^2}{\pi^2} \right]^{\frac{\alpha+1}{2}}}
\end{aligned}$$

It is easy to see that the integral

$$\int_0^\infty \frac{dt}{\left[1 + \frac{16t^2}{\pi^2} \right]^{\frac{\alpha+1}{2}}}$$

converges and let us write

$$C_{\alpha}^2 = \int_0^{\infty} \frac{dt}{\left[1 + \frac{16t^2}{\pi^2}\right]^{\frac{\alpha+1}{2}}} \quad \text{and} \quad M_{\alpha}^2 = \frac{8}{\pi} C_{\alpha}^2.$$

So we get,

$$\int_{-\pi}^{\pi} \left| f_{\alpha}(re^{it}) \right|^2 dt \leq \frac{4(1-\alpha)}{r^{\frac{\alpha+1}{2}} (1-r)^{\frac{\alpha+1}{2}}} C_{\alpha}^2.$$

Therefore, we have

$$\begin{aligned} \frac{1}{\pi} \int_0^1 \int_{-\pi}^{\pi} \left| f_{\alpha}(re^{it}) \right|^2 dt r dr &\leq \frac{4(1-\alpha)}{\pi} C_{\alpha}^2 \int_0^1 \frac{r dr}{r^{\frac{\alpha+1}{2}} (1-r)^{\frac{\alpha+1}{2}}} \\ &= \frac{4(1-\alpha)}{\pi} C_{\alpha}^2 \int_0^1 \frac{r^{\frac{1-\alpha}{2}} dr}{(1-r)^{\frac{\alpha+1}{2}}} \\ &\leq \frac{4(1-\alpha)}{\pi} C_{\alpha}^2 \int_0^1 \frac{dr}{(1-r)^{\frac{\alpha+1}{2}}} \\ &= \frac{8}{\pi} C_{\alpha}^2 = M_{\alpha}^2. \end{aligned}$$

Hence $f_{\alpha} \in L_a^2(D)$ and $\|f_{\alpha}\| \leq M_{\alpha}$.

Theorem 2.3: If $\phi: H^+ \rightarrow H^+$ is analytic and ϕ has angular derivative at some point of the real line then C_{ϕ} cannot be compact.

Proof: Let $g_{\alpha}: H^+ \rightarrow C$ be defined by

$$g_{\alpha}(w) = f_{\alpha}\left(\frac{w-i}{w+i}\right) \frac{2}{(w+i)^2}$$

So

$$\begin{aligned} &= (1-\alpha)^{\frac{1}{2}} \left(1 + \frac{w-i}{w+i}\right)^{-\left(\frac{\alpha+1}{2}\right)} \frac{2}{(w+i)^2} \\ &= (1-\alpha)^{\frac{1}{2}} (2w)^{-\left(\frac{\alpha+1}{2}\right)} \frac{2}{(w+i)^{-\left(\frac{\alpha+1}{2}\right)+2}} \end{aligned}$$

$$\begin{aligned}
&= (1-\alpha)^{\frac{1}{2}} \frac{1}{2} \left(\frac{1-\alpha}{2} \right)_{(w)}^{-\left(\frac{\alpha+1}{2}\right)} \left(\frac{\alpha+1}{2} \right)_{(w+i)}^{\left(\frac{\alpha+1}{2}-2\right)} \\
&= (1-\alpha)^{\frac{1}{2}} \frac{1}{2} \left(\frac{1-\alpha}{2} \right)_{(w)}^{-\left(\frac{\alpha+1}{2}\right)} \left(\frac{\alpha+1}{2} \right)_{(w+i)}^{\left(\frac{\alpha-3}{2}\right)}.
\end{aligned}$$

So

$$\pi \|g_\alpha\|_{L_a^2(C_+)}^2 = \|f_\alpha\|^2 \leq M_\alpha^2$$

which implies $g_\alpha \in L_a^2(H^+)$.

Without loss of generality we may assume that the point x_0 where the angular derivative exists on the real line, is the origin and $\phi(0) = 0$. So the existence of angular derivative amounts to existence of $M > 0$ such that

$$\sup_{t \in \mathbb{R}} \left| \frac{\phi(it)}{t} \right| \leq M.$$

But we can show that

$$\begin{aligned}
M_\alpha^2 &\geq \|f_\alpha\|^2 = \pi \|g_\alpha\|^2 \\
&\geq \int_0^1 \int_{\frac{1-r}{1+r}}^{\frac{1+r}{1-r}} |g_\alpha(it)|^2 \frac{(1+t)^2}{2} dt dr \\
&= (1-\alpha) 2^{-\alpha} \int_0^1 \int_{\frac{1-r}{1+r}}^{\frac{1+r}{1-r}} t^{-1-\alpha} (1+t)^{\alpha-1} dt dr \\
&\geq (1-\alpha) 2^{-\alpha} \int_0^1 \int_{\frac{1-r}{1+r}}^{\frac{1+r}{1-r}} \frac{t^{-1-\alpha}}{\left(1 + \frac{1+r}{1-r}\right)^{1-\alpha}} dt dr \\
&= \frac{1-\alpha}{2} \int_0^1 \int_{\frac{1-r}{1+r}}^{\frac{1+r}{1-r}} t^{-1-\alpha} (1-r)^{1-\alpha} dt dr \\
&= \frac{1-\alpha}{2\alpha} \int_0^1 \left[\frac{(1+r)^\alpha}{(1-r)^{2\alpha-1}} - \frac{1-r}{(1+r)^\alpha} \right] dr \\
&\geq \frac{1-\alpha}{2\alpha} \left[\int_0^1 (1-r)^{1-2\alpha} dr - \frac{1}{2} \right] = \frac{1}{4}.
\end{aligned}$$

Now

$$\begin{aligned}
\pi \|C_\phi g_\alpha\|^2 &\geq \int_0^1 \int_{\frac{1-r}{1+r}}^{\frac{1+r}{1-r}} \frac{(1-\alpha)2^{1-\alpha}}{|\phi(it)|^{\alpha+1}} \frac{1}{|i+\phi(it)|^{3-\alpha}} \frac{(1+t)^2}{2} dt dr \\
&= \int_0^1 \int_{\frac{1-r}{1+r}}^{\frac{1+r}{1-r}} \frac{(1-\alpha)2^{1-\alpha}}{|it|^{\alpha+1}} \frac{1}{|i+it|^{3-\alpha}} \frac{(1+t)^2}{2} \left| \frac{it}{\phi(it)} \right|^{\alpha+1} \left| \frac{i+it}{i+\phi(it)} \right|^{3-\alpha} dt dr \\
&\geq M^{-(\alpha+1)} \int_0^1 \int_{\frac{1-r}{1+r}}^{\frac{1+r}{1-r}} |g_\alpha(it)|^2 \left| \frac{i+it}{i+\phi(it)} \right|^{3-\alpha} \frac{(1+t)^2}{2} dt dr \\
&\geq M^{-(\alpha+2)} \int_0^1 \int_{\frac{1-r}{1+r}}^{\frac{1+r}{1-r}} |g_\alpha(it)|^2 \frac{(1+t)^2}{2} dt dr \\
&\geq \frac{M^{-(\alpha+2)}}{4}.
\end{aligned}$$

Thus we observe that $g_\alpha(w)$ converges uniformly to zero on every compact subset of H^+ as $\alpha \rightarrow 1$ and $\|g_\alpha\|$ is bounded. But $\|C_\phi g_\alpha\|$ is bounded away from zero as $\alpha \rightarrow 1$. Hence C_ϕ is not compact.

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