

Hankel-Like Operators in L_h^2

Namita Das

P. G. Department of Mathematics, Utkal University, Vanivihar, Bhubaneswar, India
E-mail: namitadas440@yahoo.co.in

(Received January 10, 2008)

Abstract: Let D be the open unit disk in the complex plane C , $dA(z)$ be the normalized area measure on D and $L_h^2(D)$ be the subspace of functions in $L^2(D, dA)$ that are harmonic. For $\phi \in L^\infty(D)$, define H_ϕ from L_h^2 into $(L_h^2)^\perp$ such that $H_\phi f = (I - P)(\phi f)$ where P is the orthogonal projection from L^2 onto L_h^2 . In this paper some algebraic properties of the Hankel-like operators H_ϕ are derived.

2000 AMS Subject Classification Number: 47B35.

1. Introduction

Let $D = \{z \in C : |z| < 1\}$. Let $dA(z)$ be the area measure on D normalized so that the area of the disk is 1. In rectangular and polar coordinates,

$$dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta.$$

Let $L^2(D, dA)$ be the Hilbert space of Lebesgue measurable functions on D with the inner product

$$\langle f, g \rangle = \int_D f(z) \overline{g(z)} dA(z), \quad f, g \in L^2(D).$$

The harmonic Bergman space L_h^2 is the space of harmonic functions on D which are in $L^2(D, dA)$. The space L_h^2 is a Hilbert space with the inner product that it inherits from $L^2(D, dA)$. The sequence of functions $\{\sqrt{n+1}z^n\}_{n=0}^\infty$ and $\{\sqrt{n+1}\bar{z}^n\}_{n=1}^\infty$ together form an orthonormal basis for $L_h^2(D)$. The Bergman space $L_a^2(D)$ is the set of those functions in $L^2(D, dA)$ that are analytic on D . The Bergman space $L_a^2(D)$ is a closed subspace of $L^2(D, dA)$. Let $(L_a^2)_0$ is the subspace of $L^2(D, dA)$ consisting of all complex conjugates of functions in $L_a^2(D)$ which vanish at the origin. That is,

$(\overline{L_a^2})_0 = \{\bar{f} : f \in L_a^2(D), f(0) = 0\}$. Thus $L_h^2 = L_a^2 \oplus (\overline{L_a^2})_0$. Let $h^\infty(D)$ be the space of bounded harmonic functions on D . Let $L^\infty(D, dA)$ denote the Banach space of Lebesgue measurable functions f on D with

$$\|f\|_\infty = \text{ess sup}\{|f(z)| : z \in D\} < \infty.$$

Let $H^\infty(D)$ be the space of bounded analytic functions on D . Let P be the orthogonal projection from L^2 onto L_h^2 . We define the Hankel-like operator $H_\phi : L_h^2 \rightarrow (L_h^2)^\perp$ with symbol $\phi \in L^\infty$ as $H_\phi f = (I - P)(\phi f)$. Notice that the operator H_ϕ is well-defined.

2. The Hankel-Like Operator H_ϕ

The following observations are immediate.

- (i) The operator H_ϕ is linear, bounded and $\|H_\phi\| \leq \|\phi\|_\infty$.
- (ii) The adjoint operator $H_\phi^* : (L_h^2)^\perp \rightarrow L_h^2$ is defined by $H_\phi^* g = P(\bar{\phi} g)$.

This is so because for $f \in L_h^2, g \in (L_h^2)^\perp$,

$$\begin{aligned} \langle f, H_\phi^* g \rangle &= \langle H_\phi f, g \rangle = \langle (I - P)(\phi f), g \rangle \\ &= \langle \phi f, g \rangle = \int_D \phi f \bar{g} dA(z) = \int_D f \overline{\bar{\phi} g} dA(z) \\ &= \langle f, P(\bar{\phi} g) \rangle. \end{aligned}$$

Proposition 2.1: *The Hankel operator $H_\phi \equiv 0$ if and only if ϕ is a constant function.*

Proof: Suppose $H_\phi \equiv 0$. Then $H_\phi f \equiv 0$ for all $f \in L_h^2$. That is, $(I - P)(\phi f) = 0$ for all $f \in L_h^2$. Hence $\phi f \in L_h^2$ for all $f \in L_h^2$. In particular, taking $f = 1$, we obtain $\phi \in L_h^2$. Thus ϕ is harmonic. Since $\phi \in L^\infty$, hence $\phi \in h^\infty$. Now $\phi f \in L_h^2$ for all $f \in L_h^2$ implies ϕ^2 is harmonic. Now ϕ is real valued, $\phi \in h^\infty(D)$, $\phi^2 \in h^\infty(D)$ implies¹ that ϕ is a constant. Otherwise (if ϕ is not real valued) since $\phi \in h^\infty(D)$, $\phi f \in L_h^2$ for all $f \in L_h^2$ it follows¹ that either ϕ is holomorphic or $\bar{\phi}$ is holomorphic.

If $\phi \in H^\infty(D)$ and $\phi f \in L_h^2$ for all $f \in L_h^2$ then it follows that $\phi \bar{p} \in L_h^2$ for all polynomial p in z . Thus

$$0 = \Delta(\phi \bar{p}) = \phi' \bar{p}'$$

where ϕ' is the derivative of ϕ with respect to z , and $\bar{p}'$ is the derivative of $\bar{p}$ with respect to $\bar{z}$. That is, either ϕ' or $\bar{p}'$ vanishes identically. Hence either ϕ or p is a constant. Since $\phi \bar{p} \in L_h^2$ for all polynomial p in z hence $\phi \equiv c$, a constant. Similarly, if

$\phi \in H^\infty(D)$ and $\phi f \in L_h^2$ for all $f \in L_h^2$ then $\phi p \in L_h^2$ for all polynomial p in z . This implies, $\phi \equiv c'$, a constant.

Proposition 2.2: If $\phi \in L^\infty$ then $\|H_\phi\| \leq \text{dist}_{L^\infty}(\phi, C)$.

Proof: It follows from proposition 2.1 that $H_\phi = H_{\phi-c}$ for all constant $c \in C$. Hence $\|H_\phi\| = \|H_{\phi-c}\| \leq \|\phi - c\|_\infty$ for all $c \in C$. That is, $\|H_\phi\| \leq \inf_{c \in C} \|\phi - c\|_\infty = \text{dist}_{L^\infty}(\phi, C)$.

Notice that the Hankel-like operator H_ϕ can also be defined with symbol $\phi \in L^2(D, dA)$. That is, given a function $\phi \in L^2(D, dA)$, we can define an operator $H_\phi : L_h^2 \rightarrow (L_h^2)^\perp$ by $H_\phi f = (I - P)(\phi f)$. Clearly, H_ϕ is densely defined for any $\phi \in L^2(D, dA)$. The operator H_ϕ in this case is unbounded in general. However, if $\phi \in L^\infty(D, dA)$, then H_ϕ is bounded with $\|H_\phi\| \leq \|\phi\|_\infty$.

Theorem 2.1: There exists no nonconstant $\phi \in L_a^2$ such that H_ϕ is of finite rank and there exist no nonconstant $\psi \in \overline{L_a^2}$ such that H_ψ is of finite rank.

Proof: Suppose $\phi \in L_a^2$, H_ϕ is of finite rank and $\text{Rank } H_\phi = n \geq 1$. Then $\text{Range } H_\phi$ is a finite dimensional subspace of $(L_h^2)^\perp$. In fact $\dim \text{Range } H_\phi = n$. Hence $H_\phi \bar{z}, H_\phi \bar{z}^2, \dots, H_\phi \bar{z}^{n+1}$ are linearly dependent. Thus there exist constants $c_1, c_2, \dots, c_{n+1}$ not all zero such that $H_\phi(c_1 \bar{z} + c_2 \bar{z}^2 + \dots + c_{n+1} \bar{z}^{n+1}) = 0$. That is, there exists a nonzero polynomial p in $\bar{z}$, $p(0) = 0$ such that $H_\phi p = 0$. Hence $\phi p \in L_h^2$. Since $\phi \in L_a^2$ and $\phi p \in L_h^2$ hence either $\phi \equiv c$, a constant or $p \equiv c'$, a constant. But p is not a constant. Hence $\phi \equiv c$. Now let $\psi \in \overline{L_a^2}$ and $\text{Rank } H_\psi = n$. Then $H_\psi z, H_\psi z^2, \dots, H_\psi z^{n+1}$ are linearly dependent. Hence there exists a nonzero polynomial q in z , $q(0) = 0$ such that $H_\psi q = 0$. That is, $\psi q \in L_h^2$. Since $\psi \in \overline{L_a^2}$, $\psi q \in L_h^2$, hence either $\psi \equiv b_1$, a constant or $q \equiv b_2$, a constant. Since q is not a constant, therefore $\psi \equiv b_1$, a constant.

Theorem 2.2: If $\phi \in C(\overline{D})$, then H_ϕ is compact.

Proof: First we shall show that the operators $H_{\bar{z}^n}$ and H_{z^n} are Hilbert-Schmidt. The reason is as follows:

$$\sum_{k=0}^{\infty} \|H_{\bar{z}^n}(\sqrt{k+1}z^k)\|^2 + \sum_{k=1}^{\infty} \|H_{\bar{z}^n}(\sqrt{k+1}\bar{z}^k)\|^2$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} (k+1) \left\| H_{\bar{z}^n} z^k \right\|^2 + \sum_{k=1}^{\infty} (k+1) \left\| H_{\bar{z}^n} \bar{z}^k \right\|^2 \\
&= \sum_{k=0}^{\infty} (k+1) \left\| (I-P)(\bar{z}^n z^k) \right\|^2 + \sum_{k=1}^{\infty} (k+1) \left\| (I-P)(\bar{z}^n \bar{z}^k) \right\|^2 \\
&= \sum_{k=0}^{\infty} (k+1) \left\| (\bar{z}^n z^k) - P(\bar{z}^n z^k) \right\|^2.
\end{aligned}$$

It is not so difficult to verify that

$$P(\bar{z}^n z^k) = \begin{cases} \frac{n-k+1}{n+1} \bar{z}^{n-k} & \text{if } k \leq n; \\ \frac{k-n+1}{k+1} z^{k-n} & \text{if } k \geq n. \end{cases}$$

Hence

$$\begin{aligned}
&\sum_{k=0}^{\infty} (k+1) \left\| \bar{z}^n z^k - P(\bar{z}^n z^k) \right\|^2 \\
&= \sum_{k=0}^n \frac{(k+1)k^2}{(k+n+1)(n+1)^2} + \sum_{k=n+1}^{\infty} \frac{n^2}{(k+n+1)(k+1)} < \infty.
\end{aligned}$$

Thus $H_{\bar{z}^n}$ is Hilbert-Schmidt. Similarly, one can check that H_{z^n} is Hilbert-Schmidt. Thus $H_{\bar{z}^n}$ and H_{z^n} are compact operators for all $n \in \mathbb{N}$. It is well known² that polynomials are dense in $C(\bar{D})$.

Hence if $\phi \in C(\bar{D})$ then there exists a sequence of polynomials p_n such that $\lim_{n \rightarrow \infty} \|p_n - \phi\|_{\infty} = 0$. Thus $\lim_{n \rightarrow \infty} H_{p_n} = H_{\phi}$. Since the set of all compact operators from L_h^2 into $(L_h^2)^{\perp}$ is closed in the norm topology and H_{p_n} are compact operators, hence H_{ϕ} is compact.

For $f \in L^2(D, dA)$, $z \in D$, let $\tilde{f}(z)$ be the Berezin transform of f defined by

$$\tilde{f}(z) = \int_D f(w) |k_z(w)|^2 dA(w) = \int_D (f \circ \phi_z)(w) dA(w)$$

and $MO(f)(z) = \left[|f|^2(z) - |\tilde{f}(z)|^2 \right]^{\frac{1}{2}}$. Let

$$\|f\|_{BMO_{\partial}} = \sup \{ MO(f)(z) : z \in D \}.$$

Let BMO_{∂} be the space of functions f with $\|f\|_{BMO_{\partial}} < +\infty$. Let $P_{L_a^2}^-$ be the orthogonal projection from L^2 onto $\overline{L_a^2} = \{ \bar{f} : f \in L_a^2 \}$. It is shown³ that if $\phi \in BMO_{\partial}$ then

$$\sup_{f \in L_a^2, \|f\|=1} \left\| P_{L_a^2}^-(\phi f) \right\| \leq \sup_{f \in L_a^2, \|f\|=1} \left\| (I - P_{L_a^2}^2)(\phi f) \right\| < \infty$$

and

$$\sup_{f \in L_a^2, \|f\|=1} \left\| P_{L_a^2}^-(\bar{\phi} f) \right\| \leq \sup_{f \in L_a^2, \|f\|=1} \left\| (I - P_{L_a^2})(\bar{\phi} f) \right\| < \infty.$$

If $f \in L^2(D, dA)$, define the operator V on L^2 as follows:

$$Vf(z) = 3(1 - |z|^2)^2 \int_D \frac{f(w)}{(1 - \bar{z}w)^4} dA(w).$$

Further, recall that given $f \in L^2(D, dA)$, the little Hankel operator $\tilde{h}_f : L_a^2 \rightarrow \overline{L_a^2}$ with symbol f is defined by $\tilde{h}_f g = P_{L_a^2}^-(fg)$. The operator $\tilde{h}_f$ is unbounded in general. However, $\tilde{h}_f$ is bounded if f is and we have $\|\tilde{h}_f\| \leq \|f\|_\infty$. If $f \in L^2(D, dA)$ then it is shown³ that $\tilde{h}_{\bar{f}}$ is bounded if and only if $Vf(z)$ is bounded in D . Moreover, there is a constant $C > 0$ such that

$$C^{-1} \|Vf\|_\infty \leq \|\tilde{h}_{\bar{f}}\| \leq C \|Vf\|_\infty$$

and $Vf(z) = 3 \langle \bar{k}_z, \tilde{h}_{\bar{f}} k_z \rangle$, where $k_z(w) = \frac{1 - |z|^2}{(1 - \bar{z}w)^2}$ is the normalized reproducing kernel of the Bergman space $L_a^2(D)$.

Theorem 2.3: *If $\phi \in BMO_\partial$ then $V\phi$ is bounded in D and there exist constants $N > 0$, $M > 0$ such that*

$$\|\phi\|_{BMO_\partial} - M(\|V\phi\|_\infty + \|V\bar{\phi}\|_\infty) \leq \|H_\phi\| + \|H_{\bar{\phi}}\| \leq N\|\phi\|_{BMO_\partial}.$$

Proof: It follows³ that

$$\begin{aligned} \|H_\phi\| + \|H_{\bar{\phi}}\| &\geq \sup_{f \in L_a^2, \|f\|=1} \left\| (I - P_{L_a^2})(\phi f) \right\| + \sup_{f \in L_a^2, \|f\|=1} \left\| (I - P_{L_a^2})(\bar{\phi} f) \right\| \\ &= \sup_{f \in L_a^2, \|f\|=1} \left[\left\| (I - P_{L_a^2})(\phi f) - P_{(L_a^2)_0}^-(\phi f) \right\| + \left\| (I - P_{L_a^2})(\bar{\phi} f) - P_{(L_a^2)_0}^-(\bar{\phi} f) \right\| \right] \\ &\geq \sup_{f \in L_a^2, \|f\|=1} \left[\left\| (I - P_{L_a^2})(\phi f) \right\| - \left\| P_{(L_a^2)_0}^-(\phi f) \right\| + \left\| (I - P_{L_a^2})(\bar{\phi} f) \right\| - \left\| P_{(L_a^2)_0}^-(\bar{\phi} f) \right\| \right] \\ &= \sup_{f \in L_a^2, \|f\|=1} \left[\|H_\phi f\| - \|\tilde{h}_\phi f\| + \|H_{\bar{\phi}} f\| - \|\tilde{h}_{\bar{\phi}} f\| \right] \\ &\geq \sup_{z \in D} \left[\|H_\phi k_z\| + \|H_{\bar{\phi}} k_z\| \right] - \|\tilde{h}_\phi\| - \|\tilde{h}_{\bar{\phi}}\| \\ &\geq \sup_{z \in D} MO(\phi)(z) - C\|V\phi\|_\infty - \tilde{C}\|V\bar{\phi}\|_\infty \end{aligned}$$

$$= \|\phi\|_{BMO_\partial} - M(\|V\phi\|_\infty + \|V\bar{\phi}\|_\infty)$$

for some positive constants $C, \tilde{C}, M$.

It also follows³ that

$$\begin{aligned} \|H_\phi\| + \|H_{\bar{\phi}}\| &= \sup_{f \in L_h^2, \|f\|=1} \|(I - P_{L_h^2})(\phi f)\| + \sup_{f \in L_h^2, \|f\|=1} \|(I - P_{L_h^2})(\bar{\phi} f)\| \\ &\leq \sup_{f \in L_a^2, \|f\|=1} \|(I - P_{L_a^2})(\phi f)\| + \sup_{f \in L_a^2, \|f\|=1} \|(I - P_{L_a^2})(\bar{\phi} f)\| \\ &\leq N\|\phi\|_{BMO_\partial}. \end{aligned}$$

The theorem follows.

One can also define another Hankel-like operator as follows: For $\phi \in L^\infty(D)$, define $S_\phi : L_h^2 \rightarrow L_h^2$ as $S_\phi f = P(J(\phi f))$, where $J : L^2 \rightarrow L^2$ such that $Jf(z) = f(\bar{z})$. We shall call S_ϕ as the little Hankel-like operator on the harmonic Bergman space L_h^2 . Let $P_{L_a^2}$ be the orthogonal projection from L^2 onto L_a^2 and $P_{(\overline{L_a^2})_0}$ be the orthogonal projection from L^2 onto $(\overline{L_a^2})_0$. Define

$$\begin{aligned} B_\phi : L_a^2 &\rightarrow L_a^2 \text{ as } B_\phi f = P_{L_a^2}(J(\phi f)); \\ D_\phi : (\overline{L_a^2})_0 &\rightarrow (\overline{L_a^2})_0 \text{ as } D_\phi \bar{f} = P_{(\overline{L_a^2})_0}(J(\phi \bar{f})); \\ h_\phi : L_a^2 &\rightarrow (\overline{L_a^2})_0 \text{ as } h_\phi f = P_{(\overline{L_a^2})_0}(J(\phi f)); \\ \Gamma_\phi : (\overline{L_a^2})_0 &\rightarrow L_a^2 \text{ as } \Gamma_\phi \bar{f} = P_{L_a^2}(J(\phi \bar{f})). \end{aligned}$$

Define $T_\phi : L_h^2 \rightarrow L_h^2$ by $T_\phi\{f, \bar{g}\} = T_\phi\{f + \bar{g}\} = B_\phi f + D_\phi \bar{g} = \{B_\phi f, D_\phi \bar{g}\}$ if $f \in L_a^2, \bar{g} \in (\overline{L_a^2})_0$.

Similarly define $R_\phi : L_h^2 \rightarrow L_h^2$ by $R_\phi\{f, \bar{g}\} = R_\phi\{f + \bar{g}\} = J h_\phi f + J \Gamma_\phi \bar{g} = \{J h_\phi f, J \Gamma_\phi \bar{g}\}$. Then $T_\phi + R_\phi = S_\phi, T_\phi = B_\phi \oplus D_\phi$ and $R_\phi = J h_\phi \oplus J \Gamma_\phi$.

Let

$$c = \sup\{\|B_\phi\|, \|D_\phi\|\} \leq \|\phi\|_\infty$$

and

$$d = \sup\{\|J h_\phi\|, \|J \Gamma_\phi\|\} \leq \|\phi\|_\infty.$$

$$\text{Now } \|T_\phi\{f, \bar{g}\}\| = \|T_\phi(f + \bar{g})\| = \|\{B_\phi f, D_\phi \bar{g}\}\| = [\|B_\phi f\|^2 + \|D_\phi \bar{g}\|^2]^{\frac{1}{2}}$$

$\leq [\|\phi\|_\infty^2 \|f\|^2 + \|\phi\|_\infty^2 \|\bar{g}\|^2]^{\frac{1}{2}} = \|\phi\|_\infty^2 \|f + \bar{g}\|^2 = \|\phi\|_\infty^2 \|\{f, \bar{g}\}\|^2$. So T_ϕ is bounded with $\|T_\phi\| \leq \|\phi\|_\infty$. Further for $f \in L_a^2, g \in (\overline{L_a^2})_0$,

$$\|B_\phi f\| = \|\{B_\phi f, 0\}\| = \|T_\phi \{f, 0\}\| \leq \|T_\phi\| \|\{f, 0\}\| = \|T_\phi\| \|f\|$$

and $\|D_\phi \bar{g}\| = \|\{0, D_\phi \bar{g}\}\| = \|T_\phi \{0, \bar{g}\}\| \leq \|T_\phi\| \|\{0, \bar{g}\}\| = \|T_\phi\| \|\bar{g}\|$. Thus $\|B_\phi\| \leq \|T_\phi\|$ and $\|D_\phi\| \leq \|T_\phi\|$. Hence $c \leq \|T_\phi\|$ and so $\|T_\phi\| = c \leq \|\phi\|_\infty$. Similarly one can show that $\|R_\phi\| = d \leq \|\phi\|_\infty$.

Theorem 2.4: If $\phi \in L^\infty(D)$, the operator S_ϕ is compact if B_ϕ, D_ϕ, h_ϕ and Γ_ϕ are compact.

Proof: Suppose T_ϕ is compact. Let $\{f_n\} \in L_a^2, \|f_n\|_2 \leq c_1$ and $\{\bar{g}_n\} \in (\overline{L_a^2})_0, \|\bar{g}_n\|_2 \leq c_2$. Then $\|f_n + \bar{g}_n\|_2^2 = \|f_n\|_2^2 + \|\bar{g}_n\|_2^2 \leq c_1^2 + c_2^2 = c^2$ (let). Hence $\{T_\phi(f_n + \bar{g}_n)\}$ has a convergent subsequence $\{T_\phi(f_{n_k} + \bar{g}_{n_k})\}$ and suppose $\{T_\phi(f_{n_k} + \bar{g}_{n_k})\}$ converges to $f + \bar{g}$. Thus it follows that

$$\|B_\phi f_{n_k} - f\|_2^2 + \|D_\phi \bar{g}_{n_k} - \bar{g}\|_2^2 = \|(B_\phi f_{n_k} + D_\phi \bar{g}_{n_k}) - (f + \bar{g})\|_2^2 \rightarrow 0.$$

Therefore $B_\phi f_{n_k} \rightarrow f$ and $D_\phi \bar{g}_{n_k} \rightarrow \bar{g}$. Conversely, let B_ϕ and D_ϕ are compact. Let $\{f_n + \bar{g}_n\} \in L_a^2 + (\overline{L_a^2})_0$ and $\|f_n + \bar{g}_n\|_2 \leq c$. This implies $\|f_n\|_2^2 + \|\bar{g}_n\|_2^2 \leq c^2$ and therefore $\|f_n\|_2 \leq c$ and $\|\bar{g}_n\|_2 \leq c$. Since B_ϕ and D_ϕ are compact, $\{B_\phi f_n\}$ and $\{D_\phi \bar{g}_n\}$ have convergent subsequences $\{B_\phi f_{n_k}\}$ and $\{D_\phi \bar{g}_{n_k}\}$. Hence $T_\phi(f_{n_k} + \bar{g}_{n_k}) = \{B_\phi f_{n_k} + D_\phi \bar{g}_{n_k}\}$ converges. Thus T_ϕ is compact if and only if B_ϕ and D_ϕ are compact. Similarly one can show that R_ϕ is compact if and only if h_ϕ and Γ_ϕ are compact. Therefore it follows that if B_ϕ, D_ϕ, h_ϕ and Γ_ϕ are compact then $T_\phi + R_\phi = S_\phi$ is compact.

The Bloch space B of D is defined to be the space of analytic functions on D such that

$$\|f\|_B = \sup\{(1 - |z|^2)|f'(z)| : z \in D\} < \infty.$$

$\|f\|_B$ is a complete semi-norm on B . B can be made into a Banach space by introducing the norm

$$\|f\| = |f(0)| + \|f\|_B.$$

The little Bloch space of D , denoted B_0 , is the closed subspace of B consisting of functions f with $(1 - |z|^2)f'(z) \rightarrow 0$ as $|z| \rightarrow 1^-$.

Proposition 2.3: If $\phi \in B_0, \phi \neq 0$, then $S_{\bar{\phi}}$ is not compact.

Proof: We have already seen that $S_{\bar{\phi}} = T_{\bar{\phi}} + R_{\bar{\phi}} = (B_{\bar{\phi}} \oplus D_{\bar{\phi}}) + (Jh_{\bar{\phi}} \oplus J\Gamma_{\bar{\phi}})$.

Notice that for $f \in L_a^2, JB_{\bar{\phi}}f = JP_{L_a^2}(J(\bar{\phi}f)) = P_{(\overline{L_a^2})}(\bar{\phi}f) = \tilde{h}_{\bar{\phi}}f$.

It is shown³, if $\phi \in B_0$, then $\tilde{h}_{\bar{\phi}}$ is compact. Hence if $\phi \in B_0, B_{\bar{\phi}}$ is compact. If $\phi \in B_0$, it is not difficult to check that $D_{\bar{\phi}} \equiv 0$. Further, if $\phi \in B_0$, then for $f \in L_a^2$,

$$h_{\bar{\phi}}f = P_{(\overline{L_a^2})_0}(J(\bar{\phi}f)) = JP_{L_a^2}J(J(\bar{\phi}f)) = JP_{L_a^2}(\bar{\phi}f) = JL_{\bar{\phi}}f,$$

where $L_{\bar{\phi}}f = P_{L_a^2}(\bar{\phi}f)$. Hence $Jh_{\bar{\phi}} = L_{\bar{\phi}}$. The operator $L_{\bar{\phi}} : L_a^2 \rightarrow L_a^2$ is compact if and only if $\phi \equiv 0$. Further if $\phi \in B_0, \bar{f} \in (\overline{L_a^2})_0$,

$$J\Gamma_{\bar{\phi}}\bar{f} = JP_{L_a^2}J(\bar{\phi}\bar{f}) = JP_{L_a^2}(\phi^+f^+),$$

where $\phi^+(z) = \bar{\phi}(\bar{z})$ and $f^+(z) = \bar{f}(\bar{z})$.

Notice that $f^+ \in L_a^2$ and $\phi^+ \in B_0$ and $J\Gamma_{\bar{\phi}}\bar{f} = JL_{\phi^+}f^+$. That is, $J\Gamma_{\bar{\phi}}Jf^+ = JL_{\phi^+}f^+$ and hence $J\Gamma_{\bar{\phi}}J = JL_{\phi^+}$.

But L_{ϕ^+} is compact if and only if $\phi^+ \equiv 0$. That is, if and only if $\phi \equiv 0$. It follows therefore that $J\Gamma_{\bar{\phi}}$ is compact if and only if $\phi \equiv 0$. Suppose $\phi \in B_0, \phi \neq 0$ and $S_{\bar{\phi}}$ is compact. Since $B_{\bar{\phi}}$ is compact, $D_{\bar{\phi}} \equiv 0$, we obtain $S_{\bar{\phi}}$ is compact if and only if $Jh_{\bar{\phi}} : L_a^2 \rightarrow (\overline{L_a^2})_0$ and $J\Gamma_{\bar{\phi}} : (\overline{L_a^2})_0 \rightarrow \overline{L_a^2}$ are compact. But $Jh_{\bar{\phi}}, J\Gamma_{\bar{\phi}}$ are compact³ if and only if $\phi \equiv 0$. This is a contradiction since $\phi \neq 0$. Hence $S_{\bar{\phi}}$ is not compact.

3. Characterization of Hankel-Like Operators

Define $L_z : L_a^2 \rightarrow L_a^2$ such that $L_zf = zf$. The operator L_z is called the Bergman shift operator. Define $M_{\bar{z}} : (\overline{L_a^2})_0 \rightarrow (\overline{L_a^2})_0$ such that $M_{\bar{z}}\bar{g} = \bar{z}\bar{g}$ for $\bar{g} \in (\overline{L_a^2})_0$. It is easy to verify that $L_z^* = L_{\bar{z}}$, where $L_{\bar{z}}f = P_{L_a^2}(\bar{z}f)$ and $M_{\bar{z}}^* = M_z$, where $M_z\bar{g} = P_{(\overline{L_a^2})_0}(z\bar{g})$.

The last statement can be verified as follows: For $\bar{g} \in (\overline{L_a^2})_0, \bar{h} \in (\overline{L_a^2})_0$,

$$\langle M_{\bar{z}}^*\bar{g}, \bar{h} \rangle = \langle \bar{g}, M_{\bar{z}}\bar{h} \rangle = \langle \bar{g}, \bar{z}\bar{h} \rangle = \int_D \bar{g}\bar{z}h dA(z) = \langle z\bar{g}, \bar{h} \rangle = \langle P_{(\overline{L_a^2})_0}(z\bar{g}), \bar{h} \rangle.$$

Notice that

$$(L_z^* \oplus M_{\bar{z}}^*)(B_\phi \oplus D_\phi) = L_z^* B_\phi \oplus M_{\bar{z}}^* D_\phi = B_\phi L_z \oplus D_\phi M_{\bar{z}} = (B_\phi \oplus D_\phi)(L_z \oplus M_{\bar{z}}).$$

Let

$$k_\xi(z) = \frac{1 - |\xi|^2}{(1 - \bar{\xi}z)^3}, \xi, z \in D.$$

Definition 3.1: A collection of points $\{\xi_i\}_{i=1}^\infty$ in D is called an η -lattice if the hyperbolic balls $\{z : \rho(z, \xi_i) < \eta\}$ cover D , and the ξ_i are separated in the sense that $\rho(\xi_i, \xi_j) \geq c_0 \eta, i \neq j$. Here c_0 is a constant associated with the η -lattice and

$$\rho(z, w) = \frac{|z - w|}{|1 - \bar{w}z|}.$$

Theorem 3.1: Let $\{\xi_i\}_{i=1}^\infty$ be an η -lattice with η sufficiently small, then for $g \in L_a^1$ (that is, g is analytic in D and $\int_D |g| dA < \infty$), there exist $\{\lambda_i\}$ in l^1 such that (i) $g = \sum_i \lambda_i k_{\xi_i}$ (ii) $\sum_i |\lambda_i| \leq \alpha \|g\|_1$, where α is a constant which depends only on η .

Proof: For the proof, the paper of N. Faour⁴ may be seen.

Theorem 3.1, is also true⁵ for $g \in L_a^2$. Thus we obtain $\text{span}\{k_{\xi_i}\}$ is dense in L_a^1 and L_a^2 . Further it is known⁴ that $\|k_\xi\|_1 \leq a, \xi \in D$ and a is a constant.

For $\phi \in L^\infty(D)$, let $S_\phi \in \mathcal{L}(L_h^2)$ be the little Hankel operator on the harmonic Bergman space L_h^2 defined by $S_\phi f = PJ(\phi f)$. We have seen above that if $S_\phi(L_a^2) \subset L_a^2$ and $S_\phi((\overline{L_a^2})_0) \subset (\overline{L_a^2})_0$, i.e., if $S_\phi = B_\phi \oplus D_\phi$ then

$$(L_z^* \oplus M_{\bar{z}}^*)(B_\phi \oplus D_\phi) = (B_\phi \oplus D_\phi)(L_z \oplus M_{\bar{z}}).$$

The converse of this statement is also true as the following theorem shows.

Theorem 3.2: Suppose $S \in \mathcal{L}(L_h^2), S(L_a^2) \subset L_a^2$ and $S((\overline{L_a^2})_0) \subset (\overline{L_a^2})_0$. That is, $S = T_1 \oplus T_2$, where $T_1 \in \mathcal{L}(L_a^2)$ and $T_2 \in \mathcal{L}((\overline{L_a^2})_0)$. Further, assume that

$$(L_z^* \oplus M_{\bar{z}}^*)(T_1 \oplus T_2) = (T_1 \oplus T_2)(L_z \oplus M_{\bar{z}}).$$

Then there exist $\phi_1, \phi_2 \in L^\infty(D)$ such that $T_1 = B_{\phi_1}$ and $T_2 = D_{\phi_2}$. If $\phi_1 \equiv \phi_2 \equiv \phi$, then $S = S_\phi$ is a little Hankel-like operator on the harmonic Bergman space L_h^2 .

Proof: We shall prove first that if $C \in \mathcal{L}(L_a^2)$ and $L_z^* C = C L_z$ then there exists $\psi \in L^\infty(D)$ such that $C = B_\psi$. Let $W = \text{span}\{k_{\xi_i}\}$, where $\{\xi_i\}$ is an η -lattice. Define the linear functional Λ on W by $\Lambda(f) = \langle C f, 1 \rangle$. Notice that $W \subset L_a^2$ and hence is contained

in L_a^1 . From theorem 3.1, given $f \in W$ there exist $\{\lambda_i\}$ in l^1 such that $f = \sum_i \lambda_i k_{\xi_i}$ and $\sum |\lambda_i| \leq \alpha \|f\|_1$. Suppose $0 < r < 1$, notice that $\|k_{\xi_i}(rz)\|_2 \leq b(r)$, a function depending only on r . Thus, with $f_r(z) = f(rz)$ we see that

$$\langle C f_r, 1 \rangle = \left\langle C \left(\sum_i \lambda_i k_{\xi_i}(rz) \right), 1 \right\rangle = \sum_i \lambda_i \langle C(k_{\xi_i}(rz)), 1 \rangle = \sum_i \lambda_i \left\langle C(k_{\xi_i}^{\frac{1}{2}}(rz)), (k_{\xi_i}^{\frac{1}{2}}(rz))^+ \right\rangle.$$

Therefore,

$$|\langle C f_r, 1 \rangle| \leq \sum_i |\lambda_i| \|C\| \sup_{\xi_i} \|k_{\xi_i}(rz)\|_1.$$

Consequently, it follows that

$$|\langle C f_r, 1 \rangle| \leq \alpha a \|f\|_1 \|C\|.$$

But $f_r \rightarrow f$ in L_a^2 . Thus, by the continuity of Λ it follows that $|\Lambda(f)| \leq \beta \|f\|_1$ for some constant β . Since $\text{span}\{k_{\xi_i}\}$ is dense in L_a^1 it follows that Λ extends by continuity to an element of $(L_a^1)^*$, and consequently, by the Hahn Banach theorem to an element of $(L^1)^* = L^\infty(D)$. Therefore, there exists $\psi \in L^\infty(D)$, such that

$$\begin{aligned} \langle C f, 1 \rangle &= \langle \psi f, 1 \rangle = \langle J(\psi f), 1 \rangle \\ &= \langle P(J(\psi f)), 1 \rangle = \langle B_\psi f, 1 \rangle. \end{aligned}$$

Moreover, $\text{span}\{k_{\xi_i}\}$ is dense⁵ in L_a^2 . Thus, it follows that $\langle B_\psi h, 1 \rangle = \langle Ch, 1 \rangle, h \in L_a^2$. Using the fact that

$$\langle C p q^+, 1 \rangle = \langle C p, q \rangle$$

(where p, q are polynomials in z , $p^+(z) = \overline{p(\bar{z})}$), it follows that $\langle C p, q \rangle = \langle B_\psi p, q \rangle$, and hence, $C = B_\psi$.

Similarly one can show that if there exists $E \in \mathcal{L}((\overline{L_a^2})_0)$ such that $M_{\bar{z}}^* E = E M_{\bar{z}}$ then there exists $\tilde{\psi} \in L^\infty(D)$ such that $E = D_{\tilde{\psi}}$.

Now since $S = T_1 \oplus T_2, T_1 \in \mathcal{L}(L_a^2), T_2 \in \mathcal{L}((\overline{L_a^2})_0)$ and

$$(L_z^* \oplus M_{\bar{z}}^*)(T_1 \oplus T_2) = (T_1 \oplus T_2)(L_z \oplus M_{\bar{z}})$$

we obtain that $L_z^* T_1 = T_1 L_z$ and $M_{\bar{z}}^* T_2 = T_2 M_{\bar{z}}$.

Hence there exist $\phi_1, \phi_2 \in L^\infty(D)$ such that $T_1 = B_{\phi_1}$ and $T_2 = D_{\phi_2}$.

If $\phi_1 \equiv \phi_2 \equiv \phi$, then $S = T_1 \oplus T_2 = B_\phi \oplus D_\phi = S_\phi$ as $S(L_a^2) \subset L_a^2$ and $S((\overline{L_a^2})_0) \subset (\overline{L_a^2})_0$. The theorem follows.

Theorem 3.3: If $\phi \in L^\infty(D)$ and L_a^2 is a reducing subspace for S_ϕ then S_ϕ is normal in $\mathcal{K}(L_h^2)$ if and only if $S_\phi|_{L_a^2}$ is normal and $S_\phi|_{(\overline{L_a^2})_0}$ is normal.

Here $T|_H$ means the restriction of the operator T to the Hilbert space H .

Proof: Since $S_\phi(L_a^2) \subset L_a^2$ and $S_\phi((\overline{L_a^2})_0) \subset (\overline{L_a^2})_0$, hence $S_\phi = B_\phi \oplus D_\phi$.

Therefore, $S_\phi^* = B_\phi^* \oplus D_\phi^* = B_{\phi^+} \oplus D_{\phi^+} = S_{\phi^+}$, where $\phi^+(z) = \overline{\phi(\bar{z})}$. Thus $S_\phi^* = S_{\phi^+}$. Now S_ϕ is normal if and only if $S_\phi^* S_\phi = S_\phi S_\phi^*$. That is, if and only if $S_{\phi^+} S_\phi = S_\phi S_{\phi^+}$. Since $S_\phi(L_a^2) \subset L_a^2$ and $S_\phi((\overline{L_a^2})_0) \subset (\overline{L_a^2})_0$, hence it follows⁶ that $S_{\phi^+}(L_a^2) \subset L_a^2$ and $S_{\phi^+}((\overline{L_a^2})_0) \subset (\overline{L_a^2})_0$ and $PS_\phi S_{\phi^+} \bar{P} = \bar{P} S_\phi S_{\phi^+} P = 0$, $PS_{\phi^+} S_\phi \bar{P} = \bar{P} S_{\phi^+} S_\phi P = 0$.

Hence S_ϕ is normal if and only if

$$\begin{pmatrix} PS_\phi S_{\phi^+} P & 0 \\ 0 & \bar{P} S_\phi S_{\phi^+} \bar{P} \end{pmatrix} = \begin{pmatrix} PS_{\phi^+} S_\phi P & 0 \\ 0 & \bar{P} S_{\phi^+} S_\phi \bar{P} \end{pmatrix}.$$

Thus $PS_\phi S_{\phi^+} P = PS_{\phi^+} S_\phi P$ and $\bar{P} S_\phi S_{\phi^+} \bar{P} = \bar{P} S_{\phi^+} S_\phi \bar{P}$.

Therefore $S_\phi S_{\phi^+} P = S_{\phi^+} S_\phi P$ and $S_\phi S_{\phi^+} \bar{P} = S_{\phi^+} S_\phi \bar{P}$. Hence S_ϕ is normal if and only if $S_\phi|_{L_a^2}$ is normal and $S_\phi|_{(\overline{L_a^2})_0}$ is normal.

References

1. W. Rudin, *Real and complex Analysis* (third edition), McGraw-Hill, New York, 1987.
2. P. L. Duren, *Theory of H^p spaces*, Academic press, New York, 1970.
3. K. Zhu, *Operator theory in function spaces*, Monographs and Textbooks in Pure and Applied Mathematics, Marcell Dekker, Inc. 139, New York and Basel, 1990.
4. N. Faour, A factorization theorem for Bergman spaces, *Indian J. Math.*, **30** (1988) 17-27.
5. R. R. Coifman and R. Rochberg, Representation theorems for holomorphic and harmonic functions in L^p , *Asterisque*, **77** (1980) 11-66.
6. N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in Hilbert space*, Monographs and studies in Mathematics, **9** Pitman, 1981.