

Existence Theorem for I-Nonexpansive Mapping

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(Received September 20, 2007)

Abstract: In this paper we find common fixed point of Mann type iterate of an I-nonexpansive maps in Banach space which satisfies Opial condition. We also furnish example in support of our results.

Mathematical Classification (2000): 49H10, 54H25.

Key Words: Uniformly Banach space quasi nonexpansive mappings Mann type iteration, common fixed point.

1. Introduction

The concept of quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real function. Diaz and Metcalf¹ and Doston² gave some results in Banach space by using quasi-nonexpansive mapping. Recently, this is used in metric space by Kirk³.

There are many results on fixed point for nonexpansive and quasi-nonexpansive mappings in Banach space and metric space due to Petryshyn and Williamson⁴, Ghosh and Debnath⁵, Temir and Gul⁶ and Zhou, Agrawal, Cho⁷ etc.

2. Preliminaries

Let K be a closed convex bounded subset of uniformly Banach space X . A self map T on K is nonexpansive if $\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in K$.

Let $F(T) = \{x \in K: Tx = x\}$ i.e. $F(T)$ be the set of fixed points of T . T is called a quasi nonexpansive mapping provided $\|Tx - f\| \leq \|x - f\| \quad \forall x, y \in K$ and $f \in F(T)$.

In this paper, we consider T_1, T_2 and I self mappings of K , where T_1, T_2 are I-nonexpansive mappings. We establish the weak convergence of sequence of Mann type iterate to a common fixed point of T_1, T_2 and I .

Let X be a normed linear space, K be a nonempty convex subset of X . Let T be a self map on K . The Mann type iterative scheme $\{x_n\}$ is defined as $x_0 = x \in K$ and

$$x_{n+1} = (1-k_n) T_1 x_n + k_n T_2 x_n$$

for every n in N , where k_n is a sequence in $(0,1)$.

A Banach space X is said to satisfy Opial condition⁸, if for each sequence $\{x_n\}$ in X , the condition

$$\overline{x_n \rightarrow x} \Rightarrow \lim_{n \rightarrow \infty} \|x_n - x\| < \lim_{n \rightarrow \infty} \|x_n - y\| \quad \forall y \in X \text{ with } y \neq x.$$

All l_p spaces for $1 < p < \infty$ have this property, however L_p space do not (unless $p=2$).

Let K be a subset of normed space $X=(X, \|\cdot\|)$ and T and I self mapping of K . then T is called I - nonexpansive on K if

$$\|Tx - Ty\| \leq \|Ix - Iy\| \quad \forall x, y \in K$$

T is called I -quasi nonexpansive on K if

$$\|Tx - f\| \leq \|Ix - f\| \quad \forall x \in K \text{ and } f \in F(T) \cap F(I).$$

3. Main Results

Theorem 1 : Let K be a closed convex bounded subset of uniformly convex Banach space satisfying Opial condition. Let T_1 and T_2 be two I quasi non - expansive mappings on K . For $x_0 \in K$ the sequence $\{x_n\}$ of Mann type iteration converges weakly to a common fixed point of T_1 T_2 and I .

Proof:- Let $F(T_1) \cap F(T_2) \cap F(I)$ is non empty set. If it contains only one point then the point is common fixed point. If it contains more than one points then

$$\begin{aligned} \|x_{n+1} - f\| &= \|(1-k_n) T_1 x_n + k_n T_2 x_n - f\| \\ &= \|(1-k_n) T_1 x_n + k_n T_2 x_n - (1-k_n + k_n)f\| \\ &\leq (1-k_n) \|T_1 x_n - f\| + k_n \|T_2 x_n - f\| \\ &\leq (1-k_n) \|I x_n - f\| + k_n \|I x_n - f\| \\ &= \|x_n - f\| \end{aligned}$$

Thus $\|x_{n+1} - f\|$ is non increasing sequence, it implies that $\lim \|x_n - f\|$ exists.

Now we will show that $\{x_n\}$ converge weakly to a point I . For weakly convergence of $\{x_n\}$, let us consider subsequences $\{x_{nk}\}$ and $\{x_{mj}\}$ which converge weakly to f and q respectively. Let $f \neq q$ is in weak limit set of sequence. Since $\lim \|x_n - f\|$ exists for $f \in F(T_1) \cap F(T_2) \cap F(I)$, by Opial condition of X , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - f\| &= \lim_{k \rightarrow \infty} \|x_{nk} - f\| \\ &< \lim_{k \rightarrow \infty} \|x_{nk} - q\| = \lim_{n \rightarrow \infty} \|x_n - q\| \\ &= \lim_{j \rightarrow \infty} \|x_{mj} - q\| < \lim_{j \rightarrow \infty} \|x_{mj} - f\| \\ &= \lim_{n \rightarrow \infty} \|x_n - f\| \end{aligned}$$

which is a contradiction, therefore $f = q$. Hence $\{x_n\}$ converges weakly to an element of $F(T_1) \cap F(T_2) \cap F(I)$, which is required common fixed point of T_1, T_2 and I .

Now we define Mann* type iteration as

$$x_{n+1} = k_{n1}T_1x_n + k_{n2}T_2x_n + k_{n3}T_3x_n + \dots + k_{ni}T_ix_n$$

where $k_{n1} + k_{n2} + k_{n3} + \dots + k_{ni} = 1$ and each k_{ni} in $(0, 1)$ and $i = 1, 2, 3, \dots, l$.

Theorem 2:- Let K be a closed convex bounded subset of uniformly convex Banach space satisfying Opial condition. Let $T_1, T_2, \dots, T_l$ be I quasi nonexpansive mappings on K . For $x_0 \in K$ the sequence $\{x_n\}$ of Mann* type iteration converges weakly to a common fixed point of $T_1, T_2, \dots, T_l$ and I .

Proof:- Similar to the proof of the theorem 1.

Example : Let $X = [0, \infty)$ and $K = [0, 1]$. The sequence $\{x_n\}$ in K is defined as $\{x_n\} = \{1/(n+1)\}$. Let T_1, T_2 & I be self mappings on K defined as follows:

$$T_1x = 2x(1-x)/(1+x), \quad T_2x = 4x/3(1+x) \quad \text{and} \quad Ix = 5x^2/6x^2+1.$$

$F(T_1) = \{0, 1/3\}$, $F(T_2) = \{0, 1/3\}$ and $F(I) = \{0, 1/3, 1/2\}$ and k_n in $(0, 1)$ is defined as $k_n = 3(1-n)/2(2-n)$, which satisfy all conditions of the Theorem 1 and hence 0 is the common fixed point of T_1, T_2 and I .

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