

Ideal Purity and Absolute Purity in Modules

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Abstract: We generalize the concept of relative divisibility of elements in submodules (Known as **RD** -purity) to that of I -purity, where I is a given left ideal of the ring. We characterize I -purity in terms of preservation of exactness by cyclic modules R/I and then in terms of exactness of a related sequence of modules. I -pure exact sequence has been shown to constitute a proper class of short exact sequences. We then consider absolute purity with respect to this purity and prove analogues of some results of Maddox¹ for the case of absolute I -purity.

Key words: Left R -modules, Ideals, Ideal purity, Finitely generated, submodule, I -pure submodule, Absolutely I -pure.

1. Introduction

The notion of purity plays a fundamental role in the theory of abelian groups as well as in module categories. In first section of this paper we define the concept of I -purity for a left ideal I of the ring R . We also derive the connection of abstract ω -purity with I -purity which is equivalent to some results of abelian groups and modules² and certain conditions equivalent to the concept of I -purity which generalize some results in purity in abelian groups³, some of them being under the additional condition that the ideal is two sided. In the second section of the present paper we define the concept of absolute I -purity and derive some consequences. Unless otherwise noted we use the notion and conventions of F. W. Anderson and K. R. Fuller⁴. In this paper R refers to a ring with identity, which need not be necessarily commutative. Also by an R module we always mean a left R -module.

2. Ideal Purity

Definition 2.1: Let R be a ring. Let M be a left R -module and L is a submodule of M . Let I be a left ideal of R . Then L is said to be I -pure in M if $IL = L \cap IM$.

Definition 2.2: A short exact sequence $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$ is said to be pure exact if $\text{Image}(\alpha)$ is an I -pure submodule of M . That is $I\alpha(L) = IM \cap \alpha(L)$, where I is left ideal of R .

We shall start with ideal purity for a two sided ideal I of R . The following proposition is an analogue of D. P. Choudhury⁵.

Proposition 2.1: *The following conditions are equivalent:*

- (i) $IL = L \cap IM$ for all two sided ideals I ,
- (ii) $L/IL \longrightarrow M/IM$ is monic for all two sided ideals I ,
- (iii) $(R/I) \otimes L \longrightarrow (R/I) \otimes M$ is monic for all two sided ideals I ,
- (iv) $IL = L \cap IM$ for all finitely generated two sided ideals I ,
- (v) $L/IL \longrightarrow M/IM$ is monic for all finitely generated two sided ideals I ,
- (vi) $(R/I) \otimes L \longrightarrow (R/I) \otimes M$ is monic for all finitely generated two sided ideals I .

Proof: (i) $\Rightarrow$ (ii) Let $\phi: L/IL \longrightarrow M/IM$ be a map defined by $\phi(l+IL) = (l+IM)$. Let $\phi(l_1+IL) = \phi(l_2+IL) \Rightarrow (l_1-l_2) \in IM \Rightarrow (l_1-l_2) \in L \cap IM \Rightarrow (l_1-l_2) \in IL \Rightarrow (l_1+IL) = (l_2+IL)$ which shows that ϕ is monic.

(ii) $\Leftrightarrow$ (iii) Since $(R/I) \otimes L \longrightarrow (R/I) \otimes M$ defined by $\sum \bar{r}_i \otimes l_i \mapsto \sum r_i l_i + IL$. Clearly it is an isomorphism. Similarly $(R/I) \otimes M \cong (R/I) \otimes (M/IM)$. Therefore $(R/I) \otimes L \longrightarrow (R/I) \otimes M$ is monic if and only if $L/IL \longrightarrow M/IM$ is monic.

(i) $\Rightarrow$ (iv) is trivial.

(ii) $\Rightarrow$ (v) is trivial.

(v) $\Leftrightarrow$ (vi) Proof is similar to (ii) $\Leftrightarrow$ (iii) for all finitely generated two sided ideals.

(v) $\Rightarrow$ (iv) Let $l \in (L \cap IM)$ then $(l+IL) \rightarrow (l+IM) = 0$ in $(M/IM) \Rightarrow (l+IL) = 0$ as the above map is monic $\Rightarrow l \in IL \Rightarrow IL = L \cap IM$.

(iv) $\Rightarrow$ (i) Let $\sum_{i=1}^n r_i m_i \in IM \cap L$ and we take $J = \sum_{i=1}^n R r_i$, which is a finitely generated

ideal of R . Since $\sum_{i=1}^n r_i m_i \in JM \cap L = JL$, therefore $\exists l_j \in L$ such that

$\sum r_i m_i = \sum r'_j l_j$ (finitely many l_j and $r'_j \in J$), but $JL \subseteq IL$. Therefore, $\sum r_i m_i \in IL$ therefore $L \cap IM \subseteq IL$. Also, $IL \subseteq L \cap IM$ holds, since $(L \subseteq M)$ and hence $IL = L \cap IM$ for all ideal I of R .

The following theorem is analogue of the corresponding results for purity in abelian groups, theorem 29.1³.

Theorem 2.1: For any left ideal I and a short exact sequence

$$(2.1) \quad 0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0,$$

each of the following conditions are equivalent:

- (i) The sequence (2.1) is I -pure,
- (ii) $0 \longrightarrow IL \xrightarrow{\alpha^*} IM \xrightarrow{\beta^*} IN \longrightarrow 0$ is exact.
- (iii) $0 \longrightarrow \frac{L}{IL} \xrightarrow{\alpha'} \frac{M}{IM} \xrightarrow{\beta'} \frac{N}{IN} \longrightarrow 0$ is exact.

Proof: (i) $\Leftrightarrow$ (ii) Suppose that $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$ is an I -pure sequence. In this proof we are assuming that α is an inclusion map i.e. $\alpha(L) = L$. Then to show that $0 \longrightarrow IL \xrightarrow{\alpha^*} IM \xrightarrow{\beta^*} IN \longrightarrow 0$ is exact, we have to show that $\text{Image}(\alpha^*) = \text{Ker}(\beta^*)$ and β^* is surjective. We are taking α^* , the restriction of α to IL and β^* is the restriction of β to IM . Now

$$(2.2) \quad \text{Image}(\alpha^*) = \alpha^*(IL) = \alpha_{IL} IL = \alpha_{IL} (L \cap IM) = L \cap IM,$$

since $IL = L \cap IM$ and α is an inclusion map. Also,

$$\text{Ker } \beta^* = \text{Ker}(\beta_{IM}) = (\text{Ker}(\beta) \cap IM) = (\text{Image}(\alpha) \cap IM) = L \cap IM, \text{ hence}$$

$$(2.3) \quad \text{Ker}(\beta^*) = L \cap IM.$$

Therefore $\text{Ker}(\beta^*) = \text{Image}(\alpha^*)$. Now to see that β^* is surjective, consider $\sum i_k n_k \in IN$ and note that $\exists m_k \in M$ such that $\beta(m_k) = n_k \forall k$. $\sum i_k n_k = \sum i_k \beta(m_k) = \beta(\sum i_k m_k) = \beta^*(\sum i_k m_k) \in \beta^*(IM)$, therefore β^* is surjective.

Conversely, Suppose that $0 \longrightarrow IL \xrightarrow{\alpha^*} IM \xrightarrow{\beta^*} IN \longrightarrow 0$ is exact, i.e. $\text{Ker } \beta^* = \text{Image } \alpha^*$, for left ideal I of R . Then we have to show that $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$ is I -pure exact. Since

$$(2.4) \quad I(\alpha(L)) = \text{Image}(\alpha^*) = \text{Ker}(\beta^*) = \{ \sum i_j m_j, \ i_j \in I, m_j \in M \mid \beta(\sum i_j m_j) = 0 \} \\ = \{ \sum i_j m_j, \ i_j \in I, m_j \in M \text{ such that } \sum i_j \beta(m_j) = 0 \}.$$

Also,

$$(2.5) \quad \alpha(L) \cap IM = \{ \sum i_j m_j \mid \sum i_j m_j = \alpha(l), l \in L \} \\ = \{ \sum i_j m_j, \ i_j \in I, m_j \in M \text{ such that } \sum i_j m_j \in \text{Im}(\alpha) = \text{Ker}(\beta) \}$$

$$\begin{aligned}
&= \{ \sum i_j m_j, \quad i_j \in I, m_j \in M, \text{ such that } \beta(\sum i_j m_j) = 0 \} \\
&= \{ \sum i_j m_j, \quad i_j \in I, m_j \in M, \text{ such that } \sum i_j \beta(m_j) = 0 \}.
\end{aligned}$$

From the equation (2.4) and (2.5) we get $I\alpha(L) = \alpha(L) \cap IM$.

(i) $\Leftrightarrow$ (iii) We consider

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & IL & \xrightarrow{\alpha} & IM & \xrightarrow{\beta} & IN \longrightarrow 0 \\
& \lambda_1 \downarrow & & \mu_1 \downarrow & & \gamma_1 \downarrow & \\
0 & \longrightarrow & L & \xrightarrow{\alpha^*} & M & \xrightarrow{\beta^*} & N \longrightarrow 0 \\
& \lambda_2 \downarrow & & \mu_2 \downarrow & & \gamma_2 \downarrow & \\
0 & \longrightarrow & \frac{L}{IL} & \xrightarrow{\alpha'} & \frac{M}{IM} & \xrightarrow{\beta'} & \frac{N}{IN} \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

The proof follows from 3×3 lemma.

Proposition 2.2: *I-purity is an inductive property.*

Proof: Suppose $M_1, M_2, M_3, \dots$ are I-pure submodules of M , and $M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots$ is an ascending chain of these I-pure submodules, then let $M' = \bigcup_{j \in J} M_j$.

Claim: M' is I-pure in M .

For this, we have to show that $IM' = M' \cap IM$. Clearly,

$$(2.6) \quad IM' \subseteq M' \cap IM.$$

Let $\sum r_\alpha m_\alpha \in (IM \cap M') \Rightarrow \sum r_\alpha m_\alpha \in IM$ and $\sum r_\alpha m_\alpha \in M' = \bigcup_{i \in I} M_i$. Suppose

$\sum r_\alpha m_\alpha = m_{i_1}$ for some $m_{i_1} \in M_{i_1} \Rightarrow \sum r_\alpha m_\alpha \in IM \cap M_{i_1} = IM_{i_1}$, since each M_i is I-pure in M and as $IM_{i_1} \subseteq IM'$ and

$$(2.7) \quad \sum r_\alpha m_\alpha \in IM' \Rightarrow IM \cap M' \subseteq IM'.$$

From (2.6) and (2.7) we get $IM' = M' \cap IM$. Hence M' is I-pure in M .

This theorem is an analogue of the corresponding results for purity in abelian groups due to Prufer, lemma 26.1³.

Theorem 2.2: *Let L and M be submodules of N such that $L \subseteq M \subseteq N$, then we have:*

(i) *If L is I-pure in M and M is I-pure in N then L is I-pure in N ,*

- (ii) If M is I -pure in N then M/L is I -pure in N/L ,
 (iii) If L is I -pure in N then L is I -pure in M ,
 (iv) If L is I -pure in N , M/L is I -pure in N/L then M is I -pure in N .

Proof: (i). If $IL = L \cap IM$ and $IM = M \cap IN$ then we show that $IL = L \cap IN$.

Since $IL = L \cap IM = L \cap (IN \cap M) = (L \cap M) \cap IN = L \cap IN$, since $L \cap M = L$. Hence, $IL = L \cap IN$.

(ii) If $IM = M \cap IN$ then to show that $I(M/L) = (M/L) \cap I(N/L)$. Clearly

$$(2.8) \quad I(M/L) \subseteq (M/L) \cap I(N/L).$$

Let $\sum r_i n_i + L \in (M/L) \cap I(N/L)$, ($r_i \in I$)

$$\Rightarrow \sum r_i n_i + L \in (M/L) \text{ and } \sum r_i n_i + L \in I(N/L),$$

$$\Rightarrow \sum r_i n_i + L = m + L \text{ for some } m \in M. \text{ Let } m = \sum r_i n_i + l \text{ where } l \in L.$$

$$\Rightarrow (m_i - l) = \sum r_i n_i \in M \cap IN = IM \Rightarrow (m - l) = \sum r'_i m'_i \text{ for some } m'_i \in M, r'_i \in I.$$

$$\Rightarrow m + L = \sum r'_i (m'_i + L) \in I(M/L). \text{ Hence}$$

$$(2.9) \quad (M/L) \cap I(N/L) \subseteq I(M/L).$$

From (2.8) and (2.9) we get $(M/L) \cap I(N/L) = I(M/L)$.

(iii) $L \cap IN = IL$ (given), therefore $L \cap IM \subseteq L \cap IN = IL \Rightarrow L \cap IM = IL$. Hence L is I -pure in M .

(iv) Since L is I -pure in N that is $L \cap IN = IL$ and M/L is I -pure in N/L , i.e.

$(M/L) \cap I(N/L) = I(M/L)$ then we show that M is I -pure in N that is $IM = M \cap IN$.

$$(2.10) \quad IM \subseteq M \cap IN.$$

Let $\sum m r_i n_i \in IN \cap M \Rightarrow \sum r_i n_i \in IN$ ($r_i \in I$) and $\sum r_i n_i \in M$. Then $\sum r_i (n_i + L) \in I(N/L)$ and $\sum r_i (n_i + L) = \sum r_i n_i + L \in (M/L) \Rightarrow \sum r_i n_i + L \in (M/L) \cap I(N/L) = I(M/L) \Rightarrow \sum r_i n_i + L = \sum r'_j m_j + L$ for some $m_j \in M$ and $r'_j \in I \Rightarrow \sum r_i n_i - \sum r'_j m_j \in L$ and also belongs to $L \cap IN = IL$ (since $m_j \in M \subseteq N$) $\Rightarrow \sum r_i n_i - \sum r'_j m_j = \sum r_k l_k$ for some $l_k \in L$ and $r_k \in I \Rightarrow \sum r_i n_i = \sum r'_j m_j + \sum r_k l_k \in IM$ ($l_k \in L \subseteq M$). Therefore

$$(2.11) \quad IN \cap M \subseteq IM.$$

From (2.10) and (2.11) we get

$$M \cap IN = IM.$$

3. Absolute I-Purity

Now we consider absolute I -purity with respect to this I -purity and prove analogues of some results of B. H. Maddox¹. At the same time, we generalize certain results in purity in abelian groups to module categories.

Definition 3.1: An R -module M is said to be absolutely I -pure if M is an I -pure submodule of every module which contains M as a submodule.

Lemma 3.1: If E' is an I -pure submodule of an absolutely I -pure left R -module E then E' is an absolutely I -pure left R -module.

Proof: Let F be a left R -module containing E' . We consider the diagram

$$\begin{array}{ccccc} & 0 & & 0 & \\ & \downarrow & & \downarrow & \\ 0 & \longrightarrow & E' & \xrightarrow{f} & F \\ & & g \downarrow & & g' \downarrow \\ 0 & \longrightarrow & E & \xrightarrow{f'} & G \end{array}$$

where G is the pushout of (f, g) . Since g and f are monic, then g' and f' be also monic. As E is absolutely I -pure, therefore E is I -pure in G . Also, E' is given to be I -pure in E , hence E' is I -pure in G . Since $(f'og) = (g'of)$. Therefore, f is an I -pure monomorphism, that is E' is I -pure in F . Hence E' is an absolutely I -pure left R -module.

Definition 3.2: A ring R is hereditary if and only if every ideal is a projective module. such rings are called Dedekind rings, if they are commutative integral domains.

Theorem 3.1: If E is a module over a Dedekind domain A then the following conditions are equivalent:

- (a) E is divisible,
- (b) E is injective,
- (c) E is absolutely I -pure, for each ideal I of R .

Proof: (b) $\Rightarrow$ (c) is trivial because every injective module is absolutely I -pure.

(a) $\Rightarrow$ (b) Follows from proposition 3, page- 134⁶.

(c) $\Rightarrow$ (a) Consider $E \subseteq E'$ (E' is injective hull of E). Let $\lambda \in R$ and $e \in E$. Then by divisibility of E' , there exists $e' \in E'$ such that $\lambda e' \in E \cap I E'$ where $I = R\lambda$. By hypothesis E is I -pure in E' . Therefore $IE = E \cap I E'$. Thus $e \in IE$, i.e. $e = \sum r_i e_i$, with $r_i \in I$ and $e_i \in E$. Therefore $e = \sum r_i \lambda e_i = \lambda (\sum r_i e_i) \in \lambda E$. So λ divides e in E , $\forall e \in E$ and for all $\lambda \neq 0 \in R$. Hence E is divisible.

Proposition 3.1: An R -module A is absolutely I -pure for all ideal I if and only if $\text{Hom}(F/C, E) \longrightarrow \text{Hom}(F/C, E/A)$ is surjective for each F/C , where F is finitely generated free module, C is a cyclic submodule of F and E is the injective hull of A .

Proof: For this, it is sufficient to show that:

(i) If A is I -pure in $E(A)$ then A is I -pure in E' for any injective module E' such that $A \subseteq E'$, where $E(A)$ is the injective hull of A , because if $A \subseteq E'$ then $E' = E(A) \oplus D$, where D is a submodule of E' , also, A is I -pure in $E(A)$ and $E(A)$ is I -pure in E' (as it is a direct summand). Therefore A is I -pure in E' .

(ii) If A is I -pure in any containing injective module, then A is I -pure in B for every $A \subseteq B$.

$$\begin{array}{c} A \xrightarrow{\subseteq} E(A) \\ \subseteq \downarrow \\ B \xrightarrow{\subseteq} E(B) \end{array}$$

By hypothesis, A is I -pure in $E(B)$ but $A \subseteq B \subseteq E(B) \Rightarrow A$ is I -pure in B , therefore A is absolutely I -pure if and only if A is I -pure in $E(A)$, i.e. $IE(A) \cap A = IA$ if and only if every module expressible as F/C , where F is a finitely generated free module and C is a cyclic submodule of it, is projective relative to the sequence

$$0 \longrightarrow A \longrightarrow E(A) \longrightarrow E(A)/A \longrightarrow 0.$$

if and only if $\text{Hom}(F/C, E(A)) \longrightarrow \text{Hom}(F/C, E(A)/A) \longrightarrow 0$ is surjective.

Corollary 3.1: If A is an absolutely I -pure R -module, then $\text{Ext}_R^1(F/B, A) = 0$ for all cyclic left ideals B of R .

Proposition 3.2: An R -module A is absolutely I -pure if and only if $\text{Ext}_R^1(M, A) = 0$, where M will belong to the class of finitely generated free by cyclic modules. That is $\text{Ext}_R^1(F/C, A) = 0$, where F is finitely generated free modules and C is cyclic.

Proof: Let E the injective envelope of A . We then have the exact sequence

$$\text{Hom}_R(F/C, E) \xrightarrow{\theta} \text{Hom}(F/C, E/A) \xrightarrow{\varphi} \text{Ext}_R^1(F/C, A) \longrightarrow 0.$$

Obviously θ is surjective if and only if $\text{Ext}_R^1(F/C, A) = 0$.

Proposition 3.3: An R -module A is absolutely I -pure if and only if for every diagram

$$\begin{array}{ccc} P' & \xrightarrow{\alpha} & P \\ \beta \downarrow & & \\ & & A \end{array}$$

with P' is cyclic and P is projective $\exists$ a homomorphism $\gamma: P \longrightarrow A$ such that $(\gamma\alpha) = \beta$.

Proof: We consider $P = \bigoplus_{\lambda} R$, that is P is a free R -module. We may assume that $P' = \alpha(x) = (a_{i_1}, 0, \dots, a_{i_k}, 0, \dots, 0, a_{i_n}, 0, \dots, 0, \dots)$ then there is a finitely generated

free direct summand $F = \bigoplus_{\{J\}} R$ of P such that $P' \subseteq F$. Therefore finding the desired γ , it is equivalent to extending β to F . But from the exact sequence

$$\text{Hom}(F, A) \xrightarrow{\theta} \text{Hom}(P', A) \xrightarrow{\varphi} \text{Ext}_R^1(F/P', A) \longrightarrow 0,$$

which gives $\text{Ext}_R^1(F/P', A) = 0$.

Proposition 3.4: (a) An arbitrary direct sum of absolutely I -pure R -module is absolutely I -pure, (b) The union of a direct family of absolutely I -pure submodules is an absolutely I -pure submodule.

Proposition 3.5: Let $\{A_i, \phi_{ij}\}$ be a direct system of R -modules, where each A_i is absolutely I -pure, each ϕ_{ij} is a monomorphism, then $A = \varinjlim A_i$ is absolutely I -pure.

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