

## An Extension of Littlewood's 'O' Tauberian Theorem

**T. Pati**

10, Bank Road, Allahabad, India  
(Department of Mathematics, University of Allahabad)  
Email: pati\_tribikram@yahoo.co.in

(Received January 15, 2007)

**Abstract:** Two general Tauberian theorems have been established, which extend the famous 'O' Tauberian theorem of Littlewood and are companions to a general Tauberian theorem recently proved by the author.

**Key words and phrases:** Summability, Tauberian theorems.

### 1. Definition and Notations

Let  $\sum_{n=0}^{\infty} a_n$  be an infinite series of real terms, and let us write

$$S_n = a_0 + a_1 + \dots + a_n \quad (n = 0, 1, 2, \dots).$$

Let  $A_n^\alpha$  be defined by

$$(1-x)^{-\alpha-1} = \sum_{n=0}^{\infty} A_n^\alpha x^n \quad (|x| < 1);$$

for  $\alpha > -1$ , we write

$$S_n^\alpha = \sum_{\nu=0}^n A_{n-\nu}^{\alpha-1} s_\nu,$$

$$s_n^\alpha = S_n^\alpha / A_n^\alpha, \quad s_n^0 = s_n.$$

Then  $s_n^\alpha$  is called the Cesaro mean of order  $\alpha$  or the  $(C, \alpha)$  mean of  $\{s_n\}$  and we say that  $\sum a_n$  or  $\{s_n\}$  is summable  $(C, \alpha)$  to  $s$  if

$$\lim_{n \rightarrow \infty} s_n^\alpha = s.$$

We write

$$\tau_n = na_n, \quad (n = 0, 1, 2, \dots)$$

and define  $\tau_n^\alpha$  as the  $(C, \alpha)$  mean of  $\{\tau_n\}$ . Thus

$$\tau_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v \quad (n = 1, 2, \dots),$$

$$\tau_0^\alpha = 0, \quad \tau_n^0 = \tau_n.$$

If  $\sum_{n=0}^{\infty} a_n x^n$  is convergent for  $0 \leq x < 1$ , we see that the sum-function  $f(x)$  is given by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = (1-x) \sum_{n=0}^{\infty} s_n x^n.$$

$$\text{If } \lim_{x \rightarrow 1-0} f(x) = s,$$

then we say that  $\sum a_n$  is summable by the Abel method, or summable  $(A)$ , to  $s$ <sup>1,2</sup>.

A generalization of both the  $(C, \alpha)$  and the Abel methods is provided by the 'product method'  $(A)(C, \alpha)$ , defined for the first time by Szász in 1952<sup>3</sup>.

If, for  $0 \leq x < 1$ , with  $s_{-1}^\alpha = 0$ ,

$$g(x) = \sum_{n=0}^{\infty} (s_n^\alpha - s_{n-1}^\alpha) x^n = (1-x) \sum_{n=0}^{\infty} s_n^\alpha x^n,$$

and  $\lim_{x \rightarrow 1-0} g(x) = s$ , then we shall say that  $\sum a_n$  is summable  $(A)(C, \alpha)$  to  $s$ . Thus the  $(A)(C, \alpha)$  summability of  $\sum a_n$  or  $\{s_n\}$  is the  $(A)$  summability of the sequence  $\{s_n^\alpha\}$ , which is the sequence of  $(C, \alpha)$  means of  $\{s_n\}$ .

It is useful to note that, using the notation ' $P \subset Q$ ' to mean that summability  $P$  implies summability  $Q$ , we have the following important results<sup>1,3</sup>:

$(C, \alpha) \subset (C, \beta), \beta > \alpha > -1; (C, \alpha) \subset (A)$  for every  $\alpha$ , however large ( $>0$ );

$(A) \not\subset (C, \alpha)$  for any  $\alpha$ , however large ( $>0$ );

$(C, 0)$  is the same as convergence;  $(A)(C, 0)$  is the same as  $(A)$ ;

$(A) \subset (A)(C, \alpha)$  for every  $\alpha > 0$ ;  $(A)(C, \alpha) \subset (A)(C, \beta), \beta > \alpha > -1$ .

A different kind of generalization of the method  $(A)$  has been introduced by Littlewood<sup>4</sup> in 1967.

Let us assume that  $\sum_{n=0}^{\infty} a_n x^n$  is convergent to  $f(x)$  for  $0 \leq x < 1$ , and that  $\int_0^1 f(t) dt$

exists as  $\lim_{\xi \rightarrow 1} \int_0^\xi f(t) dt$ .

Let us write

$$f_1(x) = \frac{1}{1-x} \int_x^1 f(t) dt.$$

Suppose that  $\int_0^1 f_1(t) dt$  exists as  $\lim_{\xi \rightarrow 1} \int_0^\xi f(t) dt$ .

Let us write

$$f_2(x) = \frac{1}{1-x} \int_x^1 f_1(t) dt,$$

and so on, so that, for positive integral  $k$ ,

$$f_k(x) = \frac{1}{1-x} \int_x^1 f_{k-1}(t) dt,$$

assuming that all the integrals  $\int_0^1 f_{k-1}(t) dt, \int_0^1 f_{k-2}(t) dt, \dots$ , which have gone before exist as

$$\lim_{\xi \rightarrow 1} \int_0^\xi f_{k-1}(t) dt, \lim_{\xi \rightarrow 1} \int_0^\xi f_{k-2}(t) dt, \dots$$

Then, following Littlewood, we call  $f_k(x)$  the 'k-tuple average' of  $f$  as  $x \rightarrow 1$

Now, if for an infinite series  $\sum_{n=0}^{\infty} a_n$ , for some positive integer  $k$ ,

$$\lim_{x \rightarrow 1} f_k(x) = s,$$

then we shall say that  $\sum_{n=0}^{\infty} a_n$  is summable  $(A)^{(k)}$  to  $s$ .

We have 'coined' this definition for convenience. If, for  $0 \leq x < 1$ , and  $\alpha > -1$ ,

$$g(x) = (1-x) \sum_{n=0}^{\infty} s_n^\alpha x^n,$$

and, for some positive integer  $k$ ,

$$\lim_{x \rightarrow 1} g_k(x) = s,$$

we shall say that  $\sum_{n=0}^{\infty} a_n$  is summable  $(A)^{(k)}(C, \alpha)$  to  $s$ .

Evidently summability  $(A)^{(0)}$  is the same as summability  $(A)$  and summability  $(A)^{(k)}(C, 0)$  is the same as summability  $(A)^{(k)}$ . We follow Landau and write ' $\theta_n = O(1)$ ' to mean that  $\{\theta_n\}$  is bounded sequence, and ' $\theta_n = o(1)$ ' to mean that  $\{\theta_n\}$  converges to zero. ' $f_n = O(g_n)$ ' means that  $f_n / g_n = O(1)$  and ' $f_n = o(g_n)$ ' means that  $f_n / g_n = o(1)$ . ' $f_n \sim g_n$ ' means that  $f_n / g_n \rightarrow 1$ , as  $n \rightarrow \infty$ . We shall write

$$\theta_n = o(1)(C, \alpha)$$

to mean that

$$\theta_n^\alpha = o(1),$$

where  $\theta_n^\alpha$  is the  $(C, \alpha)$  mean of  $\{\theta_n\}$ . Generally, we shall write  $\sum a_n$  for  $\sum_{n=0}^{\infty} a_n$ .

## 2. Introduction

A fundamental theorem due to Abel is<sup>5</sup>:

**Theorem A:** If  $\sum a_n$  is convergent to  $s$ , then  $\sum a_n$  is summable  $(A)$  to  $s$ .

The converse of Theorem A is not necessarily true, as can be seen from the example of the series

$$1 - 1 + 1 - 1 + \dots$$

which, although summable  $(A)$  to  $\frac{1}{2}$ , is not convergent.

Thus it is apparent that some restriction needs to be imposed on  $a_n$  to ensure that  $\sum a_n$  is convergent when it is known that  $\sum a_n$  is summable  $(A)$ . Tauber was the first to investigate the problem of finding such restrictions. He proved the following two theorems which are thus the first two Tauberian theorems and form the foundation of the whole of Tauberian Theory, which today occupies a very important place in Mathematics.

**Theorem B<sup>6</sup>:** If  $\sum a_n$  is summable  $(A)$  to  $s$  and

$$(2.1) \quad na_n = o(1),$$

then  $\sum a_n$  is convergent to  $s$ .

**Theorem C<sup>6</sup>:** If  $\sum a_n$  is summable  $(A)$  to  $s$  and

$$(2.2) \quad \tau_n^1 = o(1),$$

i.e.

$$\frac{1}{n+1}(a_1 + 2a_2 + \dots + na_n) = o(1),$$

then  $\sum a_n$  is convergent.

It is interesting that Theorem B is used in proving Theorem C.

It is easily seen that  $\tau_n^1 = o(1)$  is not only sufficient but also necessary for summability (A) to imply convergence to the same 'sum', because we have

$$s_n - s_n^1 = \tau_n^1$$

and it is well known that ' $s_n \rightarrow s$ ' implies that ' $s_n^1 \rightarrow s_n^1$ '. Thus ' $\tau_n^1 = o(1)$ ' is a necessary condition for convergence per se. Therefore it is somewhat misleading to state that  $\tau_n^1 = o(1)$  is necessary for summability (A) to imply convergence of  $\sum a_n$ . However, it is customary to say that  $\tau_n^1 = o(1)$  is a necessary and sufficient condition for the summability (A) of  $\sum a_n$  to  $s$  to imply the convergence of  $\sum a_n$  to  $s$ . This result is referred to in the literature as Tauber's Second Theorem and Theorem B is called Tauber's First Theorem. Tauber's theorems were established in 1897. In 1910 Hardy proved the following Tauberian theorem<sup>7</sup>.

**Theorem D:** If  $\sum a_n$  is summable  $(C, k)$ ,  $k > 0$ , to  $s$ , and, as  $n \rightarrow \infty$ ,

$$(2.3) \quad na_n = O(1),$$

then  $\sum a_n$  is convergent to  $s$ .

The question whether, in Theorem D, summability  $(C, k)$  can be replaced by summability (A) was settled by Littlewood in 1911<sup>8</sup>, when he proved.

**Theorem E:** If  $\sum a_n$  is summable (A) to  $s$  and (2.3) holds, then  $\sum a_n$  is convergent to  $s$ .

This is the famous 'O' Tauberian theorem of Littlewood and marked the beginning of nearly thirtyfive years of collaboration between the two celebrated mathematicians Hardy and Littlewood.

Soon afterwards, in 1914, the condition (2.3) in Littlewood's Theorem E was replaced by Hardy and Littlewood by the 'One-sided Tauberian Condition'. They proved<sup>9</sup>

**Theorem F:** If  $\sum a_n$  is summable (A) to  $s$  and

$$(2.4) \quad na_n > -H,$$

where  $H$  is some positive constant, then  $\sum a_n$  is convergent to  $s$ .

In his classic paper of 1911<sup>8</sup> Littlewood stated the following theorem, which has been subsumed under the general Tauberian Theory, developed some twenty years later by Wiener<sup>10</sup>, for an elegant account of which one may turn to<sup>1</sup>, where the theorems referred to above have been presented in a systematic manner.

**Theorem G:** If  $\sum a_n$  is summable (A) to  $s$  and

$$(2.5) \quad s_n = O(1),$$

then  $\sum a_n$  is summable  $(C, 1)$  to  $s$ .

A sharper result was proved in 1935 by Szász<sup>11</sup> in the form of

**Theorem H:** If  $\sum a_n$  is summable (A) to  $s$  and

$$(2.6) \quad \tau_n^1 > -H,$$

where  $H$  is some positive constant, then  $\sum a_n$  is summable (C,1) to  $s$ .

That (2.6) is lighter than (2.5) is clear from the identity:  $s_n - s_n^1 = \tau_n^1$ , which shows that (2.5) implies (2.6) and it is easily seen that the converse is not necessarily true.

Recently the present author has proved the following still more general theorem<sup>12,13</sup>.

**Theorem I:** If  $\sum a_n$  is summable (A)(C,p) to  $s$  for some  $p \geq 0$ , and

$$(2.7) \quad \tau_n^p > -H,$$

where  $H$  is a positive constant, then  $\sum a_n$  is summable (C,p) to  $s$ .

In 1967 Littlewood<sup>4</sup> gave a generalization of Theorem F of Hardy and Littlewood by proving

**Theorem J:** If  $\sum a_n$  is summable (A)<sup>(k)</sup> for some positive integer  $k$ , to  $s$ , then (2.4) is a Tauberian condition for the convergence of  $\sum a_n$  to  $s$ .

### 3. Aim of the Paper

The object of this paper is to prove the following two theorems of which the first is more general than both our Theorem I and Littlewood's Theorem J, and the second is more general than Littlewood's 'O' Tauberian Theorem E.

**Theorem 1:** If, for some positive integer  $k$ ,  $\sum a_n$  is summable (A)<sup>(k)</sup> (C,p) to  $s$ ,  $p \geq 0$ , and (2.7) holds, then  $\sum a_n$  is summable (C,p).

**Theorem 2:** If, for some positive integer  $k$ ,  $\sum a_n$  is summable (A)<sup>(k)</sup> (C,p) to  $s$ ,  $p \geq 0$ , and

$$(3.1) \quad \tau_n^p = O(1),$$

then  $\sum a_n$  is summable (C,  $p + \delta - 1$ ) for every  $\delta > 0$ , to  $s$ .

From Theorem 2 we deduce the following corollaries.

**Corollary 1:** If, for some positive integer  $k$ ,  $\sum a_n$  is summable (A)<sup>(k)</sup> (C,1) to  $s$  and

$$(3.2) \quad \tau_n^1 = O(1),$$

then  $\sum a_n$  is summable (C,  $\delta$ ) for every  $\delta > 0$ , to  $s$ .

**Corollary 2:** *If, for some positive integer  $k$  and  $0 < p < 1$ ,  $\sum a_n$  is summable  $(A)^{(k)}(C, p)$  to  $s$  and*

$$(3.3) \quad \tau_n^p = O(1),$$

*then  $\sum a_n$  is convergent to  $s$ .*

**Corollary 3:** *If, for some positive integer  $k$ ,  $\sum a_n$  is summable  $(A)^{(k)}$  to  $s$  and (2.3) holds, then  $\sum a_n$  is summable  $(C, \delta - 1)$  to  $s$  for every  $\delta > 0$ .*

**Remarks:** The Tauberian condition in Theorem 2 is stronger than the Tauberian condition in Theorem 1. However, in compensation, the result of Theorem 2 is sharper than that of Theorem 1.

In Corollary 1 of Theorem 2 our hypothesis:  $\sum a_n$  is summable  $(A)^{(k)}(C, 1)$  is lighter than that of Szász's Theorem H:  $\sum a_n$  is summable  $(A)$ , while our Tauberian condition is stronger than that of Theorem H. Our result:  $\sum a_n$  is summable  $(C, \delta)$ ,  $\delta > 0$ , is stronger than that of Theorem H:  $\sum a_n$  is summable  $(C, 1)$ . In Corollary 2 the hypothesis:  $\sum a_n$  is summable  $(A)^{(k)}(C, p)$ ,  $0 < p < 1$ , is evidently lighter than the hypothesis of Littlewood's Theorem E, and our Tauberian condition is lighter than the 'O' Tauberian condition of Littlewood. We still get the convergence of  $\sum a_n$ . Corollary 3 is more general than the sharper form of Theorem E which is obtained by replacing convergence by summability  $(C, \delta - 1)$  by virtue of the result<sup>1</sup>, (Theorem 45): If  $\sum a_n$  is convergent and  $na_n = O(1)$ , then  $\sum a_n$  is summable  $(C, \delta - 1)$  for every  $\delta > 0$ .

#### 4. Certain Lemmas

**Lemma 1**<sup>14,15,7</sup>: For  $\alpha > -1$ ,

$$(i) \quad n(s_n^\alpha - s_{n-1}^\alpha) = \tau_n^\alpha;$$

$$(ii) \quad (\alpha + 1)(s_n^\alpha - s_n^{\alpha+1}) = \tau_n^{\alpha+1}.$$

**Lemma 2**<sup>2</sup>: *All the Cesàro methods of positive order are equivalent for bounded sequences. More precisely, if  $s_n = O(1)$  and  $s_n^\alpha \rightarrow s$  as  $n \rightarrow \infty$ , for some  $\alpha > 0$ , then  $s_n^\beta \rightarrow s$  as  $n \rightarrow \infty$  for every  $\beta > 0$ .*

**Lemma 3**<sup>1</sup>: (Notes on Chapter V, 5.8). *If  $s_n^\alpha \rightarrow s(C, \beta)$ , i.e., the  $(C, \beta)$  mean of  $\{s_n^\alpha\} \rightarrow s$ , then  $s_n^{\alpha+\beta} \rightarrow s$  for  $\alpha \geq 0$ ,  $\beta \geq 0$ , and conversely.*

**Lemma 4**<sup>1</sup>: *If  $s_n^\alpha \rightarrow s$  as  $n \rightarrow \infty$ ,  $\alpha > -1$ , then  $s_n^{\alpha+\delta} \rightarrow s$  as  $n \rightarrow \infty$  for every  $\delta > 0$ .*

This is the familiar 'consistency theorem' for Cesàro summability.

### 5. Proof of Theorem 1

Without loss of generality, we assume that  $s = 0$ . Then our hypothesis is that, as  $x \rightarrow 1$ ,  $g_k(x) = o(1)$ , where  $g_k(x)$  is the 'k-tuple average' of

$$g(x) = (1-x) \sum_{n=0}^{\infty} s_n^p x^n, \quad 0 \leq x < 1.$$

Now

$$g(x) = \sum_{n=0}^{\infty} (s_n^p - s_{n-1}^p) x^n.$$

By Lemma 1(i),

$$n(s_n^p - s_{n-1}^p) = \tau_n^p > -H$$

for some positive constant, by hypothesis. Hence, by the result of Theorem J, we get

$$\sum_{n=0}^{\infty} (s_n^p - s_{n-1}^p) \quad (s_{-1}^p = 0)$$

is convergent i.e.,

$$s_n^p = o(1),$$

which means that  $\sum a_n$  is summable  $(C, p)$  to zero.

This completes the proof of Theorem 1.

### 6. Proof of Theorem 2

Since  $\tau_n^p = O(1) \Rightarrow \tau_n^p > -H$  for some positive constant  $H$ , by Theorem 1 we obtain

$$(6.1) \quad s_n^p = o(1).$$

Hence, by Lemma 4,

$$s_n^{p+1} = o(1)$$

and by Lemma 1 (ii),

$$\tau_n^{p+1} = (p+1)(s_n^p - s_n^{p+1}) = o(1).$$

By Lemma 3, this is equivalent to

$$\tau_n^p = o(1)(C, 1).$$

But, by hypothesis,

$$\tau_n^p = O(1).$$

Hence, by Lemma 2,

$$\tau_n^p = o(C, \delta) \text{ for every } \delta > 0.$$

This, by Lemma 3, is equivalent to

$$\tau_n^{p+\delta} = o(1).$$

By Lemma 1 (ii) this means that

$$(6.2) \quad s_n^{p+\delta-1} - s_n^{p+\delta} = o(1).$$

Now, by (6.1),

$$s_n^p = o(1),$$

and hence, by Lemma 4,

$$s_n^{p+\delta} = o(1) \text{ for every } \delta > 0.$$

Hence, from (6.2), we get

$$s_n^{p+\delta-1} = o(1),$$

which is the same as the statement that  $\sum a_n$  is summable  $(C, p + \delta - 1)$  to zero.

This completes the proof of Theorem 2.

## 7. Proof of Corollaries

For proving Corollary 1, we take  $p = 1$  in Theorem 2.

For proving Corollary 2, we take  $\delta = 1 - p$  ( $0 < p < 1$ ) in Theorem 2.

For proving Corollary 3, we take  $p = 0$  in Theorem 2.

## References

1. G. H. Hardy, *Divergent Series*, Oxford, 1949.
2. A. Peyerimhoff, *Lectures on Summability*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 1969.
3. Otto Szász, Products of Summability methods, *Proc. Amer. Math. Soc.*, **3** (1952) 257-263.
4. J. E. Littlewood, A theorem about successive derivatives of a function and some Tauberian theorems, *Jour. London Math. Soc.*, **42** (1967) 169-179.
5. E. C. Titchmarsh, *The Theory of Functions*, Oxford, 1939.
6. A. Tauber, Ein Satz aus der Theorie unendlichen Reihen, *Monatshefte f. Math.*, **8** (1897) 273-277.
7. G. H. Hardy, Theorem relating\* to summability and convergence of slowly oscillating series, *Proc. London Math. Soc.* (2), **8** (1910) 301-320.

8. J. E. Littlewood, The converse of Abel's theorem on power series, *Proc. London Math. Soc.* (2), **9** (1910) 434-448.
9. G. H. Hardy And J. E. Littlewood, Tauberian theorems concerning power series and Dirichlet series whose coefficients are positive, *Proc. London Math. Soc.* (2), **13** (1914) 174-191.
10. N. Wiener, Tauberian Theorems, *Annals of Math.* (2), **33** (1932) 1-100.
11. Otto Szász, Generalization of two theorems of Hardy and Littlewood on power series, *Duke Math. Jour.*, **1** (1935) 105-111.
12. T. Pati, Tauberian Theorems, Prof. R. S. Misra Memorial Lecture, *Proc. Third Conference of the International Academy of Physical Sciences*, Allahabad (1999) 15-27.
13. T. Pati, Tauberian Theorems, *Proc. National Conference on Recent Developments in Sequences, Summability and Fourier Analysis*, Ed. Dipty Rath and S. Nanda, Narosa Publishing House, New Delhi.
14. E. G. Kogbetliantz, Sur les séries absolument sommables par la méthode des moyennes arithmétiques, *Bull. Des Sciences Math.*, (2), **49** (1925) 234-256.
15. E. G. Kogbetliantz, Sommaton des séries et integrales divergentes par les moyennes arithmétiques et typiques. *Mémorial des Sciences Math.*, **51** (1931) Paris.