

Submanifolds Immersed in a Generalized Para Structure Metric Manifold

Kalpana Singh

Department of Mathematics and Astronomy, University of Lucknow, Lucknow, India

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Abstract: Almost complex and almost contact submanifolds have been studied by Mishra¹ and others. In this paper submanifolds of a generalized para structure metric manifold have been studied and certain results related to such submanifolds have been obtained.

1. Preliminaries

Let M^n be a differentiable manifold of class C^∞ and there exists on M^n a tensor field F of type $(1,1)$, $r(C^\infty)$ 1 - forms $\overset{\alpha}{A}$, $r(C^\infty)$ vector fields $\overset{T}{\alpha}$ and scalar functions $\overset{\beta}{p}$, $\alpha, \beta = 1, 2, \dots, r$, satisfying

$$(1.1a) \quad F^2 = -a^2 I_n - \eta \overset{\alpha}{A} \otimes \overset{T}{\alpha},$$

$$(1.1b) \quad F\left(\overset{T}{\alpha}\right) = \overset{\beta}{p} \overset{T}{\alpha},$$

where a and η are constants. Then the structure $\left\{F, \overset{\alpha}{A}, \overset{\beta}{T}, \overset{\beta}{p}, \alpha, \beta = 1, 2, \dots, r\right\}$ is known as generalized para structure and M^n is known as generalized para structure manifold.

In generalized para structure manifold M^n , the following results hold:

$$(1.2) \quad \overset{\alpha}{A} \circ F = \overset{\beta}{p} \overset{\alpha}{A},$$

$$(1.3) \quad \eta \overset{\alpha}{A}\left(\overset{T}{\beta}\right) = a^2 \overset{\alpha}{\delta} + \overset{\alpha}{p} \overset{\gamma}{o} \overset{\gamma}{p},$$

where δ_{β}^{α} denotes kronecker delta.

Suppose that M^n is endowed with a Riemannian metric g . If

$$(1.4) \quad F(X, Y) = g(FX, Y),$$

be skew symmetric then M^n is called generalized para structure metric manifold. On generalized para structure metric manifold, we have

$$(1.5) \quad g(FX, FY) - a^2 g(X, Y) - \eta A(X) A(Y) = 0,$$

where

$$(1.6) \quad A(X) = g\left(T, X\right).$$

2. Submanifolds

Let M^{2n-r+1} be a submanifold of M^{2n+1} and let $b : M^{2n-r+1} \rightarrow M^{2n+1}$ be the inclusion map such that

$$q \in M^{2n-r+1} \rightarrow bq \in M^{2n+1}.$$

The map b induces the Jacobian map B such that

$$B : V_{(2n-r+1)} \rightarrow V_{(2n+1)},$$

where $V_{(2n-r+1)}$ and $V_{(2n+1)}$ are tangent spaces at q of M^{2n-r+1} and bq of M^{2n+1} respectively.

Thus for X in $V_{(2n-r+1)}$ at $q \Rightarrow BX$ in $V_{(2n+1)}$ at bq .

If g^* is the induced metric tensor in M^{2n-r+1} , then

$$(2.1) \quad (g(BX, BY))ob = g^*(X, Y).$$

Let $N_x, x = 1, 2, \dots, r$ be the set of r mutually orthogonal normals to M^{2n-r+1} . Then

$$(2.2a) \quad g(BX, N_x) = 0,$$

and

$$(2.2b) \quad g(N_x, N_y) = \delta_{xy}.$$

Let D be the induced Riemannian connection in M^{2n-r+1} . Then Gauss and Weingarten equations are given by

$$(2.3) \quad E_{BX} BY = BD_X Y + \overset{x}{H}(X, Y) N_x,$$

where $\overset{x}{H}$ are symmetric bilinear functions in M^{2n-r+1} and

$$(2.4) \quad E_{BX} N_x = -B \overset{y}{H}^*(X) + \overset{y}{\ell}(X) N_y,$$

where

$$(2.5a) \quad g^*(H^*(X), Y) = H(X, Y),$$

$$(2.5b) \quad \overset{y}{\ell} + \overset{x}{\ell} = 0.$$

Theorem 2.1: The necessary and sufficient conditions that M^{2n-r+1} be generalized para structure metric submanifold with the structure $\left\{ F^*, \overset{\alpha}{A}', \overset{\beta}{T}', \overset{\beta}{p}, \alpha, \beta = 1, 2, \dots, r \right\}$ in M^{2n+1} are

$$(2.6a) \quad \left(\overset{\alpha}{A}(BX) \right)_{ob} = \overset{\alpha}{A}'(X),$$

and

$$(2.6b) \quad F(BX) = BF^* X,$$

where

$$(2.7) \quad \overset{\alpha}{T} = B \overset{\alpha}{T}'.$$

Proof: Let us put

$$(2.8a) \quad F(BX) = BF^* X + \overset{x}{q}(X) N_x,$$

$$(2.8b) \quad \overset{\alpha}{T} = B \overset{\alpha}{T}' + \overset{x}{\phi} N_x,$$

$$(2.8c) \quad \overset{x}{F}(N) = -B \overset{y}{P} + \overset{y}{\xi} N_y.$$

Pre multiplying (2.8a) by F , we get

$$F^2(BX) = F(BF^* X) + \overset{x}{q}(X) F(N_x).$$

Using (1.1) and (2.8c), we obtain

$$-a^2 BX - \eta \overset{\alpha}{A}(BX) \overset{\alpha}{T} = BF^{*2} X + q(F^* X) \overset{x}{N} + q(X) \overset{x}{\left\{ -B P + \overset{y}{\xi} N \right\}}.$$

Using (2.8b), we get

$$-a^2 BX - \eta \overset{\alpha}{A}(BX) \left(\overset{\alpha}{B} T' + \overset{x}{\phi} N \right) = BF^{*2} X + q(F^* X) \overset{x}{N} + q(X) \overset{x}{\left\{ -B P + \overset{y}{\xi} N \right\}},$$

or

$$(2.9) \quad -a^2 BX' - \eta \overset{\alpha}{A}(BX) \left(\overset{\alpha}{B} T' + \overset{x}{\phi} N \right) = BF^{*2} X - B q(X) \overset{x}{P} + \left\{ q(F^* X) + q(X) \overset{x}{\xi} \right\} \overset{x}{N}.$$

Now pre multiplying $\overset{\alpha}{T} = \overset{\alpha}{B} T'$ by F and using (1.1b), we obtain

$$\overset{\beta}{p} \overset{\alpha}{T} = BF^* \overset{\alpha}{T'}.$$

Also from equation (2.8b), we get

$$(2.10) \quad \overset{\beta}{p} \left(\overset{\alpha}{B} T' + \overset{x}{\phi} N \right) = BF^* \overset{\alpha}{T'}.$$

From (1.5), we have

$$g(FBX, FBY) - a^2 g(BX, BY) - \eta \overset{\alpha}{A}(BX) \overset{\alpha}{A}(BY) = 0.$$

Using (2.8a) and the relation $(g(BX, BY))ob = g^*(X, Y)$, we get

$$(2.11) \quad g^*(F^* X, F^* Y) + q(X) \overset{y}{q}(Y) \overset{y}{\delta} - a^2 g^*(X, Y) - \eta \left\{ \overset{\alpha}{A}(BX) ob \right\} \left\{ \overset{\alpha}{A}(BY) ob \right\} = 0.$$

Equations (2.9), (2.10) and (2.11) imply

$$(2.12a) \quad F^{*2} = -a^2 I_n - \eta \overset{\alpha}{A}' \otimes \overset{\alpha}{T'},$$

$$(2.12b) \quad F^* \overset{\beta}{p} \overset{\alpha}{T'} = \overset{\beta}{p} \overset{\alpha}{T'},$$

and

$$(2.12c) \quad g^*(F^* X, F^* Y) - a^2 g^*(X, Y) - \eta \overset{\alpha}{A}'(X) \overset{\alpha}{A}'(Y) = 0,$$

where

$$\overset{\alpha}{A}'(X) = g^* \overset{\alpha}{(T', X)},$$

if and only if

$$(2.13a) \quad (A(BX))_{ob} = A'(X),$$

$$(2.13b) \quad \eta \underset{\alpha}{A'(X)} \underset{\alpha}{\phi}^x = q(F^* X)^x + q(X)^y \underset{y}{\xi}^x,$$

$$(2.13c) \quad q(X)^x P = 0,$$

$$(2.13d) \quad \underset{\beta}{\phi}^x = 0,$$

and

$$(2.13e) \quad q(X)^x q(Y)^y \underset{x}{\delta}^y = 0.$$

Since $\underset{\alpha}{A}(BX) \neq 0$, equations (2.13) are consistent if and only if

$$(2.14a) \quad (\underset{\alpha}{A}(BX))_{ob} = \underset{\alpha}{A}'(X),$$

$$(2.14b) \quad q(X)^x = 0,$$

$$(2.14c) \quad \underset{\alpha}{\phi}^x = 0.$$

Substituting from (2.14) in (2.8a) and (2.8b), we obtain (2.6b) and (2.7).

3. Some Other Results

In this section we shall prove some more theorems on the submanifold M^{2n-r+1} of the manifold M^{2n+1} .

Theorem 3.1: Let M^{2n-r+1} be a generalized para structure metric submanifold of a generalized para structure metric manifold M^{2n+1} . Then

$$(3.1a) \quad \underset{x}{A}(N) = 0,$$

and

$$(3.1b) \quad F(N)^x = -\underset{x}{\xi}^z \underset{z}{o} \underset{y}{\xi}^y N.$$

Proof: Using the relation $F(X, Y) = -F(Y, X)$, we have

$$\underset{x}{F}(N, BX) = -\underset{x}{F}(BX, N),$$

or

$$g(F N, BX) = -g(FBX, N).$$

By virtue of (2.8c) and (2.6b), we get

$$g(-B P + \xi N, BX) = -g(BF^* X, N).$$

Using (2.1) and (2.2a), we obtain

$$g^*(P, X) = 0 \Rightarrow P = 0.$$

Pre multiplying (2.8c) by F and using (1.1), we get

$$-a^2 N - \eta A(N) T = \xi F(N).$$

Using (2.7) and (2.8c), we get

$$-a^2 N - \eta A(N) B T' = \xi \{-B P + \xi N\},$$

which yields (3.1a) and

$$(3.2) \quad \xi \circ \xi = -a^2 \delta.$$

From (1.5), we have

$$g(F N, F N) - a^2 g(N, N) - \eta A(N) A(N) = 0.$$

Using (2.2b), (2.8c) and (3.1a), we get

$$g(-B P + \xi N - B P + \xi N) = a^2 \delta,$$

$$\text{which implies } \xi \circ \xi \delta = a^2 \delta.$$

Let us put $z = y$, then we get

$$\xi \circ \xi \delta = a^2 \delta \Rightarrow \xi = a^2 \delta.$$

Hence from (2.8c), we get

$$(3.3) \quad F(N) = a^2 \delta N,$$

which is equivalent to $F(N) = -\xi_o \xi N$,

by making use of the relation (3.2).

Theorem 3.2: Let N and N^* be Nijenhuis tensor in the generalized para structure metric manifolds M^{2n-r+1} and M^{2n+1} respectively. Then

$$N(BX, BY) = BN^*(X, Y).$$

Proof: We have

$$F(F[BX, BY]) = F(F(E_{BX}BY - E_{BY}BX)).$$

Using (2.3) and the fact that $\tilde{H}$ are symmetric, we get

$$F(F[BX, BY]) = F(F(BD_XY - BD_YX)).$$

Also (2.6) implies

$$\begin{aligned} F(F[BX, BY]) &= F(BF^*(D_XY) - BF^*(D_YX)) \\ &= BF^{*2}(D_XY) - BF^{*2}(D_YX) \\ &= BF^{*2}[X, Y] \text{ and so on.} \end{aligned}$$

Hence, we have

$$\begin{aligned} N(BX, BY) &= F(F([BX, BY])) + [F(BX), F(BY)] - F([F(BX), (BY)]) - F([BX, F(BY)]) \\ &= BF^{*2}[X, Y] + B[F^*X, F^*Y] - BF^*[F^*X, Y] - BF^*[X, F^*Y] \\ &= B(F^{*2}[X, Y] + [F^*X, F^*Y] - F^*[F^*X, Y] - F^*[X, F^*Y]) \\ &= BN^*(X, Y). \end{aligned}$$

Theorem 3.3: Let M^{2n-r+1} be a generalized para structure submanifold of the generalized para structure metric manifold M^{2n+1} and the affine connection E in M^{2n+1} be such that

$$(3.4) \quad (E_X F)(Y) = 0.$$

Then the condition that the induced connection D in M^{2n-r+1} also satisfies the similar condition, i.e.

$$(3.5) \quad (D_X F^*)(Y) = 0,$$

is

$$(3.6a) \quad \overset{x}{H}(X, F^*Y) = -\overset{z}{\xi} \overset{y}{o} \overset{x}{\xi} \overset{x}{H}(X, Y),$$

or equivalently

$$(3.6b) \quad \overset{x}{H}(X, F^*Y) = a^2 \overset{x}{\delta} \overset{x}{H}(X, Y).$$

Proof: We have

$$E_{BX}(F(BY)) = (E_{BX}F)(BY) + F(E_{BX}BY).$$

In consequence of (3.4), (2.6b) and (2.3), we get

$$\begin{aligned} BD_x(F^*Y) + \overset{x}{H}(X, F^*Y) N_x &= F \left\{ BD_xY + \overset{x}{H}(X, Y) N_x \right\} \\ \Rightarrow D_x(F^*Y) &= F^*(D_xY) \end{aligned}$$

or

$$\begin{aligned} (D_xF^*)(Y) &= 0 \\ \Leftrightarrow \overset{x}{H}(X, F^*Y) N_x &= \overset{x}{H}(X, Y) F(N_x) \\ \Leftrightarrow \overset{x}{H}(X, F^*Y) &= a^2 \overset{x}{\delta} \overset{x}{H}(X, Y), \end{aligned}$$

by using the equation (3.3)

$$\Leftrightarrow \overset{x}{H}(X, F^*Y) = -\overset{z}{\xi} \overset{x}{\delta} \overset{x}{\xi} \overset{x}{H}(X, Y).$$

References

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