

On $*gs$ -Homeomorphism in Topological Spaces

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Abstract: In this paper we introduce a new class of closed maps namely $*gs$ -closed maps which settled in between the class of ψ -closed maps¹ and the class of gs -closed maps². We also introduce and study new class of homeomorphisms called $*gs$ -homeomorphisms and $*gs^*$ -homeomorphisms. Further we show that the set of all $*gs^*$ -homeomorphisms form a group under the operation composition of maps.

Key Words: $*gs$ -closed maps, $*gs^*$ -closed maps, $*gs$ -homeomorphisms, $*gs^*$ -homeomorphisms.

1. Introduction

The notion of homeomorphism plays an important role in topology. Let X, Y denote topological spaces. A homeomorphism is a bijective map $f: X \rightarrow Y$ when both f and f^{-1} are continuous. Malghan³ and Devi, Maki and Balachandran² introduced the concept of generalized closed maps and semi-generalized closed maps respectively in topological spaces. Shikha and Manoj⁴ introduced the concept of $*gs$ -closed sets in topological spaces. In this paper we introduce a new class of closed maps namely $*gs$ -closed maps and show that the class of gs -closed maps contains the class of closed maps (or the class of semi-closed maps or the class of ψ -closed maps) and contained in the class of gs -closed maps. Further we introduce and study $*gs$ -homeomorphisms and $*gs^*$ -homeomorphisms and then show that the set of all $*gs^*$ -homeomorphisms forms a group under the operation of composition of functions.

2. Preliminaries

Throughout this paper (X, τ) , (Y, σ) and (Z, η) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of space (X, τ) the $cl(A)$, $int(A)$ and A^c denote the closure of A , the interior of A and the complement of A in X respectively.

- (i) $*gs-cl(A)$ is the smallest $*gs$ -closed set containing A .
- (ii) A is $*gs$ -closed iff $*gs-cl(A) = A$.

Proof : Follows from definitions.

Proposition 3.6: For any two subsets A and B of (X, τ)

- (i) If $A \subseteq B$ then $*gs-cl(A) \subseteq *gs-cl(B)$.
- (ii) $*gs-cl(A \cap B) \subseteq *gs-cl(A) \cap *gs-cl(B)$.

Proof: Immediately follows from definitions.

Theorem 3.1: If A is $*gs$ -closed set and B is closed set then $A \cap B$ is $*gs$ -closed set.

Proof: Follows immediately since every closed set is $*gs$ -closed.

Definition 3.3: Let (X, τ) be a topological space and $A \subseteq X$ then $*gs$ -interior of A ($*gs-int(A)$) is defined as the union of all $*gs$ -open sets contained in A .

Lemma 3.1: For any $A \subseteq X$, $int(A) \subseteq *gs-int(A) \subseteq A$.

Proof: Since every open set is $*gs$ -open, proof is straight forward.

Proposition 3.7: A map $f: (X, \tau) \rightarrow (Y, \sigma)$ is $*gs$ -closed map iff $*gs-cl(f(A)) \subseteq f(cl(A))$ for every subset A of (X, τ) .

Proof: Suppose that f is $*gs$ -closed map and $A \subseteq X$ then $f(cl(A))$ is $*gs$ -closed in (Y, σ) . We have $f(A) \subseteq f(cl(A))$ and by proposition (3.5) and (3.6), $*gs-cl(f(A)) \subseteq *gs-cl(f(cl(A))) = f(cl(A))$.

Conversely let A be any closed set in (X, τ) then $A = cl(A)$ and so $f(A) = f(cl(A)) \supseteq *gs-cl(f(A))$ by hypothesis. We have $f(A) \subseteq *gs-cl(f(A))$ by proposition (3.5) so $f(A) = *gs-cl(f(A))$ i.e. $f(A)$ is $*gs$ -closed by proposition (3.5) and hence f is $*gs$ -closed map.

Theorem 3.2: A map $f: (X, \tau) \rightarrow (Y, \sigma)$ is $*gs$ -closed map iff for each subset A of (Y, σ) and each open set U containing $f^{-1}(A)$ there exists a $*gs$ -open set V of (Y, σ) such that $A \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof: Let f is $*gs$ -closed map. Let $A \subseteq Y$ and U be an open subset of (X, τ) such that $f^{-1}(A) \subseteq U$ then $V = (f(U^c))^c$ is a $*gs$ -open set containing A such that $f^{-1}(V) \subseteq U$.

Conversely let U be a closed set in X then $f^{-1}((f(U))^c) \subseteq U^c$ and U^c is open in X . By assumption there exists a $*gs$ -open set V of (Y, σ) s.t. $(f(U))^c \subseteq V$ and $f^{-1}(V) \subseteq U^c$ so $U \subseteq (f^{-1}(V))^c$. Hence $V^c \subseteq f(U) \subseteq f(f^{-1}(V))^c \subseteq V^c$ i.e. $f(U) = V^c$, since V^c is $*gs$ -closed so $f(A)$ is $*gs$ -closed i.e. f is $*gs$ -closed map.

Proposition 3.8: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a closed map and $g: (Y, \sigma) \rightarrow (Z, \eta)$ be a $*gs$ -closed map then their composition $g \circ f: (X, \tau) \rightarrow (Z, \eta)$ is $*gs$ -closed map.

Proof: Follows from definitions.

Remark 3.4: If $f: (X, \tau) \rightarrow (Y, \sigma)$ be a *gs-closed map and $g: (Y, \sigma) \rightarrow (Z, \eta)$ be a closed map then their composition $gof: (X, \tau) \rightarrow (Z, \eta)$ need not be a *gs-closed map as seen from the following example.

Example 3.11: $X = Y = Z = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$, $\sigma = \{\emptyset, \{a, c\}, Y\}$, $\eta = \{\emptyset, \{b\}, \{a, c\}, Z\}$. Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = b, f(b) = b, f(c) = c$ and $g: (Y, \sigma) \rightarrow (Z, \eta)$ by identity mapping then f is *gs-closed map and g is closed map but their composition $gof: (X, \tau) \rightarrow (Z, \eta)$ is not a *gs-closed map since for the closed set $\{b, c\}$ in (X, τ) , $(gof)(\{b, c\}) = g(f\{b, c\}) = g(\{b, c\}) = \{b, c\}$ which is not a *gs-closed set in (Z, η) .

Theorem 3.3: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ and $g: (Y, \sigma) \rightarrow (Z, \eta)$ be two maps such that their composition $gof: (X, \tau) \rightarrow (Z, \eta)$ be a *gs-closed map then the following statements are true:

- (i) If f is continuous and surjective then g is *gs-closed map.
- (ii) If g is *gs-irresolute and injective then f is *gs-closed map.

Proof: (i) Let U be a closed set in (Y, σ) . Since f is continuous so $f^{-1}(U)$ is closed in X and since gof is *gs-closed so $(gof)(f^{-1}(U))$ i.e. $g(U)$ is *gs-closed set in Z (f is surjective). Thus g is *gs-closed map.

(ii) Omitted.

Definition 3.4: A map $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be a *gs-open map if the image $f(U)$ is *gs-open in (Y, σ) for each open set U in (X, τ) .

Proposition 3.9: For any bijective $f: (X, \tau) \rightarrow (Y, \sigma)$ the following statements are equivalent:

- (i) $f^{-1}: (Y, \sigma) \rightarrow (X, \tau)$ is *gs-continuous.
- (ii) f is a *gs-open map.
- (iii) f is a *gs-closed map.

Proof: (i) $\Rightarrow$ (ii) let U be an open set of (X, τ) . By assumption $(f^{-1})^{-1}(U) = f(U)$ is *gs-open in (Y, σ) and so f is *gs-open map.

(ii) $\Rightarrow$ (iii) let U be a closed set of (X, τ) then U^c is open in (X, τ) . By assumption $f(U^c)$ is *gs-open in (Y, σ) i.e. $f(U^c) = (f(U))^c$ is *gs-open in (Y, σ) and so $f(U)$ is *gs-closed in (Y, σ) . Hence f is *gs-closed map.

(iii) $\Rightarrow$ (i) Let U be closed set in (X, τ) . By assumption $f(U)$ is *gs-closed in Y . But $f(U) = (f^{-1})^{-1}(U)$ so f^{-1} is *gs-continuous.

Definition 3.5: A map $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be a *gs*-closed (open) if the image $f(U)$ is *gs-closed (open) set in (Y, σ) for every *gs-closed (open) set U in (X, τ) .

Propositon 3.10: Every *gs*-closed map is *gs-closed map but converse is not necessarily true as seen from the following example.

Example 3.12: $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$. Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by identity mapping then f is *gs-closed map but not *gs*-closed map.

Proposition 3.11: A map $f: (X, \tau) \rightarrow (Y, \sigma)$ is $*gs$ -closed iff $*gs-cl(f(A)) \subseteq f(*gs-cl(A))$ for every subset A of (X, τ) .

Proof: Similar to proposition (3.7).

Proposition 3.12: For any bijection $f: (X, \tau) \rightarrow (Y, \sigma)$ the following are equivalent:

- (i) $f^{-1}: (Y, \sigma) \rightarrow (X, \tau)$ is $*gs$ -irresolute.
- (ii) f is a $*gs^*$ -open map and
- (iii) f is a $*gs^*$ -closed map.

Proof: Similar to proposition (3.9).

4. $*gs$ -Homeomorphisms and $*gs^*$ - Homeomorphisms

In this section we introduce the following definitions.

Definition 4.1 : A bijection $f: (X, \tau) \rightarrow (Y, \sigma)$ is called $*gs$ -homeomorphism if f is both $*gs$ -continuous and $*gs$ -closed map.

Proposition 4.1: If $f: (X, \tau) \rightarrow (Y, \sigma)$ be a bijective and $*gs$ -continuous map then following are equivalent :

- (i) f is a $*gs$ -open map.
- (ii) f is a $*gs$ -homeomorphism.
- (iii) f is a $*gs$ -closed map.

Proof: Follows from definitions.

Proposition 4.2: Every semi -homeomorphism (B) and so homeomorphism is $*gs$ -homeomorphism but converse is not necessarily true as seen from the following example.

Example 4.1: $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$, $\sigma = \{\emptyset, \{b\}, \{a, b\}, Y\}$. Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$ then f is $*gs$ -homeomorphism but not semi-homeomorphism (B) and so not homeomorphism.

Proposition 4.3: Every ψ -homeomorphism is $*gs$ -homeomorphism.

Proof: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ is ψ -homeomorphism then it is bijective, ψ -continuous and ψ -open map. But every ψ -continuous map is $*gs$ -continuous [] and every ψ -closed map is $*gs$ -closed by proposition (3.2) so f is $*gs$ -homeomorphism.

The converse of the above proposition is not necessarily true as seen from the following example.

Example 4.2: $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, \{a, c\}, X\}$, $\sigma = \{\emptyset, \{b\}, \{c\}, \{b, c\}, Y\}$. Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by identity mapping then f is $*gs$ -homeomorphism but not ψ -homeomorphism.

Proposition 4.4: sg -homeomorphism and $*gs$ -homeomorphism are independent.

Proof: Since sg -continuity and $*gs$ -continuity are independent so proof is follows immediately.

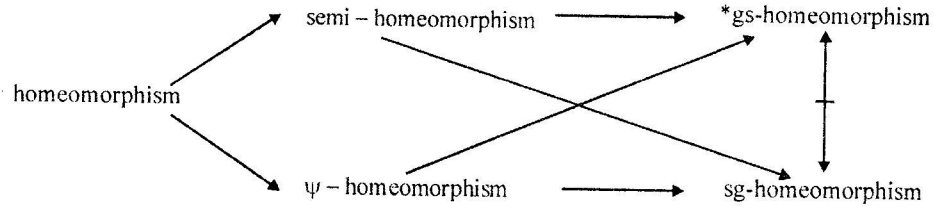


Diagram (4.1)

Definition 4.2: A bijection $f: (X, \tau) \rightarrow (Y, \sigma)$ is called a $^*gs^*$ -homeomorphism if f and f^{-1} are $^*gs$ -irresolute.

We denote the family of all $^*gs^*$ -homeomorphism of a topological space (X, τ) onto itself by $^*gs^*-h(X, \tau)$.

Remark 4.1: It is obvious that every $^*gs^*$ -homeomorphism is $^*gs$ -homeomorphism but not conversely.

Proposition 4.5: If $f: (X, \tau) \rightarrow (Y, \sigma)$ and $g: (Y, \sigma) \rightarrow (Z, \eta)$ are $^*gs^*$ -homeomorphism then their composition $gof: (X, \tau) \rightarrow (Z, \eta)$ is also $^*gs^*$ -homeomorphism.

Proof: Let U be $^*gs$ -open set in (Z, η) then $g^{-1}(U)$ is $^*gs$ -open set in Y and so $f^{-1}(g^{-1}(U)) = (gof)^{-1}(U)$ is $^*gs$ -open in (X, τ) so gof is $^*gs$ -irresolute.

Again let V be $^*gs$ -open set in X then by hypothesis $f(V)$ is $^*gs$ -open in Y and then $g(f(V)) = (gof)(V)$ is $^*gs$ -open in Z so $(gof)^{-1}$ is $^*gs$ -irresolute by proposition (3.9). Hence gof is a $^*gs^*$ -homeomorphism.

Theorem 4.1: The set $^*gs^*-h(X, \tau)$ is a group under the composition of maps.

Proof: Define a binary operation $\cdot : ^*gs^*-h(X, \tau) \times ^*gs^*-h(X, \tau) \rightarrow ^*gs^*-h(X, \tau)$ by $f \cdot g = gof$ for all f and $g \in ^*gs^*-h(X, \tau)$. Then by proposition (4.5), $gof \in ^*gs^*-h(X, \tau)$. Again composition of maps is associative and the identity map $I : (X, \tau) \rightarrow (X, \tau)$ belonging to $^*gs^*-h(X, \tau)$ is identity element of $^*gs^*-h(X, \tau)$. If $f \in ^*gs^*-h(X, \tau)$ then $f^{-1} \in ^*gs^*-h(X, \tau)$ s.t. $f \circ f^{-1} = f^{-1} \circ f = I$ so inverse exist for all element of $^*gs^*-h(X, \tau)$. Thus $(^*gs^*-h(X, \tau), \circ)$ is a group under the operation of composition of maps.

Theorem 4.2: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a $^*gs^*$ -homeomorphism then f induces an isomorphism from the group $^*gs^*-h(X, \tau)$ onto the group $^*gs^*-h(Y, \sigma)$.

Proof: Define $\theta_f : ^*gs^*-h(X, \tau) \rightarrow ^*gs^*-h(Y, \sigma)$ by $\theta_f(h) = f \circ h \circ f^{-1}$ for every $h \in ^*gs^*-h(X, \tau)$. Then θ_f is a bijection. Again for all $h_1, h_2 \in ^*gs^*-h(X, \tau)$, $\theta_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \theta_f(h_1) \circ \theta_f(h_2)$ so θ_f is a homeomorphism and so it is an isomorphism induced by f .

Theorem 4.3: $^*gs^*$ -homeomorphism is an equivalence relation in the collection of all topological spaces.

Proof: Follows from proposition (4.5).

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