

Linear Connections on $F_a(2\nu + 3, 3)$ Structure Manifold

Rashmi Mishra

Department of Mathematics and Astronomy, Lucknow University, Lucknow, India

(Received April 26, 2007)

Abstract: Structure defined by a tensor field F of type $(1,1)$ satisfying $F^{2\nu+3} + F = 0$ is given by Demetropoulou – Psomopoulou¹ and others. In the present paper, we have defined a structure called $F_a(2\nu + 3, 3)$, on a manifold and studied linear connections on such manifolds. Certain results related to such structure have been obtained.

1. Preliminaries

Let M^n be an n -dimensional differentiable manifold of class C^∞ . Suppose there exists on M^n a tensor field $F(\neq 0)$ of type $(1,1)$ and of class C^∞ satisfying

$$(1.1) \quad F^{2\nu+3} + a^2 F^3 = 0,$$

where ν is a positive integer and a is a complex number not equal to zero. Let us put

$$(1.2) \quad l = -\frac{F^{2\nu}}{a^2} \quad \text{and} \quad m = I + \frac{F^{2\nu}}{a^2},$$

where I denotes the unit tensor field. Then in view of the equations (1.1) and (1.2), it can be easily shown that

$$(1.3) \quad l^2 = l, \quad m^2 = m, \quad l + m = I, \quad lm = ml = 0.$$

Thus for the manifold M^n equipped with the tensor field F satisfying the equation (1.1) at the operators l and m defined by equation (1.2), a point of the tangent space of M^n are complementary projection operators. Thus there exist complementary distributions L^* and M^* corresponding to the projection operators l and m respectively. If the rank of F is constant everywhere and equal to r then the dimensions of L^* and M^* are r and $(n-r)$ respectively. Let us call such a structure on M^n as $F_a(2\nu + 3, 3)$ - structure of rank r .

Let us define the two operators $\bar{\nabla}$ and $\tilde{\nabla}$ in terms of an arbitrary connection ∇ on M^n by

$$(1.4) \quad \bar{\nabla}_X Y = l \nabla_X (lY) + m \nabla_X (mY),$$

and

$$(1.5) \quad \tilde{\nabla}_X Y = l \nabla_{lX} (lY) + m \nabla_{mX} (mY) + l[mX, lY] + m[lX, mY].$$

Then it can be easily shown that $\bar{\nabla}$ and $\tilde{\nabla}$ are linear connections on the manifold M^n .

Theorem 1.1: For the manifold M^n with $F_a(2\nu+3,3)$ -structure and for the operators l and m given by equation (1.2), we have

$$(1.6) \quad \begin{aligned} (i) \quad Fl = lF = F, \quad (ii) \quad Fm = mF = 0, \\ (iii) \quad \left(\frac{F^\nu}{a}\right)^2 l = -l, \quad (iv) \quad \left(\frac{F^\nu}{a}\right)^2 m = 0. \end{aligned}$$

Thus, $\left(\frac{F^\nu}{a}\right)$ acts on L^* as a complex structure and on M^* as a null operator.

Proof : Proof of the first two parts of the theorem follows easily by virtue of the equations (1.1) and (1.2). Also

$$\begin{aligned} \left(\frac{F^{2\nu}}{a^2}\right)l &= \left(\frac{F^{2\nu}}{a^2}\right)\left(-\frac{F^{2\nu}}{a^2}\right) = -\left(\frac{F^{2\nu}}{a^2}\right)\left(\frac{F^{2\nu}}{a^2}\right) \\ &= \frac{F^{2\nu}}{a^2} = -l \text{ by equation (1.2)} \end{aligned}$$

$$\text{Thus } \left(\frac{F^\nu}{a}\right)^2 l = -l.$$

The last part of the theorem can also be proved in a similar manner.

2. Integrability Conditions

Let $N(X, Y)$ be the Nijenhuis tensor of the manifold M^n admitting $F_a(2\nu+3,3)$ -structure. Then

$$(2.1) \quad N(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y].$$

Theorem 2.1: For arbitrary vector fields X and Y on M^n , we have

$$(2.2) \quad \begin{aligned} (i) \quad N(mX, mY) &= lN(mX, mY) = F^2l[mX, mY], \\ (ii) \quad mN(X, Y) &= m[FX, FY], \\ (iii) \quad mN\left(\frac{-F^{2\nu-1}}{a^2}X, \frac{-F^{2\nu-1}}{a^2}Y\right) &= m[lX, lY]. \end{aligned}$$

Proof : Replacing X by mX and Y by mY in equation (2.1), we get

$$(2.3) \quad N(mX, mY) = [FmX, FmY] - F[FmX, mY] - F[mX, FmY] + F^2[mX, mY].$$

Since $Fm = mF = 0$, hence the above equation (2.3) takes the form

$$(2.4) \quad N(mX, mY) = F^2[mX, mY].$$

Since $l+m = I$, hence we can write

$$N(mX, mY) = F^2l[mX, mY] + F^2m[mX, mY],$$

or

$$(2.5) \quad N(mX, mY) = F^2l[mX, mY],$$

as $Fm = mF = 0$. Also

$$lN(mX, mY) = F^2l^2[mX, mY].$$

Since $l^2 = l$, hence we can write above equation as

$$(2.6) \quad lN(mX, mY) = F^2l[mX, mY].$$

Thus in view of the equations (2.5) and (2.6), it follows that

$$N(mX, mY) = lN(mX, mY) = F^2l[mX, mY],$$

which proves the first part of the theorem (2.1). Other parts of the theorem can also be proved in a similar manner.

It is well known that the distribution M^* is integrable if and only if

$$(2.7) \quad l[mX, mY] = 0,$$

for arbitrary vector fields X, Y on M^n . Hence, we have

Theorem 2.2: A necessary and sufficient condition for the distribution M^* to be integrable is that

$$(2.8) \quad N(mX, mY) = 0,$$

or equivalently

$$(2.9) \quad lN(mX, mY) = 0.$$

Also the distribution L^* is integrable if and only if

$$(2.10) \quad m[lX, lY] = 0.$$

We also have another theorem:

Theorem 2.3 : A necessary and sufficient condition for the distribution L^* to be integrable is that

$$(2.11) \quad mN\left(\frac{-F^{2\nu-1}}{a^2}X, \frac{-F^{2\nu-1}}{a^2}Y\right) = 0,$$

for arbitrary vector fields X, Y on M^n .

Proof : Proof follows easily by virtue of equations (1.2), (1.3) and (2.2).

3. Distribution's Parallelism

In this section we have the following definitions :

Definition 3.1: The distribution L^* is ∇ -parallel if for $X \in L^*$ and $Y \in T(M^n)$ the vector field $\nabla_Y X \in L^*$, or equivalently ∇_Y is a transformation of L^* for each $Y \in T(M^n)$, where $T(M^n)$ denotes the tangent bundle of the manifold M^n .

Definition 3.2: The distribution L^* is ∇ -half parallel if for $X \in L^*$ and $Y \in T(M^n)$ the vector field $(\Delta F)(X, Y) \in L^*$, where

$$(3.1) \quad (\Delta F)(X, Y) = F\nabla_X Y - F\nabla_Y X - \nabla_{FX} Y + \nabla_Y (FX),$$

F being (1,1) tensor field on M^n satisfying equation (1.1).

In a similar manner parallelism and half parallelism of the distribution M^* can also be defined.

Theorem 3.1: In the $F_a(2\nu+3,3)$ -structure manifold M^n the distributions L^* and M^* are $\bar{\nabla}$ -parallel as well as $\tilde{\nabla}$ -parallel.

Proof : Since $lm = ml = 0$, we have by equations (1.4) and (1.5)

$$m\bar{\nabla}_X Y = 0, \quad m\tilde{\nabla}_X Y = 0, \quad X \in T(M^n), \quad Y \in L^*.$$

Consequently, L^* is $\bar{\nabla}$ -parallel as well as $\tilde{\nabla}$ -parallel.

A similar result for M^* can also be proved in a similar manner.

Theorem 3.2 : In the $F_a(2\nu+3,3)$ -structure manifold M^n , the distributions L^* and M^* are ∇ -parallel if and only if the connection ∇ and $\bar{\nabla}$ are equal.

Proof : If L^* and M^* are ∇ -parallel, then

$$m\nabla_X(lY) = 0, \quad l\nabla_X(mY) = 0, \quad \text{for } X, Y \in T(M^n).$$

Therefore, since $l+m = I$

$$\nabla_X(lY) = l\nabla_X(lY), \quad \nabla_X(mY) = m\nabla_X(mY).$$

Since by equations (1.4) and (1.5), ∇ and $\bar{\nabla}$ are equal. The converse of the theorem can also be proved easily.

Theorem 3.3: In the $F_a(2\nu+3,3)$ - structure manifold M^n , the distribution L^* is $\bar{\nabla}$ -half parallel if $m\nabla_{FX}(mY) = 0$ for $X \in L^*$ and $Y \in T(M^n)$.

Proof: Since $mF = 0$, according to equation (3.1)

$$m(\Delta F)(X, Y) = -m\bar{\nabla}_{FX}Y + m\bar{\nabla}_Y(FX) \text{ for } X \in L^* \text{ and } Y \in T(M^n).$$

Thus, by equation (1.4),

$$m(\Delta F)(X, Y) = -m\nabla_{FX}(mY),$$

which proves the theorem.

Theorem 3.4: In the manifold M^n equipped with $F_a(2\nu+3,3)$ - structure the distribution M^* is $\bar{\nabla}$ -half parallel if $F\nabla_X(lY) = 0$ for $X \in M^*$ and $Y \in T(M^n)$.

Proof : According to equation (3.1) since $lF = Fl$,

$$l(\Delta F)(X, Y) = F\bar{\nabla}_X Y - F\bar{\nabla}_Y X - l\bar{\nabla}_{FX}Y + l\bar{\nabla}_Y(FX),$$

for $X \in M^*$, $Y \in T(M^n)$. Thus, by equations (1.4) and (1.3) we have

$$l(\Delta F)(X, Y) = F\nabla_X(lY),$$

which proves the theorem.

Theorem 3.5: In the manifold M^n equipped with $F_a(2\nu+3,3)$ - structure the distribution L^* is $\tilde{\nabla}$ -half parallel if the vector field $[mY, FX] \in L^*$ for $X \in L^*$ and $Y \in T(M^n)$.

Proof: From equation (3.1) and since $mF = 0$, we have

$$m(\Delta F)(X, Y) = -m\tilde{\nabla}_{FX}Y + m\tilde{\nabla}_Y(FX), \text{ for } X \in L^*, Y \in T(M^n).$$

Thus by equations (1.5) and (1.3), we have

$$m(\Delta F)(X, Y) = -m[FX, mY],$$

which proves the theorem.

4. Geodesic in the Manifold M^n

Let C be a curve in M^n , T be a tangent vector field and ∇ be an arbitrary connection on M^n . Then we have

Definition 4.1: The curve C is ∇ -geodesic if $\nabla_T T = 0$ along C .

Applying the above definition to the connection $\bar{\nabla}$ and $\tilde{\nabla}$, we have the following results:

Theorem 4.1: The curve C is $\bar{\nabla}$ -geodesic if the vector field $\nabla_T T - \nabla_T(mT) \in M^*$ and $\nabla_T(mT) \in L^*$.

Proof: In view of equation (1.4), we have

$$\bar{\nabla}_T T = l \nabla_T(lT) + m \nabla_T(mT),$$

and since $l + m = 1$,

$$\bar{\nabla}_T T = l(\nabla_T T - \nabla_T(mT)) + m \nabla_T(mT),$$

which proves the theorem.

Theorem 4.2: The curve C is $\tilde{\nabla}$ -geodesic if the vector fields $\nabla_{lT}(lT) + [mT, lT] \in M^*$ and $\nabla_{mT}(mT) + [lT, mT] \in L^*$.

Proof: Proof follows easily like that of the theorem (4.1) using the definition of $\tilde{\nabla}$ from the equation (1.5).

Theorem 4.3: The $(1,1)$ tensor fields l and m are $\bar{\nabla}$ -covariantly constant as well as $\tilde{\nabla}$ -covariantly constant.

Proof: We have

$$\bar{\nabla}_X(lY) = (\bar{\nabla}_X l)Y + l(\bar{\nabla}_X Y).$$

Thus, by (1.4)

$$(\bar{\nabla}_X l)Y = 0, \quad Y \in T(M^n).$$

Similarly

$$\bar{\nabla}_X m = 0, \quad \tilde{\nabla}_X l = 0, \quad \tilde{\nabla}_X m = 0.$$

References

1. Demetropoulou D. Psomopoulou, Linear connections on manifold admitting $F_D(2\nu + 3, 3)$ -structure, *Tensor N.S.*, **47** (1988) 235-239.
2. S. Ishihara and K. Yano, Integrability conditions of a structure f satisfying $f^3 + f = 0$, *Quarterly J. Math. Oxford*, **15** (1964) 217-222.
3. R. S. Mishra, *Structures on a differentiable manifold and their applications*, Chandrama Prakashan, 50-A, Balrampur House, Allahabad, India, 1984.
4. K. D. Singh and R. K. Vohra, Linear connections on $f(3, -1)$ manifold, *Comptes Rendus del' Academie Bulgare des Sciences*, **26** (1973) 1305-1307.
5. K. Yano, On a structure defined by a tensor field f of type $(1,1)$ satisfying $f^3 + f = 0$, *Tensor N.S.*, **14** (1963) 99-109.