

## Little Hankel Operators and Inner Functions

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**Abstract:** Let  $L_a^2(D)$  be the Bergman space of holomorphic functions in the unit disc  $D = \{z \in \mathbb{C} : |z| < 1\}$ . For  $\phi \in L^\infty(D)$ , let  $S_\phi$  be the little Hankel operator on  $L_a^2(D)$ . A function  $G \in L_a^2(D)$  is called an inner function (norm expansive multiplier) in the Bergman space if  $\|G\|_{L_a^2} = 1$  and  $\|Gg\|_{L_a^2} \geq \|g\|_{L_a^2}$  for all  $g \in H^\infty(D)$ . Let  $A$  be the set of those  $\phi \in L^\infty(D)$  for which there exists an expansive multiplier  $G \in L_a^2(D)$  such that  $\phi G \in \overline{(L_a^2(D))}^\perp$ . In this paper we show that  $\phi \in A$  if and only if  $\ker S_\phi \neq \{0\}$  and if  $\phi \notin A$  then  $\overline{S_\phi(L_a^2 \cap L_a^2)} = L_a^2$  for any  $G \in L_a^2(D)$  that is a (norm) expansive multiplier in  $L_a^2(D)$ .

**Key words:** Bergman space, Hankel operators, inner functions.

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### 1. Introduction

Let  $D = \{z \in \mathbb{C} : |z| < 1\}$  be the open unit disc in the complex plane  $\mathbb{C}$ . Let  $dA(z)$  be the area measure on  $D$  normalized so that the area of the disc  $D$  is 1. In rectangular and polar coordinates,  $dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta$ . Let  $L^2(D, dA)$  denote the Hilbert space of complex-valued, absolutely square-integrable, Lebesgue measurable functions  $f$  on  $D$  with the inner product  $\langle f, g \rangle = \int f(z) \overline{g(z)} dA(z)$ . Let  $L^\infty(D, dA)$  denote the space of measurable functions  $f$  on  $D$  with  $\|f\|_\infty = \text{ess sup} \{|f(z)| : z \in D\} < \infty$ . Let  $H^\infty(D)$  be the space of bounded analytic functions on  $D$ . Let  $L_a^2(D)$  (the subscript "a" stands for analytic) be the subspace of  $L^2(D, dA)$  consisting of analytic functions. The space  $L_a^2(D)$  is called the Bergman space. Since point evaluation at  $z \in D$  is a bounded linear functional

on the Hilbert space  $L_a^2(D)$ , the Riesz representation theorem implies that there exists a unique function  $K_z$  in  $L_a^2(D)$  such that

$$f(z) = \int_D f(w) \overline{K_z(w)} dA(w),$$

for all  $f$  in  $L_a^2(D)$ . Let  $K(z, w)$  be the function on  $D \times D$  defined by  $K(z, w) = \overline{K_z(w)}$ . The function  $K(z, w)$  is thus the reproducing kernel for the Bergman space  $L_a^2(D)$  and is called the Bergman kernel. It can be shown that the sequence of functions  $\{e_n(z)\}_{n \geq 0} = \{\sqrt{n+1} z^n\}_{n \geq 0}$  form the standard orthonormal basis for  $L_a^2(D)$  and  $K(z, w) = \sum_{n=1}^{\infty} e_n(z) \overline{e_n(w)}$ . The Bergman kernel is independent of the choice of orthonormal basis and  $K(z, w) = \frac{1}{(1 - \bar{z}w)^2}$ . Since  $L_a^2(D)$  is a closed subspace of

$L^2(D, dA)^1$ ; there exists an orthogonal projection  $P$  from  $L^2(D, dA)$  onto  $L_a^2(D)$ . For  $\phi \in L^\infty(D)$ , we define the Toeplitz operator  $T_\phi$  on  $L_a^2(D)$  by  $T_\phi f = P(\phi f)$ ,  $f \in L_a^2(D)$ . The big Hankel operator  $H_\phi$  is a mapping from  $L_a^2(D)$  into  $(L_a^2(D))^\perp$  defined by  $H_\phi f = (I - P)(\phi f)$ ,  $f \in L_a^2(D)$ . The little Hankel operator  $h_\phi$  is a mapping from  $L_a^2(D)$  into  $\overline{L_a^2(D)}$  defined by  $h_\phi f = \bar{P}(\phi f)$ ,  $f \in L_a^2(D)$ , where  $\bar{P}$  is the projection from  $L^2(D, dA)$  onto  $\overline{L_a^2(D)}$ . There are also many equivalent ways of defining little Hankel operators. For example, define the map  $S_\phi$  from  $L_a^2(D)$  into  $L_a^2(D)$  by  $S_\phi f = P(J(\phi f))$ , where  $J$  is the mapping from  $L^2(D)$  into  $L^2(D)$  defined by  $J(h(z)) = h(\bar{z})$ . It is easy to see that the operator  $S_\phi$  is unitarily equivalent to the little Hankel operator  $h_\phi$  for some  $\phi \in L^\infty(D)$ .

Let  $L_h^2(D)$  be the subspace of functions in  $L^2(D)$  that are harmonic. The space  $L_h^2(D)$  is a closed subspace<sup>2</sup> of  $L^2(D)$  with orthonormal basis  $\{\sqrt{n+1} \bar{z}^n\}_{n=1}^{\infty} \cup \{\sqrt{n+1} z^n\}_{n=0}^{\infty}$ . Moreover  $L_h^2 = L_a^2 \oplus (\bar{L}_a^2)_0$ , where  $(\bar{L}_a^2)_0$  is the space of all complex conjugates of functions in  $L_a^2(D)$  which vanish at the origin. It is to be noted that functions of the type  $|z|^{2n} z^m \in (\bar{L}_a^2)^\perp$  and  $|z|^{2n} \bar{z}^m \in (L_a^2)^\perp$  for  $m > 0$ . Note that the orthogonal complement of  $L_a^2$  in  $L^2(D, dA)$  is much bigger than the space  $(\bar{L}_a^2)_0$ . For example, the function  $f(z) = |z|^2 \bar{z}$  is in  $L^2(D, dA)$ . It can easily be checked that  $f \in (L_a^2)^\perp$ , but  $f$  is not an element of  $(\bar{L}_a^2)_0$ .

Let  $T$  denote the unit circle in  $\mathbb{C}$ . Let  $L^2(T)$  be the space of complex-valued, absolutely square integrable, Lebesgue measurable functions on  $T$ . Let  $H^2(T)$  be the corresponding Hardy space of functions on  $T$  with vanishing negative Fourier coefficients. For  $\phi \in L^\infty(T)$ , the space of essentially bounded measurable functions on  $T$ , we define the Hankel operator  $L_\phi$  from  $H^2(T)$  into  $H^2(T)$  as  $L_\phi f = \tilde{P}(\tilde{J}(\phi f))$ , where  $\tilde{J}$  is the mapping from  $L^2(T)$  into  $L^2(T)$  defined by  $\tilde{J}(h(z)) = h(\bar{z})$  and  $\tilde{P}$  is the projection from  $L^2(T)$  onto  $H^2(T)$ . For  $\phi \in L^\infty(T)$ , the Toeplitz operator  $B_\phi$  is defined from  $H^2(T)$  into itself as  $B_\phi f = \tilde{P}(\phi f)$ . If  $H$  is a Hilbert space, let  $\mathcal{L}(H)$  be the set of all bounded linear operators from  $H$  into itself.

## 2. Kernel of Big Hankel Operators

**Theorem 2.1:** *If  $\phi$  is analytic in  $D$  then  $H_{\bar{\phi}}$  is a finite rank big Hankel operator if and only if  $\phi$  is a constant.*

**Proof:** Sufficiency is obvious. For the necessity, suppose  $H_{\bar{\phi}}$  is a finite rank operator, where  $\phi$  is analytic on  $D$ . Then

$$\ker H_{\bar{\phi}} = \{f \in L_a^2(D) : (I - P)(\bar{\phi}f) = 0\} = \{f \in L_a^2(D) : \bar{\phi}f \in L_a^2(D)\}$$

has finite codimension and is invariant under multiplication by  $z$ . By the result of Axler and Bourdon<sup>3</sup>, there exists a polynomial  $q$  whose roots lie in  $D$  such that  $\ker H_{\bar{\phi}} = qL_a^2(D)$ . Let  $\phi(z) = \sum c_k z^k$ , then  $\bar{\phi}(z)q(z) \in L_a^2(D)$  implies either  $\phi$  is a constant or  $q = 0$ . But  $q \neq 0$  since  $H_{\bar{\phi}}$  has finite rank, so the result follows.

Theorem 2.3 below describes the kernel of a big Hankel operator. The following theorem establishes that on the Bergman space  $L_a^2(D)$  the Toeplitz operator  $T_\phi$  represents a multiplication operator on a closed subspace of  $L_a^2(D)$  if and only if the symbol  $\phi$  is analytic.

**Theorem 2.2:** *Let  $S$  be a subspace of  $L^\infty(D)$  such that for  $\phi \in S$  there exists a closed subspace  $M \neq \{0\}$  of  $L_a^2(D)$  for which  $T_\phi f = \phi f$ , for all  $f \in M$ . Then  $S \subset H^\infty(D)$ .*

For proof, the paper of N. Das<sup>4</sup> may be seen.

**Theorem 2.3:** *For  $\phi \in L^\infty(D)$  the Kernel of a big Hankel operator  $H_\phi$  is trivial.*

**Proof:** The Kernel of a big Hankel operator is given by

$$\ker H_\phi = \{f \in L_a^2(D) : (I - P)(\phi f) = 0\} = \{f \in L_a^2(D) : \phi f \in L_a^2(D)\}.$$

Now if  $\ker H_\phi \neq 0$  then  $\phi \in H^\infty$ , by Theorem 2.2. Therefore the operator  $H_\phi$  is zero and  $\ker H_\phi = L_a^2(D)$ . Thus if  $H_\phi \neq 0$  then  $\ker H_\phi = \{0\}$ .

An operator  $A \in \mathcal{L}(H)$  is said to be hyponormal if  $[A] \geq 0$  where  $[A] = A^*A - AA^*$ .

**Theorem 2.4:** If  $[T_\phi]$  is hyponormal and  $\phi$  is not a constant function then  $\ker H_{\bar{\phi}} = \{0\}$ .

**Proof:** For  $f$  in  $L_a^2$ ,  $\langle f, [T_\phi]f \rangle = \|T_\phi f\|^2 - \|T_{\bar{\phi}} f\|^2 = \|\phi f\|^2 - \|H_\phi f\|^2 - (\|\bar{\phi} f\|^2 - \|H_{\bar{\phi}} f\|^2) = \|H_{\bar{\phi}} f\|^2 - \|H_\phi f\|^2$ . Thus a vector  $f$  in  $L_a^2$  is in  $\ker H_{\bar{\phi}}$  if and only if  $0 = \|H_{\bar{\phi}} f\|^2 = \|H_\phi f\|^2 + \langle f, [T_\phi]f \rangle$ . If  $[T_\phi]$  is hyponormal, then  $[T_\phi] \geq 0$ , and therefore this equation holds if and only if  $f$  is in  $\ker H_\phi$  and  $f$  is in  $\ker [T_\phi]$ . It follows therefore that  $\ker H_\phi \cap \ker [T_\phi] = \ker H_{\bar{\phi}}$ . Thus if  $\ker H_\phi = \{0\}$  then  $\ker H_{\bar{\phi}} = \{0\}$ . If  $\ker H_\phi = L_a^2$  then  $\ker [T_\phi] = \ker H_{\bar{\phi}}$ . If  $\ker H_{\bar{\phi}} = L_a^2$  then it follows that  $\phi$  and  $\bar{\phi}$  both belong to  $H^\infty(D)$  and hence  $\phi$  is a constant. This contradicts our hypothesis. Thus by theorem 2.3,  $\ker H_{\bar{\phi}} = \{0\}$ .

### 3. Common Zero Set and Invariant Subspaces

It is shown by N.S. Faour<sup>5</sup> that if  $T_z$  is the Toeplitz operator on the Bergman space  $L_a^2(D)$  with symbol  $z$  then an operators  $S$  on  $L_a^2(D)$  is a little Hankel operator if and only if

$$T_z^* S = S T_z.$$

**Definition 3.1:** A function  $G \in L_a^2(D)$  ( $G \in H^2$ ) is called an inner function in  $L_a^2(D)$  (respectively  $H^2$ ) if  $|G|^2 - 1$  is orthogonal to  $H^\infty$ . The above definition for inner function on the Bergman space can be made<sup>6</sup>.

**Remark 3.1:** A function  $G \in L_a^2(D)$  is called a (norm) expansive multiplier if  $\|G\|_{L_a^2} = 1$  and  $\|Gg\|_{L_a^2} \geq \|g\|_{L_a^2}$  for all  $g \in H^\infty$ . If  $G$  is an expansive multiplier then  $G$  is inner. The converse is also true<sup>6</sup>.

**Theorem 3.1:** If  $S_\phi$  is a little Hankel operator on  $L_a^2(D)$  and if  $\ker S_\phi = \{f \in L_a^2(D) : f = 0 \text{ on } b\}$ , where  $b = \{b_j\}_{j=1}^\infty$  is an infinite sequence of points in  $D$  then there exists an inner function  $G_b \in L_a^2(D)$  such that  $\ker S_\phi = G_b L_a^2(D) \cap L_a^2(D)$ .

inner function in  $H^\infty(T)$  and  $h \in H^\infty(T)$  (for proof, see the paper of S. C. Power<sup>10</sup>) and the theorem follows.

**Corollary 3.1:** If  $\phi = \psi_1 + \psi_2$ , where  $\psi_1 \in (\overline{L_a^2})^\perp$  and  $\psi_2(z) = \sum_{k=0}^\infty \hat{\psi}_2(-k)\bar{z}^k \in \overline{H^\infty}$  then  $\ker S_\phi$  is nontrivial if  $\tilde{\psi}_2 * \tilde{\phi}_1 = z\bar{u}h$ , where  $u$  is an inner function in  $H^\infty(T)$  and  $h \in H^\infty(T)$ ,  $\tilde{\psi}_2(e^{i\theta}) = \sum_{k=0}^\infty \hat{\psi}_2(-k)e^{-ik\theta}$ ,  $\tilde{\phi}_1(e^{i\theta}) = \sum_{k=0}^\infty \frac{1}{k+1}e^{-ik\theta}$  and  $*$  denotes convolution.

**Proof:** The proof follows from theorem 3.6, since  $S_\psi = 0$  if  $\psi \in (\overline{L_a^2})^\perp$ .

**Corollary 3.2:** If  $\phi = \bar{\Phi}_1\Phi_2$ , where  $\Phi_1$  and  $\Phi_2$  are bounded analytic functions on  $D$  and  $\bar{\Phi}_1(z) = \sum_{k=0}^\infty \hat{\Phi}_1(-k)\bar{z}^k$  and  $\Phi_2(z) = \sum_{k=0}^\infty \hat{\Phi}_2(k)z^k$  then  $\ker S_\phi$  is nontrivial if  $\tilde{\psi}_1\tilde{\Phi}_2 = z\bar{u}h$ , where  $u$  is an inner function in  $H^\infty(T)$  and  $h \in H^\infty(T)$ ,  $\tilde{\psi}_1(e^{i\theta}) = \sum_{k=0}^\infty \frac{1}{k+1}\hat{\Phi}_1(-k)e^{-ik\theta}$ , and  $\tilde{\Phi}_2(e^{i\theta}) = \sum_{k=0}^\infty \hat{\Phi}_2(k)e^{ik\theta}$ .

**Proof:** The proof follows easily by observing that  $S_\phi = S_{\bar{\Phi}_1\Phi_2} = S_{\bar{\Phi}_1}T_{\Phi_2} = D_2F_{\tilde{\psi}_1\tilde{\Phi}_2}D_2$ , where  $T_{\Phi_2}$  is the Toeplitz operator on the Bergman space  $L_a^2(D)$  with symbol  $\Phi_2$  and  $F_{\tilde{\psi}_1\tilde{\Phi}_2}$  is the operator on  $L_a^2(D)$  having a classical Hankel matrix with respect to the standard orthonormal basis of  $L_a^2(D)$  with symbol  $\tilde{\psi}_1\tilde{\Phi}_2$ .

#### 4. Norm Expansive Multipliers and Kernel of Hankel Operators

Let  $A$  be the set of those  $\phi \in L^\infty(D)$  for which there exists an expansive multiplier  $G \in L_a^2(D)$  such that  $\phi G \in (\overline{L_a^2(D)})^\perp$ .

**Theorem 4.1:** The following conditions are equivalent:

- (1)  $\phi \in A$ .
- (2)  $\ker S_\phi \neq \{0\}$ .
- (3)  $\overline{\text{ran } S_\phi} \neq L_a^2$ .

**Proof:** (1)  $\Leftrightarrow$  (2) Suppose  $\phi \in A$ . Then by definition of  $A$  there exists an inner function  $G \in L_a^2(D)$  such that  $\phi G \in (\overline{L_a^2(D)})^\perp$ . Thus  $G \in \ker S_\phi$  and hence  $\ker S_\phi \neq \{0\}$ . Now suppose  $\ker S_\phi \neq \{0\}$ . Since  $\ker S_\phi$  is an invariant subspace of  $z$ , there

Let  $W$  be the operator mapping from the standard basis  $\{z^n\}_{n \in \mathbb{N}}$  of  $H^2$  onto the basis  $\{\sqrt{n+1}z^n\}_{n \in \mathbb{N}}$  of  $L_a^2(D)$ ,

$$W : \sum_{n=0}^{\infty} f_n z^n \rightarrow \sum_{n=0}^{\infty} f_n \sqrt{n+1} z^n.$$

The map  $W$  is unitary and it induces a map from  $\mathcal{L}(H^2)$  into  $\mathcal{L}(L_a^2)$  that sends  $T \in \mathcal{L}(H^2)$  to  $WTW^* \in \mathcal{L}(L_a^2)$ . Note the matrix of the Hankel operator  $L_\chi$ ,  $\chi \in L^\infty(T)$  with respect to the standard orthonormal basis of  $H^2$  is the same as the matrix of  $F_\chi = WL_\chi W^*$  with respect to the standard orthonormal basis of  $L_a^2$ . We shall always refer  $F_\chi$  as the operator on  $L_a^2(D)$  having a classical Hankel matrix with respect to the standard orthonormal basis of  $L_a^2(D)$ .

For  $\tilde{\psi} = \sum_{k=-\infty}^{\infty} \hat{\tilde{\psi}}(k) e^{ik\theta} \in L^\infty(T)$ , let  $F_{\tilde{\psi}}$  (with symbol  $\tilde{\psi}$ ), be the operator on  $L_a^2(D)$  having a classical Hankel matrix  $(\hat{\tilde{\psi}}(-(i+j)))$  with respect to the standard orthonormal basis of  $L_a^2(D)$ . Here  $\hat{\tilde{\psi}}(k)$  denotes the  $k$ th Fourier coefficient of  $\tilde{\psi}$ .

**Theorem 3.2:** Let  $\phi(z) = \sum_{k=0}^{\infty} \hat{\phi}(-k) \bar{z}^k \in \overline{H^\infty}$ . Then  $\ker S_\phi$  is nontrivial if  $\tilde{\phi} * \tilde{\phi}_1 = z\bar{u}h$ , where  $\tilde{\phi}(e^{i\theta}) = \sum_{k=0}^{\infty} \hat{\phi}(-k) e^{-ik\theta}$ ,  $\tilde{\phi}_1(e^{i\theta}) = \sum_{k=0}^{\infty} \frac{1}{k+1} e^{-ik\theta}$ ,  $u$  is an inner function in  $H^\infty(T)$  and  $h \in H^\infty(T)$ . Here  $*$  denotes convolution.

**Proof:** The matrix of  $S_\phi$  with respect to the standard orthonormal basis  $\{e_n\}_{n=0}^{\infty} = \{\sqrt{n+1}z^n\}_{n=0}^{\infty}$  of  $L_a^2(D)$  is given by

$$\langle S_\phi e_j, e_i \rangle = \frac{\sqrt{i+1}\sqrt{j+1}}{i+j+1} \hat{\phi}(-(i+j)), \quad i, j \geq 0.$$

Thus  $S_\phi = D_2 F_{\tilde{\psi}} D_2$ , where  $\tilde{\psi}(e^{i\theta}) = \sum_{k=0}^{\infty} \frac{1}{k+1} \hat{\phi}(-k) e^{-ik\theta}$ , i.e.  $\tilde{\psi}$  is the convolution on the circle of  $\tilde{\phi} = \sum_{k=0}^{\infty} \hat{\phi}(-k) e^{-ik\theta}$  with the function  $\tilde{\phi}_1(e^{i\theta}) = \sum_{k=0}^{\infty} \frac{1}{k+1} e^{-ik\theta}$ ,  $F_{\tilde{\psi}}$  is the operator on  $L_a^2(D)$  having a classical Hankel matrix with respect to the standard orthonormal basis of  $L_a^2(D)$  with symbol  $\tilde{\psi}$  and  $D_2 e_j = \sqrt{j+1} e_j$  for all  $j \geq 0$ . Now if  $g$  belongs to  $\ker F_{\tilde{\psi}}$  then  $D_1 g$  belongs to  $\ker S_\phi$  where  $D_1 e_j = \frac{1}{\sqrt{j+1}} e_j$  for all  $j \geq 0$ . Further, if  $f \in \ker S_\phi$  then  $D_2 f \in \ker F_{\tilde{\psi}}$ . Thus  $\ker S_\phi$  is nontrivial if and only if  $\ker F_{\tilde{\psi}}$  is nontrivial. But  $\ker F_{\tilde{\psi}}$  is nontrivial if and only if  $\tilde{\psi}$  is of the form  $z\bar{u}h$ , where  $u$  is an

$\{b_\zeta\}$  is dense in  $L_a^1$  and  $L_a^2$ , the functional  $\Omega$  can be continuously extended to  $L_a^1$ . Since  $(L^1)^* = L^\infty$ , it follows from Hahn-Banach theorem that  $\Omega$  can be extended to  $(L^1)^*$  without increasing the norm. Thus it follows that there exists  $\psi \in L^\infty$  such that  $\int \psi f dA = 0$  for all  $f \in L^1$ . Hence  $\psi = 0$ . This implies  $GG_1 = 0$  and therefore either  $G = 0$  or  $G_1 = 0$ . But this is not possible as both  $G$  and  $G_1$  are inner functions. Further note that<sup>8,11</sup> if  $N$  is a subspace of  $L_a^2(D)$ ,  $Z(N) = \{z \in D : f(z) = 0 \forall f \in N\} = \{\xi_j\}$ ,  $G$  the corresponding canonical divisor and  $G_n \in H^\infty$  is the canonical divisor of the finite subset  $\{\xi_1, \dots, \xi_n\}$  then  $G_n(z) \rightarrow G(z)$  uniformly on each compact subset of  $D$  and  $\|G_n - G\|_2 \rightarrow 0$  as  $n \rightarrow \infty$ .

### References

1. K. Zhu, *Operator theory in function spaces*, Monographs and Text-books in Pure and Applied Mathematics, Marcell Dekker, Inc. **139**, New York and Basel, 1990.
2. J. B. Conway, *Subnormal operators*, Pitman Research Notes in Math. **51**, Boston-London, 1981.
3. S. Axler and P. Bourdon, Finite co-dimensional invariant subspaces of Bergman spaces, *Trans. Amer. Math. Soc.*, **306** (1988) 805-817.
4. N. Das, Norm of Toeplitz operators on the Bergman space, *Indian J. pure applied Math.* **33(2)** (2002) 255-267.
5. N. S. Faour, A theorem of Nehari type, *Illinois J. Math.*, **35** (1991) 533-535.
6. B. Korenblum and M. Stessin, On Toeplitz – invariant subspaces of the Bergman space, *J. Funct. Anal.*, **111** (1993) 76-96.
7. P. L. Duren, D. Khavinson, H.S. Shapiro and C. Sundberg, Contractive zero-divisors in Bergman spaces, *Pacific J. Math.*, **157** (1993) 37-56.
8. H. Hedenmalm, A factorization theorem for square area-integrable analytic functions, *J. Reine. Angew. Math.*, **422** (1991) 45-68.
9. N. Das, The kernel of a Hankel operator on the Bergman space, *Bulletin of London mathematical society*, **31** (1999) 75-80.
10. S. C. Power, *Hankel operators on Hilbert space*, Research Notes in Mathematics **64**, Pitman, Boston, 1982.
11. P. L. Duren, D. Khavinson, H.S. Shapiro and C. Sundberg, Invariant subspaces in Bergman spaces and the biharmonic equation, *Michigan Math. J.*, **41** (1994) 247-259.

exists<sup>7,8,11</sup> an inner function  $G \in \ker S_\phi$ . Thus  $\phi G \in (\overline{L_a^2(D)})^\perp$  and  $\phi \in A$  as every inner function is an expansive multiplier.

(2)  $\Leftrightarrow$  (3) Note  $S_\phi^* = S_{\phi^+}$  and  $f \in \ker S_\phi$  if and only if  $f^+ \in \ker S_{\phi^+}$ , where  $f^+(z) = \overline{f(\bar{z})}$  and  $\phi^+(z) = \overline{\phi(\bar{z})}$ . Thus  $\ker S_\phi \neq \{0\}$  if and only if  $\ker S_{\phi^+} \neq \{0\}$ . Further,  $\ker S_{\phi^+} = \ker S_\phi^* = (\overline{\text{ran} S_\phi})^\perp$ . Hence  $\ker S_{\phi^+} \neq \{0\}$  if and only if  $\overline{\text{ran} S_\phi} \neq L_a^2$ . The theorem follows.

**Theorem 4.2:** If  $\phi \notin A$  then  $\overline{S_\phi(GL_a^2 \cap L_a^2)} = L_a^2$  for any  $G \in L_a^2(D)$  that is a (norm) expansive multiplier in  $L_a^2(D)$ .

**Proof:** Suppose the function  $\phi \notin A$ . By theorem 4.1,  $\ker S_\phi = \{0\}$ . Hence there exists no inner function  $G$  such that  $G \in \ker S_\phi$ . Thus for all inner functions  $G$  we have  $\phi G \notin (\overline{L_a^2})^\perp$ . Thus  $S_{\phi G} \neq 0$ . This follows from the fact that if  $\psi \in L^2$  then  $S_\psi = 0$  if and only if  $\psi \in (\overline{L_a^2})^\perp$ . The function  $\phi G$  need not be bounded as inner functions can be unbounded. Therefore the operator  $S_{\phi G}$  is densely defined. The operator  $S_{\phi G}$  is a closed operator and  $\ker S_{\phi G}$  is closed. Thus  $\ker S_{\phi G} \subset H^\infty$  is an invariant subspace of  $T_z$ .

Now if  $\ker S_{\phi G} \neq \{0\}$  then there exists<sup>8,11</sup> an inner function  $G_1 \in H^\infty$  such that  $S_{\phi G} G_1 = 0$ . Therefore  $\phi G G_1 \in (\overline{L_a^2})^\perp$ . Note  $G G_1 \in L_a^2$ . Further,  $\phi G G_1 \in (\overline{L_a^2})^\perp$  implies  $\langle \phi G G_1, \bar{h} \rangle = 0$  for all  $h \in L_a^2$ . Hence  $\langle G G_1, \bar{\phi} \bar{h} \rangle = \langle G G_1, P J(\phi^+ h^+) \rangle = \langle G G_1, S_{\phi^+} h^+ \rangle = 0$  for all  $h \in L_a^2$ , where  $h^+(z) = \overline{h(\bar{z})}$  and  $\phi^+(z) = \overline{\phi(\bar{z})}$ . It can easily be verified that as  $\phi \notin A$ , the function  $\phi^+ \notin A$ . By theorem 4.1,  $\ker S_{\phi^+} = \{0\}$ . Hence  $\text{ran} S_{\phi^+}$  is dense in  $L_a^2$ . Thus it follows that  $G G_1 \in (\overline{L_a^2})^\perp$ . This implies  $G G_1 = 0$  and therefore either  $G = 0$  or  $G_1 = 0$ . But this is not possible as both  $G$  and  $G_1$  are inner functions. It follows therefore that  $\ker S_{\phi G} = \{0\}$ . Thus  $\overline{\text{ran} S_{\phi G}} = L_a^2$ . That is,  $\overline{S_\phi(GL_a^2 \cap L_a^2)} = L_a^2$ .

**Remark 4.1:** It may be noted that there does not exist inner functions  $G$  and  $G_1$  in  $L_a^2$  such that  $G G_1 \in (\overline{L_a^2})^\perp$ . Suppose to the contrary that  $G$  and  $G_1$  are two inner functions such that  $G G_1 \perp L_a^2$ . Let  $b_\varsigma(z) = \frac{1-|\varsigma|^2}{(1-\varsigma z)^3}$ ,  $\varsigma, z \in D$ . It is known<sup>5</sup> that  $\text{span} \{b_\varsigma\}$  is dense in  $L_a^1$  and  $L_a^2$ . Further  $\|b_\varsigma\|_1 \leq c$ ,  $\varsigma \in D$  and  $c$  is a constant. Define  $\Omega: L_a^2 \rightarrow \mathbb{C}$  as  $\Omega(f) = \int f \overline{G G_1} dA$ . Note that  $\Omega$  is a bounded linear functional on  $L_a^2$ . Since  $\text{span}$

