

Applications of Fibonacci Numbers

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Abstract. The objective of the present paper is to discover the mathematical delights that awaits new explores in various fields.

1. Introduction

12th century Italian mathematician Leonard of Pisa or Fibonacci was known in his time and is still recognized today as the “greatest European mathematician of the middle ages”.

He was born in the year 1170 and died in the year 1240 and there is now a statue commemorating him located at the Leaning Tower end in Pisa. His full name was Leonardo of Pisa, or Leonardo Pisano in Italian since he was born in Pisa. He called himself Fibonacci, which was short for Filius Bonacci, standing for “son of Bonacci”, which was his father’s name. Leonardo’s father was a custom officer in the North African town of Bugia. So Fibonacci grew up with a North African education and later traveled extensively around the Mediterranean coast. He then met with many merchants and learned of their systems of doing arithmetic. He soon realized the advantages of the “Hindu-Arabic” system over all the others. He was one of the first to introduce the Hindu-Arabic number system into Europe – the system we now use today-based of ten digits with its decimal point and a symbol for zero: 1, 2, 3, 4, 5, 6, 7, 8, 9 and 0.

2. Origin of Fibonacci Numbers

His book on how to do arithmetic in the decimal system, called ‘*Liber abbaci*’ completed in 1202 persuaded many of the European mathematicians of his day to use his “new” system. In the ‘*Liber abbaci*’ Leonardo states and gives solution to the famous Rabbit problem. The use of the Fibonacci sequence goes back to the year 1202, when Leonardo of Pisa wondered how many pairs of rabbits would be produced in the n^{th} generation if he used the following rule. Start with a single pair. Each pair produces a new pair in the next generation and also in the generation following before it dies.

Leonardo traces the progress of the rabbits and obtained the sequence of numbers which is the solution to this rabbit problem thus became known as the Fibonacci sequence i.e. 1, 2, 3, 5, 8, 13, 21...

The sequence in which each number is the sum of the two preceding numbers is known as the Fibonacci series. The ratio of successive pairs tends to the so-called *golden ratio* approaches $(1+5^{1/2})/2$ Greeks, who used it in their architecture, knew this.

In the later half of the nineteenth century, a French mathematician Edouard Lucas went on to study the same recurrence with starting values 2 and 1. This version of the numbers became known as *Lucas numbers*.

Lucas was the person who made the term Fibonacci numbers popular.

3. Fibonacci Sequence

If we start with any two positive numbers and generate a sequence of numbers in which every new term is created by adding together the two preceding terms, we get a sequence called the Fibonacci sequence. If we start with the first two terms as 1 and 1, we get the following sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ...

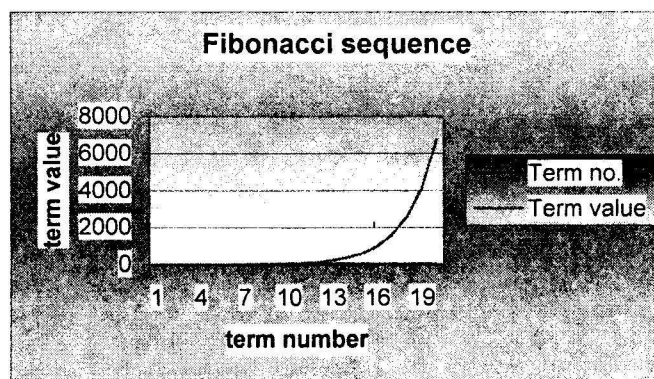
This is the basic Fibonacci sequence generated by the starting terms 1 and 1. Starting with any other pair of terms will produce a sequence which looks totally different but which has similar properties.

While a first glances at this sequence of numbers shows nothing exciting. We discover a number of interesting properties as we study it in great details.

This sequence does not seem at first glance to have any notable or interesting features, except perhaps that the term size seems to grow rather slowly at first but very rapidly later. After the first few terms, the growth rate is almost constant and approximately 61.8 percent per term. i.e. the consecutive terms are in a geometric progression with a progression ratio of about 1.618...

Term Number: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

Term Value: 1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 1597 ...



4. Properties of Fibonacci Sequences

What is Special about the Fibonacci Ratio 1.618...

What is so special about the ratio of 1.618...! How does this particular ratio originate from the repeated addition of pairs of consecutive numbers and can we deduce the correct value mathematically!

The term value in the Fibonacci sequence rises after every five terms by an order of magnitude as $1.618^5 = 11.09$

Term number	Term Value
5	5
10	55
15	610
20	6,765
25	75,025
30	832,040
35	9,227,465
40	102,334,155
45	1,134,903,170
50	12,586,269,025
55	139,583,862,445
60	1,548,008,755,920

If we calculate the ratios of consecutive terms, we get the following table.

Term no.	Term	Ratio	Deviation from final
1	1		
2	1	1	- 0.618034...
3	2	2	+0.381966...
4	3	1.5	- 0.118034...
5	5	1.666...	+0.04863262...
6	8	1.6	- 0.01803398...
7	13	1.625	+0.00696599...
8	21	1.615385...	- 0.00264943...
9	34	1.619048...	+0.00101364...
10	55	1.617647...	- 0.00038695...
11	89	1.618182...	+0.00014782...
12	144	1.617977...	- 0.00005650...
13	233	1.618056...	+0.00002158...
14	377	1.618026...	- 0.00000822...
15	610	1.618033...	+0.00000310...
16	987	1.618034...	- 0.00000119...
17	1597	1.618034...	+0.00000048...
18	2584	1.618034...	- 0.00000024...
19	4181	1.618034...	+0.00000005...
20	6765	1.618034...	- 0.00000002...

Here we see the interesting pattern in the deviation of successive ratios from the final ratio. The ratio approaches its final value. Digits will remain constant are highlighted. The deviation of each ratio from the final ratio alternates the sign but becomes smaller in magnitude as the sequence advances. i.e. we observe that while the ratio approaches closer and closer to its final value, it keeps overshooting its target and never really reaches it. So we can assume that this pattern will continue forever and the ratio will reach its final value only after an infinite number of terms.

One more interesting property of this sequence is that this oscillating sequence converges to 1.618034... irrespective of the values of the first two terms. The first two terms need not be integers and they need not bear any relationship with each other, still they converge to the same ratio. As the ultimate ratios of consecutive terms are concerned, the first two terms seem to have no significance.

Mathematics behind Fibonacci sequence

Term number	Term value
1	a
2	b
3	a+b
4	a+2b
5	2a+3b
6	3a+5b

The coefficients of a and b are the terms of the basic Fibonacci sequence. If t_n is the nth term of the basic sequence and T_n is nth term of that of a, b sequence, we can see that $T_n = a \cdot t_{n-2} + b \cdot t_{n-1}$

Let the ratio of the consecutive terms are A, $A \cdot R$, and $A \cdot R^2$ where A is any number and R is the ratio to be calculated.

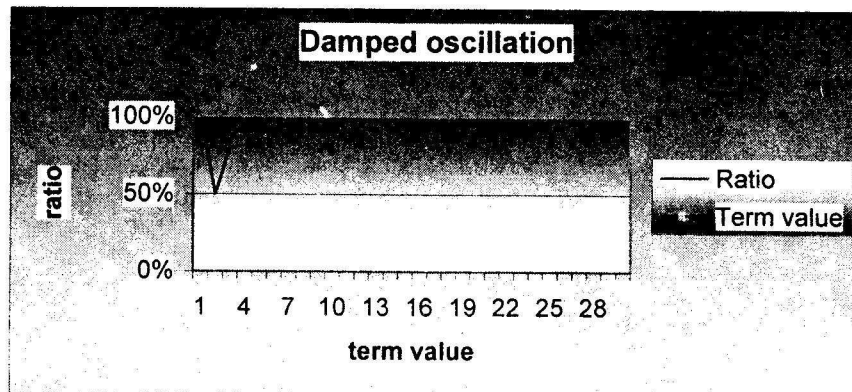
Therefore, $A \cdot R^2 = A + (A \cdot R)$, hence, $R^2 = 1 + R$. Solving this quadratic equation, we get $R = (1 + \sqrt{5})/2 = 1.618034...$, which is the same number that we observed as Fibonacci ratio.

Damped Oscillation of Fibonacci Ratios

We see that the ratios oscillate as in a damped oscillation around the final value i.e. 1.618033988749895...

Term value	Ratio	Term value	Ratio
1		987	0.6180344
1	1	1597	0.6180338
2	0.5	2584	0.618034
3	0.6666	4181	0.6180339
5	0.6	6765	0.6180339
8	0.625	10946	0.6180339
13	0.6153486	17711	0.6180339
21	0.6190476	28657	0.6180339
34	0.617647	46368	0.6180339
55	0.6181818	75025	0.6180339
89	0.6179775	121393	0.6180339

144	0.6180555	12146825	0.6180339
233	0.6180257	24286125	0.6180339
377	0.6180371	12389686	0.6180339
610	0.6180327	36675811	0.6180339



The Golden Section and the Golden Rectangle

A special value, closely related to the Fibonacci series, is called the golden section. This value is obtained by taking the ratio of successive terms in the Fibonacci numbers. If we plot a graph of these values, we will see that they seem to be tending to a limit. This limit is the positive square root of a quadratic equation and is called golden section or golden ratio.

Take the consecutive terms of the series, a , b , and $a+b$, then, $b/a = (a+b)/b = a/b + 1$

We define the golden section, ϕ , to be the limit of b/a , so

$$\phi = 1/\phi + 1$$

$$\phi^2 - \phi - 1 = 0$$

$$0 = (1 + \sqrt{5})/2 = 1.618034\dots$$

Rectangles that having the proportions of the golden rectangle are more beautiful than other rectangles. (We don't know why!). Many golden rectangles and golden sections appear in famous paintings, sculpture and architecture.

Reciprocal of the Fibonacci numbers:

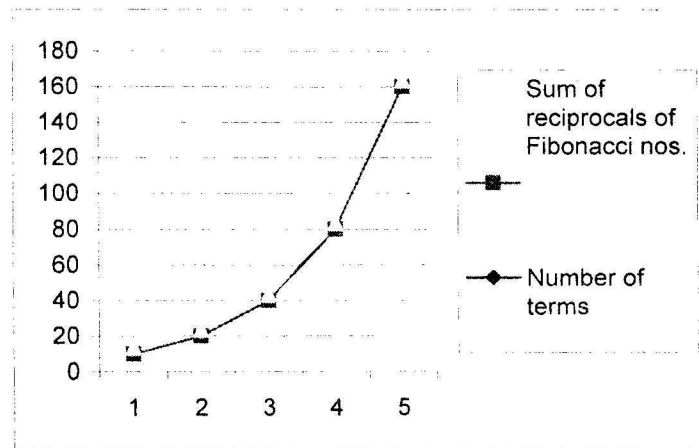
Can we convert the Fibonacci sequence into a series! A direct conversion will lead to a divergent series $1+1+2+3+5+\dots$ with a sum equals to infinity. However if we take the reciprocals of the terms of the basic Fibonacci sequence, we get

$$1/1 + 1/1 + 1/2 + 1/3 + 1/5 + 1/8 + \dots$$

Since the terms of this sequence are eventually in a geometric progression with the ratio 1.618034... the series should be convergent with a finite sum.

Number of terms	Sum of reciprocals of Fibonacci nos.
10	3.33046904763159
20	3.359646489010237
40	3.359885650431898
80	3.359885666243178
160	3.359885666243178
320	3.359885666243178

This sum appears to be an irrational number.



5. Applications of Fibonacci Numbers

The Fibonacci and Lucas numbers appear in numerous mathematical problems. There is an entire journal dedicated to such topics. Fibonacci Quarterly is a mathematical periodical devoted entirely to the subject of mathematics related to Fibonacci sequences. It covers topics involving many branches of mathematics, including linear algebra and probability.

In computer science, there is a data structure called Fibonacci heap, which is at the heart of many fast algorithms that manipulate graphs.

The golden section is denoted by the Greek letter *phi*. The Greek mathematicians recognized it as a significant value and Greek architects used the ratio 1:*phi* as an integral part of their designs, the most famous of which is the Parthenon in Athens.

Phi also occurs in geometry. For example, it is the ratio of the side of a regular pentagon to its diagonal. If we draw all the diagonals then they cut each other with the golden ratio too. The resulting pentagon describes a star, which forms part of many of the flags of the world.

The Fibonacci numbers are studied as part of number theory and have applications in the counting of mathematical objects such as sets, permutations and sequences.

The Fibonacci numbers appear in nature as well. The Fibonacci numbers are Nature's numbering system. They appear everywhere in Nature, from the leaf arrangements in plants, to the pattern of the florets of a flower, the bracts of a pinecone, or the scales of a pineapple. The Fibonacci numbers are therefore applicable to the growth of every living thing, including a single cell, a grain of wheat, a hive of bees, and even all of mankind.

Plants do not know about this sequence. They just grow in the most efficient ways. Many plants show the Fibonacci numbers in the arrangements of the leaves around the stem. Some pinecones and fir cones also show the numbers. Sunflower contains the number 89, or even 144. Palm trees show the numbers in the rings on their trunks. The florets in the head of a sunflower appear to have two systems of spirals, both beginning from the center. Counting these spirals on one particular sunflower shows that they are 55 in the clockwise and 34 in the counterclockwise direction. Other plants, which exhibit similar structure, include pineapples, daisies, cauliflowers, pinecones, ocotillos, and Joshua trees etc.

Nature is not trying to use the Fibonacci numbers: they are appearing as a by-product of a deeper physical process.

Probably most of us have never taken the time to examine very carefully the number or arrangements of petals on a flower. If we were to do so, we would find that the number of petals on a flower is a Fibonacci number.

Leonardo da Vinci painted Mona Lisa's face to fit perfectly into a golden rectangle, and structured the rest of the painting around similar rectangles.

The golden ratio retains a mystique, partly because excellent approximations of it turn up in many unexpected places in nature. The spiral inside a nautilus shell is remarkably close to the golden section, and the ratio of the lengths of the thorax and abdomen in most bees is nearly the golden ratio. Even a cross section of the most common form of human DNA fits nicely into a golden decagon. The golden ratio also appears in many unexpected contexts in mathematics, and they continue to spark interest in the mathematical community. The Fibonacci numbers have been found in many other areas as well. In the area of Physics, there are indications that the golden ratio and Fibonacci numbers are related to the structure of atoms and also to the spacing of planets in the solar system.

6. Fibonacci's Rabbits

The original problem that Fibonacci investigated in 1202 was about how fast rabbits could breed in ideal circumstances. Suppose a newly born pair of rabbits, one male and one female are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. Suppose our pair of rabbits never die and that the female always produces one new pair (one male, one female) every month from the second month on. Fibonacci posed a question:

How many pairs will there be in one year!

1. At the end of the first month, they mate, but there is still only 1 pair.
2. At the end of second month, the female produces a new pair, so now there are 2 pairs of rabbits in the field.
3. At the end of the third month the original female produces a second pair, making 3 pairs in the field.

4. At the end of the fourth month, the original female produces another new pair; the female born two months ago produces her first pair also, making 5 pairs.

The number of pairs of rabbits in the field at the start of each month is

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

This is nothing but Fibonacci sequence.

Now can we redefine Fibonacci's age-old sequence and how! If instead of bearing one pair of baby rabbits per month, the mature rabbits bear two pairs of baby rabbits per month! The answer is a fascinating sequence that has led to the development of multi-Fibonacci sequences of order q , where q is the number of rabbit pairs per family.

Take $q = 2$

Pair sequence	Month
1	1
1	2
3	3
5	4
11	5
21	6
43	7

Let us call this sequence the "new sequence"; each term can be obtained by adding the preceding term to twice the one before it, i.e.

$$T_n = T_{n-1} + 2T_{n-2}.$$

Using a similar process, sequences can be developed for situations when these multi-Fibonacci sequences when 3, 4, 5 and 6 rabbits per family are born.

F1		F2		F3		F4		F5	
n	T_n	n	U_n	n	V_n	n	W_n	n	Z_n
0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1	2	1
3	3	3	4	3	5	3	6	3	7
4	5	4	7	4	9	4	11	4	13
5	11	5	19	5	29	5	41	5	55
6	21	6	40	6	65	6	96	6	133
7	43	7	97	7	181	7	301	7	463

General Formula

For sequence F1: $T_{n-1} + 2T_{n-2}$

For sequence F2: $U_{n-1} + 3U_{n-2}$

For sequence F3: $V_{n-1} + 4V_{n-2}$

For sequence F4: $W_{n-1} + 5W_{n-2}$

For sequence F5: $Z_{n-1} + 6Z_{n-2}$.

Multi-Fibonacci Sequences are vast improvement over Fibonacci sequence. The unit digits of the Fibonacci sequence repeat every 60 terms. Multi-Fibonacci sequences have unit digit repeat periods.

Rabbit Pairs	Sequence	Unit Digit Repeat Period
1	Fibonacci	60
2	F1	04
3	F2	24
4	F3	06
5	F4	03
6	F5	20

Fibonacci type sequences develop from multiple rabbit births. These sequences have interesting properties of their own, which opens new avenues for future study.

Basic Program For Fibonacci Sequence

This program calculates the Fibonacci sequence starting with terms 1 and 1, up to 60 terms and then prints out every fifth term. The purpose is to show how the magnitude in Fibonacci sequence grows in a geometric progression.

```

T60= 1548008755920
10 CLS
15 a# = 0; b# = 1
20 FOR I = 1 TO 60
25 c# = b#: b# = a# + b#: a# = c#
30 IF i/5 = Int (i/5) THEN 40 ELSE 50
40 PRINT i, a#
50 NEXT

```

Conclusion

According to Bertrand Russell, "Mathematics, rightly viewed possesses not only truth, but supreme beauty. The mathematician's patterns must be beautiful; the ideas like the colors or the words must fit together in a harmonious way. Russell was thinking a highly abstract form of beauty, an inner beauty. It is a beauty of logical form and structure, of elegance of proof; a beauty, which can be glimpsed only after a long apprenticeship, has been served.

I have tried to provide brief applications of Fibonacci's Sequences. There are many applications of Fibonacci sequences in many other fields too.

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